GRADUATE STUDIES IN MATHEMATICS 244

Introduction to Complex Manifolds

John M. Lee

American Mathematical Society

10.1090/gsm/244

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Providence, Rhode Island

2020 *Mathematics Subject Classification*. Primary 32Qxx; Secondary 32-01, 32Q15, 53-01, 53C55, 53C56.

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Library of Congress Cataloging-in-Publication Data

Names: Lee, John M., 1950- author.
Title: Introduction to complex manifolds / John M. Lee.
Description: Providence, Rhode Island : American Mathematical Society, [2024] | Series: Graduate studies in mathematics, 1065-7339 ; volume 244 | Includes bibliographical references and index.
Identifiers: LCCN 2024011524 | ISBN 9781470476953 (hardcover) | ISBN 9781470477820 (paperback) | 9781470477813 (ebook)
Subjects: LCSH: Complex manifolds. | Manifolds (Mathematics) | AMS: Several complex variables and analytic spaces – Complex manifolds. | Several complex variables and analytic spaces – Instructional exposition (textbooks, tutorial papers, etc.). | Several complex variables and analytic spaces – Complex manifolds – Kähler manifolds. | Differential geometry – Instructional exposition (textbooks, tutorial papers, etc.). | Differential geometry – Global differential geometry – Hermitian and Kählerian manifolds. | Differential geometry – Global differential geometry – Other complex differential geometry.
Classification: LCC QA649 .L44 2024 | DDC 515/.946–dc23/eng/20240409
LC record available at https://lccn.loc.gov/2024011524

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Preface

Complex manifold theory is one of the most beautiful branches of geometry, in which algebraic topology, differential geometry, algebraic geometry, homological algebra, complex analysis, and partial differential equations (PDEs) come together in deep and surprising ways to create a rich theory. This book is meant to be an introduction to the concepts, techniques, and principal results about complex manifolds (mainly compact ones), aimed primarily at graduate students and researchers who already have a solid background in differential geometry.

What are complex manifolds? They are defined in exactly the same way as smooth manifolds, except the local coordinate charts are required to take their values in $\mathbb{C}^n$ and to overlap holomorphically. This might sound like a minor tweak to the definition of smooth manifolds, but in fact the requirement of holomorphicity changes everything. For example, on a connected compact complex manifold, the only global holomorphic functions are the constants, and the space of holomorphic sections of a holomorphic vector bundle is always finite-dimensional. Whereas every smooth manifold can be smoothly embedded in some Euclidean space, only certain complex manifolds can be holomorphically embedded in $\mathbb{C}^n$ or in complex projective space. There is a deep interplay between differential geometry and complex analysis, especially for Kähler manifolds, the ones on which the metric structure and the holomorphic structure play together nicely.

Complex manifolds have profound applications in many areas of mathematics. Here are a few examples:

- Riemann surfaces (1-dimensional complex manifolds) are essential for understanding global properties of holomorphic functions in one complex variable.
- Complex surfaces (2-dimensional complex manifolds) play a central role in attempts to classify 4-dimensional smooth manifolds.

- Complex manifolds defined by algebraic equations are among the central objects of interest in algebraic geometry, and the study of their differential geometry has contributed important advances in algebraic geometry.

- Calabi–Yau manifolds are complex manifolds that play a crucial role in string theory.

Because complex manifold theory is so intimately connected with algebraic geometry and complex analysis as well as differential geometry, there are many paths one might follow to get to know the subject. This book is about the differential geometry of complex manifolds, so the techniques and results in it are all firmly situated in differential geometry. Although I survey quite a few of the connections with other subjects, especially complex algebraic geometry, my choice of topics is heavily influenced by my desire to focus on techniques that will be familiar to those with a good background in differential geometry but not necessarily in commutative algebra or analysis of several complex variables. I have tried to include enough points of contact with algebraic geometry that the book can serve as a useful jumping-off point for differential geometers who decide to delve into the algebraic geometry literature, while at the same time serving to introduce algebraic geometers and complex analysts to a differential-geometric viewpoint on their subject.

I have chosen to use the Kodaira embedding theorem—which characterizes those compact complex manifolds that admit holomorphic embeddings into projective spaces—as a unifying theme for the book, because it draws on most of the important techniques in complex manifold theory and it illustrates one of the most profound differences between smooth manifolds and complex ones. Many of the definitions, theorems, and techniques introduced in the book are collected together to lay the groundwork for proving that profound theorem. But not everything is here for that purpose—I also hope to offer readers a strong general background that will prepare them for more advanced study in any aspect of complex geometry.

There are many other good introductory books on complex manifolds. Some excellent examples are [**Bal06**, **Dem12**, **Huy05**, **Mor07**, **Wel08**, **Zhe00**] and the first few chapters of [**GH94**]. What distinguishes this book is an approach that readers of my previous graduate texts will find familiar—instead of aiming for comprehensive coverage of all the aspects of the subject (which would be impossible in any case), I aim for two overarching goals: first, to make the explanations of definitions and concepts user-friendly, well motivated, and accessible; and second, to write the proofs with enough detail and rigor that students will hopefully not be left wondering how to bridge the gaps. This approach results in explanations that may be more wordy than some readers are used to; but in my experience it really helps beginners start to feel comfortable with a new subject.

Prerequisites

The main prerequisite is familiarity with the foundational results on topological, smooth, and Riemannian manifolds. Because this subject draws on so many of those results, there would be no point in trying to summarize all the requisite differential-geometric background here. All of the background material on manifolds that a reader needs to understand this book (and more) is contained in my three previous graduate textbooks [**LeeTM**, **LeeSM**, **LeeRM**], and I draw freely on them throughout this book (with specific references whenever appropriate).

Familiarity with elementary complex analysis is also a prerequisite, but only at the level of a typical undergraduate course on complex analysis in one variable. Any decent undergraduate complex analysis textbook will serve as a reference, such as [**BC13**, **MH98**, **Gam01**].

Beyond these subjects, the reader should also have a basic familiarity with algebraic topology—particularly singular homology and cohomology at the level of [**Hat02**] or [**Mun84**]. I give references for the main results that I use in the text.

For readers who are well versed in the prerequisite material, this book should be essentially self-contained, with one major exception: all of the results on Hodge theory rest on a fundamental Fredholm theorem for elliptic partial differential equations (Thm. 9.14), which is stated here without proof because developing the machinery for proving it would carry us too far afield into the weeds of PDE theory. The theorem is easy to state and easy to use, so readers can accept it on faith; or, for those who are curious about the proof, I offer several references where proofs can be found.

Exercises and Problems

Like my other graduate texts, this book includes both exercises (integrated into the text) and problems (collected at the ends of the chapters). The exercises are mostly routine verifications, and are there to provide the reader with opportunities to check how well they have digested the material; while the problems are mostly more difficult (some considerably so), and are designed to challenge the reader to grapple more deeply with the ideas in the text. As was the case with my previous books, I have not and do not intend to prepare written solutions to the problems or the exercises, because I do not want to deprive readers of the opportunity to get "stuck" on a problem and do the productive work of finding their own ways forward. In any case, most of these are not problems that have a single "right answer."

Typographical Conventions

This book generally follows the same typographical conventions as my previous graduate texts. Mathematical terms are typeset in ***bold italics*** when they are officially defined, to make them easy to spot on the page. The exercises in the text are indicated with the symbol ►, and numbered consecutively with the theorems to make them easy to find. The symbol □ marks the ends of proofs, and also marks the ends of statements of corollaries that follow so easily that they do not need proofs. The symbol // marks the ends of numbered examples. End-of-chapter problems are numbered 1-1, 1-2, 1-3, etc., with hyphens instead of dots, to make it easier to distinguish problem references from exercise references.

Acknowledgements

As always, I owe a lot to my students who have given me feedback on early drafts of this text, especially Shahriar Talebi. I also want to thank Jim Isenberg, who gave the whole book a close reading and contributed invaluable suggestions. Finally, I want to thank Ina Mette at the AMS, who has been wonderfully encouraging and patient as I have struggled to bring this project to fruition.

I welcome feedback from readers about any aspects of the book, especially if you find mistakes or unclear passages. There will be an updated list of corrections on my website. I hope you enjoy the book.

Seattle, Washington, USA

John M. Lee
johnmlee@uw.edu
www.math.washington.edu/~lee/

10.1090/gsm/244/01

Chapter 1

The Basics

If you are familiar with the prerequisites for reading this book, you are probably already familiar with the notion of a *smooth manifold*—a topological manifold equipped with an atlas of coordinate charts whose transition functions are all smooth. (Precise definitions will be found farther down in this chapter.)

You may also have encountered variations on that theme—different classes of manifolds that can be defined by modifying the compatibility condition for charts. For example, a ***C^k manifold*** is one equipped with an atlas whose transition functions are all of class C^k (meaning k times continuously differentiable), and a ***real-analytic manifold*** is one with an atlas whose transition functions are all real-analytic (meaning they are equal to the sum of a convergent power series in a neighborhood of each point).

Another variation on that theme, and the one to which this book is devoted, is a *complex manifold*—this is a topological manifold equipped with an atlas whose transition functions are all holomorphic. While the other classes of manifolds mentioned above are really just slight variations on the theme of smooth manifolds, it turns out that nearly everything changes when we move into the holomorphic category, as you will soon see. That is why the subject of complex manifolds is worth an entire book of its own.

In this chapter we introduce the main definitions, and describe some examples and basic properties of complex manifolds.

Definitions

The most basic type of manifold is a ***topological manifold***: this is a second-countable Hausdorff topological space with the property that every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ for some fixed n, called the

dimension of the manifold. (In this book, all manifolds are understood to be manifolds without boundary unless otherwise specified.)

By adding extra structure to a topological manifold, we can obtain other types of manifolds. Differential geometry is concerned primarily with ***smooth manifolds***, which are topological manifolds endowed with *smooth structures*, defined as follows: If M is a topological manifold of dimension n, a ***coordinate chart*** (often called just a ***chart***) for M is a pair (U, φ), where U is an open subset of M and φ is a homeomorphism from U to an open subset of $\mathbb{R}^n$. An ***atlas*** for M is a collection of charts whose domains cover M. Given two charts (U, φ) and (V, ψ) with overlapping domains, their ***transition functions*** are the composite maps $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ and their inverses $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$. Two charts are said to be ***smoothly compatible*** if their domains are disjoint or their transition functions are smooth as maps between open subsets of $\mathbb{R}^n$. (Here and throughout the book, ***smooth*** means infinitely differentiable or of class C^∞.) A ***smooth atlas*** for M is an atlas with the property that any two charts in the atlas are smoothly compatible with each other. Finally, a ***smooth structure*** for M is a smooth atlas that is ***maximal***, meaning that it is not properly contained in any larger smooth atlas; to say that $\mathscr{A}$ is a maximal smooth atlas just means that every chart that is smoothly compatible with every chart in $\mathscr{A}$ is already in $\mathscr{A}$.

The definition of a complex manifold is, at first glance, just a minor modification of the definition of smooth manifolds. The main change is that we require each transition function to be ***holomorphic***, meaning that it is continuous and each of its complex-valued component functions has complex partial derivatives with respect to each of the independent complex variables $z^1, \dots, z^n$. (We will explore properties of holomorphic functions in more depth below; for now, it suffices to know that they are smooth and that compositions of holomorphic functions are holomorphic.) To apply this requirement to the transition functions for topological manifolds, we choose the following standard identification between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$:

$$(x^1, y^1, \dots, x^n, y^n) \leftrightarrow (x^1 + iy^1, \dots, x^n + iy^n).$$

(As in [**LeeSM**] and [**LeeRM**], we index coordinate functions with upper indices to be consistent with the Einstein summation convention, described later in this chapter.) With this identification, it makes sense to ask whether a map between open subsets of $\mathbb{R}^{2n}$ is holomorphic.

Now suppose M is a $2n$-dimensional topological manifold. If (U, φ) and (V, ψ) are two coordinate charts for M, we say they are ***holomorphically compatible*** if $U \cap V = \varnothing$ or both transition functions are holomorphic under our standard identification of $\varphi(U \cap V)$ and $\psi(U \cap V)$ as open subsets of $\mathbb{C}^n$. A ***holomorphic atlas*** for M is an atlas with the property that any two charts in the atlas are holomorphically compatible with each other, and a ***holomorphic structure*** for M is a maximal holomorphic atlas. An ***n-dimensional complex manifold*** (or ***holomorphic manifold***) is a topological manifold of dimension $2n$ endowed with a given holomorphic structure. A complex manifold of dimension 1 is called a ***complex curve***, and one of dimension 2 is called a ***complex surface***. A complex manifold of dimension 3 or higher is sometimes called a ***complex threefold***, ***fourfold***, etc. When it is necessary to distinguish between the dimension of an n-dimensional complex manifold and the dimension of its underlying topological $2n$-manifold, we call n the ***complex dimension*** (denoted by $\dim_{\mathbb{C}} M$) and $2n$ the ***real dimension*** (denoted by $\dim_{\mathbb{R}} M$). Any one of the charts in the maximal holomorphic atlas is called a ***holomorphic coordinate chart***, and the complex-valued coordinate functions $(z^1, \dots, z^n)$ (where $z^j = x^j + iy^j$) are called ***holomorphic coordinates***. We denote the complex conjugate of z^j by $\bar{z}^j = x^j - iy^j$.

Holomorphic structures on manifolds are traditionally called ***complex structures***, but that term risks confusion with complex structures on vector bundles, to be discussed below.

Because all holomorphic functions are smooth (see Thm. 1.21 below), a holomorphic atlas is also a smooth atlas and thus determines a unique smooth structure on M; thus every complex manifold is also a smooth manifold in a canonical way. On the other hand, it is important to note that a given even-dimensional smooth manifold may have many different holomorphic structures that induce the given smooth structure (see Problem 1-4), or it may have none at all. The simplest example of an even-dimensional smooth manifold that carries no holomorphic structure is $\mathbb{S}^4$; see the discussion following Theorem 1.63 for more detail.

Proposition 1.1. *Let M be a topological manifold.*

(a) *Every holomorphic atlas $\mathcal{A}$ for M is contained in a unique maximal holomorphic atlas, called the* **holomorphic structure determined by $\mathcal{A}$**.

(b) *Two holomorphic atlases for M determine the same holomorphic structure if and only if their union is a holomorphic atlas.*

Proof. The proof is essentially identical to that of its smooth counterpart [**LeeSM**, Prop. 1.17]. □

To turn a set into a complex manifold using the definitions directly, it would be necessary to go through the separate steps of constructing a topology, verifying that it is a manifold, and then constructing a holomorphic structure for it. But in most cases the following shortcut can be used.

Lemma 1.2 (Complex Manifold Chart Lemma). *Let M be a set, and suppose we are given a collection $\{U_\alpha\}_{\alpha \in A}$ of subsets of M together with maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$, such that the following properties are satisfied:*

(i) *For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{C}^n$.*

(ii) *For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{C}^n$.*

(iii) *When* $U_\alpha \cap U_\beta \neq \varnothing$, *the map* $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ *is holomorphic.*

(iv) *Countably many of the sets* U_α *cover* M.

(v) *Whenever* p, q *are distinct points in* M, *either there exists some* U_α *containing both* p *and* q *or there exist disjoint sets* U_α, U_β *with* $p \in U_\alpha$ *and* $q \in U_\beta$.

Then M *has a unique structure as a complex manifold such that each* $(U_\alpha, \varphi_\alpha)$ *is a holomorphic chart.*

► **Exercise 1.3.** Prove this lemma by verifying that the proof of Lemma 1.35 of [**LeeSM**] goes through in this setting.

Some Examples

Before we go much further, we should have a few examples of complex manifolds to think about. We will introduce many more examples in Chapter 2.

Example 1.4 (Complex *n*-Space). It follows from Proposition 1.1(a) that $\mathbb{C}^n$ has a canonical holomorphic structure determined by the holomorphic atlas consisting of the single coordinate chart $(\mathbb{C}^n, \mathrm{Id}_{\mathbb{C}^n})$. Similarly, the canonical holomorphic structure on every open subset $U \subseteq \mathbb{C}^n$ is defined by the single chart (U, Id_U). When working with $\mathbb{C}$, $\mathbb{C}^n$, or their open subsets, we always use this holomorphic structure, typically without further comment. Here are some specific open subsets that will play important roles in what follows:

- For any $p \in \mathbb{C}^n$ and any $r > 0$, the ***(open) ball of radius r around p*** is the set $B_r(p) = \{z \in \mathbb{C}^n : |z - p| < r\}$, where $|\cdot|$ denotes the norm associated with the ***Euclidean inner product*** on $\mathbb{C}^n \approx \mathbb{R}^{2n}$, which can be written in complex coordinates as $\langle z, w\rangle = z \cdot \overline{w} = \sum_{j=1}^n z^j \overline{w}^j$. The ***unit ball*** of real dimension $2n$, denoted by $\mathbb{B}^{2n}$, is the open ball of radius 1 about the origin in $\mathbb{C}^n$.
- An open ball in $\mathbb{C}$ is called a ***disk***, and the notation is modified accordingly. Thus $D_r(p)$ represents the disk of radius r about $p \in \mathbb{C}$, and the ***unit disk*** is the disk $D_1(0)$, denoted by $\mathbb{D}$.
- A ***polydisk*** is a Cartesian product of open disks, that is, an open subset of the form $D_{r_1}(p^1) \times \cdots \times D_{r_n}(p^n) \subseteq \mathbb{C}^n$ for a point $p = (p^1, \dots, p^n) \in \mathbb{C}^n$ and positive real numbers $r_1, \dots, r_n$. When the radii are all equal, we use the notation $D_r^n(p)$ for the polydisk $D_r(p^1) \times \cdots \times D_r(p^n)$. //

Example 1.5 (Open Submanifolds). Somewhat more generally, if M is a complex n-manifold and U is an open subset of M, we can define a canonical holomorphic structure on U consisting of all holomorphic charts for M whose domains are contained in U. With this holomorphic structure, U is a complex n-manifold, called an ***open submanifold of M***. //

Example 1.6 (Complex Vector Spaces). If V is a finite-dimensional complex vector space, any choice of ordered basis $(b_1, \dots, b_n)$ defines an isomorphism $B: \mathbb{C}^n \to V$ by

$$B(z^1, \dots, z^n) = z^j b_j. \tag{1.1}$$

(Here and throughout the book, we use the ***Einstein summation convention***: each index name that appears twice in the same monomial term, once as an upper index and once as a lower one, is understood to be summed over all possible values of that index, typically from 1 to the dimension of the space. In formula (1.1), since V has dimension n, the implied summation is from 1 to n.) Interpreting B^{-1} as a global chart thus defines a holomorphic structure on V. Since the transition map between any two such charts is an invertible complex-linear transformation and therefore holomorphic along with its inverse, this structure is independent of the choice of basis. We will call this the ***standard holomorphic structure on V***. //

Example 1.7 (0-Manifolds). A topological 0-manifold is just a countable discrete space. Each point has a unique map to $\mathbb{C}^0 = \{0\}$, and the transition functions between these maps are vacuously holomorphic, so every 0-manifold has a canonical holomorphic structure. //

Example 1.8 (Product Manifolds). If $M_1, \dots, M_k$ are complex manifolds, their Cartesian product $M_1 \times \cdots \times M_k$ (with the product topology) is a complex manifold whose dimension is the sum of the dimensions of the factors, with products of holomorphic coordinate maps providing holomorphic coordinates. //

Example 1.9 (Complex Projective Spaces). The next examples are, after $\mathbb{C}^n$ itself, the most important complex manifolds of all. For any nonnegative integer n, we define the ***complex projective space of dimension n***, denoted by $\mathbb{CP}^n$, to be the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$, which we can identify with the quotient of $\mathbb{C}^{n+1} \smallsetminus \{0\}$ by the equivalence relation defined by $w \sim w'$ if and only if $w' = \lambda w$ for some nonzero complex number λ. We endow $\mathbb{CP}^n$ with the quotient topology. By this definition, $\mathbb{CP}^0$ is a single point.

We denote the equivalence class of a point $w = (w^0, w^1, \dots, w^n) \in \mathbb{C}^{n+1} \smallsetminus \{0\}$ by $[w] = [w^0, \dots, w^n]$. The complex numbers $(w^0, \dots, w^n)$ are traditionally called ***homogeneous coordinates*** of the point $[w]$; but be careful about using this terminology, because they are not actually coordinates in the usual sense. The same point $[w]$ is represented by any homogeneous coordinates of the form $(\lambda w^0, \dots, \lambda w^n)$ with $\lambda \neq 0$, so there is not a one-to-one correspondence between points and homogeneous coordinates, even in a small neighborhood of a point.

We can construct honest coordinates for $\mathbb{CP}^n$ as follows. For each $\alpha = 0, \dots, n$, let $U_\alpha \subseteq \mathbb{CP}^n$ be the open subset $U_\alpha = \{[w] \in \mathbb{CP}^n : w^\alpha \neq 0\}$, and define a map

$\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ by

$$\varphi_\alpha([w^0, \dots, w^n]) = \left(\frac{w^0}{w^\alpha}, \dots, \frac{w^{\alpha-1}}{w^\alpha}, \frac{w^{\alpha+1}}{w^\alpha}, \dots, \frac{w^n}{w^\alpha} \right).$$

It is continuous by the characteristic property of the quotient topology [**LeeTM**, Thm. 3.70], and it is a homeomorphism because it has a continuous inverse given by

$$\varphi_\alpha^{-1}(z^1, \dots, z^n) = [z^1, \dots, z^{\alpha-1}, 1, z^\alpha, \dots, z^n].$$

Thus each $(U_\alpha, \varphi_\alpha)$ is a coordinate chart, called ***affine coordinates for*** $\mathbb{CP}^n$. Coupled with the facts that $\mathbb{CP}^n$ is Hausdorff and second-countable (Exercise 1.10), this shows that $\mathbb{CP}^n$ is a topological manifold of real dimension $2n$. It is compact and connected, because it is the image of the surjective continuous map $q : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ given by $q(w^0, \dots, w^n) = [w^0, \dots, w^n]$, where $\mathbb{S}^{2n+1}$ is the set of unit vectors in $\mathbb{C}^{n+1}$.

For $\alpha < \beta$, the transition function between these charts can be computed explicitly as

$$\varphi_\alpha \circ \varphi_\beta^{-1}(z^1, \dots, z^n) = \left(\frac{z^1}{z^\alpha}, \dots, \frac{\widehat{z^\alpha}}{z^\alpha}, \dots, \frac{1}{z^\alpha}, \dots, \frac{z^n}{z^\alpha} \right),$$

where the hat indicates that the term in position α is omitted, and the $1/z^\alpha$ term is in position β; the formula for $\alpha > \beta$ is similar. These transition functions are all holomorphic, so they turn $\mathbb{CP}^n$ into a complex manifold of dimension n. //

► **Exercise 1.10.** Verify that $\mathbb{CP}^n$ is Hausdorff and second-countable.

Example 1.11 (Projectivization of a Vector Space). For some purposes, it is useful to construct projective spaces starting with different complex vector spaces in place of $\mathbb{C}^{n+1}$ itself. Suppose V is an n-dimensional complex vector space with $n > 0$. The ***projectivization of V***, denoted by $\mathbb{P}(V)$, is the set of 1-dimensional complex subspaces of V, endowed with the quotient topology obtained from the equivalence relation on $V \smallsetminus \{0\}$ given by $v_1 \sim v_2$ if $v_2 = \lambda v_1$ for some $\lambda \in \mathbb{C} \smallsetminus \{0\}$. A choice of basis for V yields an isomorphism $V \cong \mathbb{C}^n$ that descends to a bijection $\mathbb{P}(V) \to \mathbb{CP}^{n-1}$, which we can use to give $\mathbb{P}(V)$ the structure of a complex manifold. In the next chapter, we will see that the holomorphic structure obtained in this way is independent of the choice of basis (see Exercise 2.10). //

Example 1.12 (Complex Grassmannians). Suppose V is an n-dimensional complex vector space with $n > 0$, and k is a nonnegative integer less than or equal to n. Let $\mathrm{G}_k(V)$ be the set of k-dimensional complex-linear subspaces of V, called a ***complex Grassmannian***. (The case $k = 1$ is exactly the projective space $\mathbb{P}(V)$.) We can construct complex coordinates on $\mathrm{G}_k(V)$ as follows. Choose a subspace $P \subseteq V$ of dimension k and a complementary $(n-k)$-dimensional subspace Q, and write $V = P \oplus Q$. Then the graph of each complex-linear map $X : P \to Q$ is a

k-dimensional subspace $\Gamma(X) \subseteq V$, and every subspace whose intersection with Q is trivial is the graph of a unique such map. Let $U_Q \subseteq \mathrm{G}_k(V)$ denote the set of such subspaces. By choosing bases for P and Q, we obtain a bijection from U_Q to the vector space $\mathrm{M}((n-k) \times k, \mathbb{C})$ of complex $(n-k) \times k$ matrices, whose matrix entries we can use as coordinates on U_Q. The argument in [**LeeSM**, Example 1.36] (adapted in an obvious way to the complex case) shows that when two such charts overlap, the matrix entries in the new chart are rational functions of the original ones, so any two such charts overlap holomorphically. The arguments of that example also show that hypotheses (iv) and (v) of the chart lemma are satisfied, so $\mathrm{G}_k(V)$ is a complex manifold of dimension $(n-k)k$. Problem 1-5 shows that it is compact. //

Holomorphic Maps

We define holomorphic maps between complex manifolds in the same way as one defines smooth maps between smooth manifolds: if M and N are complex manifolds, a ***holomorphic map*** from M to N is a map $f : M \to N$ with the property that for every $p \in M$ there exist holomorphic coordinate charts (U, φ) for M and (V, ψ) for N whose domains contain p and $f(p)$, respectively, such that $f(U) \subseteq V$ and the composite map $\psi \circ f \circ \varphi^{-1}$ is holomorphic as a map from $\varphi(U)$ to $\psi(V)$. The function $\widehat{f} = \psi \circ f \circ \varphi^{-1}$ is called the ***coordinate representation of f*** with respect to the given holomorphic coordinates. As is the case in smooth manifold theory (see [**LeeSM**, pp. 15–16]), one often uses a coordinate map to temporarily identify an open subset of a manifold with an open subset of $\mathbb{C}^n$, and uses the same notation for a map and its coordinate representation.

When the codomain of a map f is $\mathbb{C}^k$ (or an open subset of $\mathbb{C}^k$) with its canonical holomorphic structure, we can always use the identity map as a holomorphic coordinate chart on $\mathbb{C}^k$, so being holomorphic is equivalent to the requirement that for each $p \in M$, there is a holomorphic chart (U, φ) for M whose domain contains p such that $f \circ \varphi^{-1}$ is holomorphic from $\varphi(U)$ to $\mathbb{C}^k$. It is standard practice to reserve the term ***holomorphic function*** for holomorphic maps whose codomains are open subsets of $\mathbb{C}$ (scalar-valued holomorphic functions) or $\mathbb{C}^k$ (vector valued holomorphic functions); the terms *holomorphic map* and *holomorphic mapping* can refer to maps between arbitrary complex manifolds.

If M is a complex manifold, the notation $\mathscr{O}(M)$ means the set of all holomorphic functions from M to $\mathbb{C}$. This applies, in particular, to any open submanifold of M: if $U \subseteq M$ is open, $\mathscr{O}(U)$ is the set of holomorphic functions from U to $\mathbb{C}$.

A bijective holomorphic map with holomorphic inverse is called a ***biholomorphism***, and a biholomorphism from a complex manifold to itself is called an ***automorphism***. More generally, a map $F : M \to N$ is called a ***local biholomorphism*** if every $p \in M$ has a neighborhood U such that $F|_U$ is a biholomorphism onto an open subset of N.

The following facts about holomorphic maps are proved just like their smooth analogues [**LeeSM**, Props. 2.6 and 2.10 and Example 2.14(b)].

Proposition 1.13.

(a) *The restriction of a holomorphic map to an open subset is holomorphic.*

(b) *If a map f has the property that each point in the domain has a neighborhood U on which the restriction $f|_U$ is holomorphic, then f is holomorphic.*

(c) *Every constant map between complex manifolds is holomorphic.*

(d) *The identity map of every complex manifold is holomorphic.*

(e) *The inclusion map of every open submanifold is holomorphic.*

(f) *Every holomorphic coordinate chart is a biholomorphism onto its image.*

(g) *Every composition of holomorphic maps between complex manifolds is holomorphic.*

Two complex manifolds are said to be ***biholomorphic*** if there is a biholomorphism between them. For example, if V is an n-dimensional complex vector space, any choice of basis determines a complex-linear isomorphism between V and $\mathbb{C}^n$, so all such vector spaces are biholomorphic to $\mathbb{C}^n$. Similarly, a choice of basis yields a biholomorphism between $\mathbb{P}(V)$ and $\mathbb{CP}^{n-1}$, and between $\mathrm{G}_k(V)$ and $\mathrm{G}_k(\mathbb{C}^n)$ for each k. It is easy to check that being biholomorphic is an equivalence relation on the class of all complex manifolds. The main subject matter of this book is properties of complex manifolds that are preserved by biholomorphisms.

Because holomorphic maps are smooth, biholomorphic manifolds are automatically diffeomorphic. However, the converse might not be true: Example 1.31 and Problem 1-4 describe complex manifolds that are diffeomorphic but not biholomorphic.

Covering Manifolds and Quotient Manifolds

In this section, we discuss some ways to produce new complex manifolds from old ones. Recall that a ***covering map*** is a surjective continuous map $\pi : M \to N$ between connected and locally path-connected topological spaces such that every point of N has a neighborhood U that is ***evenly covered***, meaning that $\pi^{-1}(U)$ is a disjoint union of connected open subsets each of which is mapped homeomorphically onto U by π. A covering map $\pi : M \to N$ is said to be ***normal*** if for some $x \in M$, the induced subgroup $\pi_*\big(\pi_1(M, x)\big) \subseteq \pi_1(N, \pi(x))$ is a normal subgroup (meaning it is invariant under conjugation). Equivalently, π is normal if the group of covering automorphisms (homeomorphisms $\varphi : M \to M$ satisfying $\pi \circ \varphi = \pi$) acts transitively on each fiber $\pi^{-1}(y)$. A discussion of the properties of covering maps can be found in [**LeeTM**, Chaps. 11 & 12].

Suppose $\pi : M \to N$ is a covering map. If M and N are smooth manifolds and π is a local diffeomorphism, then it is called a ***smooth covering map***. Properties of smooth covering maps are discussed in [**LeeSM**, pp. 91–95]. Similarly, if M and N are complex manifolds and π is a local biholomorphism, it is called a ***holomorphic covering map***.

► **Exercise 1.14.** Suppose $\pi : M \to N$ is a holomorphic covering map. Show that every point of M is in the image of a ***holomorphic local section*** of π, that is, a holomorphic map $\sigma : U \to M$ defined on an open subset $U \subseteq N$ such that $\pi \circ \sigma = \mathrm{Id}_U$.

The next proposition shows that every covering space of a connected complex manifold is a complex manifold in a natural way.

Proposition 1.15 (Coverings of Complex Manifolds are Complex Manifolds). *Suppose M is a connected complex manifold and $\pi : E \to M$ is a (topological) covering map. Then E is a topological manifold and has a unique holomorphic structure such that π is a holomorphic covering map.*

Proof. Proposition 4.40 in [**LeeSM**] shows that E is a topological manifold and has a unique smooth structure such that π is a smooth covering map. We can define holomorphic charts on E as follows: Given a point $p \in E$, let U be an evenly covered neighborhood of $\pi(p)$. After shrinking U if necessary, we can find a holomorphic coordinate map $\varphi : U \to \mathbb{C}^n$. Let $\widetilde{U}$ be the connected component of $\pi^{-1}(U)$ containing p, and define $\widetilde{\varphi} = \varphi \circ \pi : \widetilde{U} \to \mathbb{C}^n$. The argument in the proof of [**LeeSM**, Prop. 4.40] shows that when two such charts $\big(\widetilde{U}, \widetilde{\varphi}\big)$ and $\big(\widetilde{V}, \widetilde{\psi}\big)$ overlap, in a neighborhood of each point the transition function can be expressed as $\widetilde{\psi}^{-1} \circ \widetilde{\varphi}^{-1} = \psi^{-1} \circ \varphi^{-1}$, which in this case is holomorphic. Then π is a local biholomorphism because its coordinate representation is the identity with respect to the holomorphic coordinates $\big(\widetilde{U}, \widetilde{\varphi}\big)$ on E and (U, φ) on M.

If $\widetilde{E}$ is the same topological space E with another holomorphic structure such that $\pi : \widetilde{E} \to M$ is a holomorphic covering map, then because π is a local biholomorphism, each of the charts constructed above must be a holomorphic chart for $\widetilde{E}$, so the holomorphic structure of $\widetilde{E}$ is the same as the one constructed above. □

Under certain circumstances, we can also put holomorphic structures on manifolds *covered by* complex manifolds. Suppose Γ is a ***discrete Lie group*** (i.e., a countable group with the discrete topology). Recall that an action of Γ on a manifold M is ***free*** if $g \cdot x = x$ for some $g \in \Gamma$ and $x \in M$ implies g is the identity; and it is ***proper*** if the map $\Gamma \times M \to M \times M$ given by $(g, x) \mapsto (g \cdot x, x)$ is a proper map, meaning that the preimage of every compact set is compact. (See [**LeeSM**, pp. 543–544].) If M is a complex manifold, the action is ***holomorphic*** if the map $x \mapsto g \cdot x$ is holomorphic for each $g \in \Gamma$.

Theorem 1.16 (Holomorphic Quotient Manifold Theorem). *Suppose Γ is a discrete Lie group acting holomorphically, freely, and properly on a complex manifold M. Then the quotient space M/Γ has a unique complex manifold structure such that the quotient map $q: M \to M/\Gamma$ is a holomorphic normal covering map.*

Proof. Smooth manifold theory shows that M/Γ has a unique smooth manifold structure such that q is a smooth normal covering map [**LeeSM**, Thm. 21.13]. To define a complex manifold structure on M/Γ, let $U \subseteq M/\Gamma$ be any evenly covered open set, and choose a smooth local section $\sigma: U \to M$. Because M is a complex manifold, $\sigma(U)$ has a covering by holomorphic charts $(U_\alpha, \varphi_\alpha)$, and for each such chart we can define $(\sigma^{-1}(U_\alpha), \varphi_\alpha \circ \sigma))$ as a chart for M/Γ. For a fixed local section σ, all of these charts are holomorphically compatible with each other. If $\widetilde{\sigma}: U \to M$ is any other local section, there is an element $g \in \Gamma$ such that $\widetilde{\sigma}(x) = g \cdot \sigma(x)$ for all $x \in U$; and the fact that $x \mapsto g \cdot x$ is a biholomorphism of M with inverse $x \mapsto g^{-1} \cdot x$ guarantees that the charts obtained from $\widetilde{\sigma}$ will be holomorphically compatible with those obtained from σ. □

A ***complex Lie group*** is a complex manifold G endowed with a group structure such that the multiplication map $m: G \times G \to G$ and the inversion map $i: G \to G$ are holomorphic. Here are some simple examples; we will see more in the next chapter (see Example 2.26).

- Every countable discrete group is a 0-dimensional complex Lie group.
- Every finite-dimensional complex vector space is a complex Lie group under addition.
- The group $\mathrm{GL}(n, \mathbb{C})$ of invertible $n \times n$ complex matrices is a complex Lie group of dimension n^2, with the matrix entries as global holomorphic coordinates. The component functions of the multiplication map are holomorphic polynomials in the matrix entries, and those of the inversion map are holomorphic rational functions.
- Given any n-dimensional complex vector space V, the group $\mathrm{GL}(V)$ of complex linear automorphisms of V becomes a Lie group isomorphic to $\mathrm{GL}(n, \mathbb{C})$ once we choose a basis for V, and the resulting holomorphic structure is independent of the choice of basis.

Corollary 1.17. *Suppose G is a connected complex Lie group and $\Gamma \subseteq G$ is a discrete subgroup. The left coset space G/Γ is a complex manifold, and the quotient map $\pi: G \to G/\Gamma$ is a holomorphic normal covering map. If Γ is also a normal subgroup, then G/Γ is a complex Lie group and π is a group homomorphism.*

Proof. The left coset space G/Γ is the quotient of G by the action of Γ by right translation. This action is holomorphic by the definition of a complex Lie group, and the proof of Theorem 21.17 in [**LeeSM**] shows that it is free and proper. Thus

Theorem 1.16 above shows that G/Γ has the structure of a complex manifold and π is a holomorphic normal covering map.

If Γ is a normal subgroup, then elementary group theory shows that G/Γ is a group and π is a homomorphism. To see that the group operations in G/Γ are holomorphic, just note that given any pair of points $p, q \in G/\Gamma$, we can choose neighborhoods U of p and V of q on which there exist holomorphic local sections $\sigma : U \to G$ and $\tau : V \to G$. Then the multiplication map $\widetilde{m} : G/\Gamma \times G/\Gamma \to G/\Gamma$ can be written in a neighborhood of (p, q) as $\pi \circ m \circ (\sigma \times \tau)$:

$$\begin{array}{ccc} G \times G & \xrightarrow{\;m\;} & G \\ {\scriptstyle \sigma\times\tau}\big\uparrow & & \big\downarrow{\scriptstyle \pi} \\ G/\Gamma \times G/\Gamma & \xrightarrow[\widetilde{m}]{} & G/\Gamma. \end{array}$$

This is a composition of holomorphic maps and thus holomorphic. A similar argument applies to inversion. □

Example 1.18 (Complex Tori). Suppose V is an n-dimensional complex vector space, considered as an abelian complex Lie group. A ***lattice*** in V is a discrete additive subgroup $\Lambda \subseteq V$ generated by $2n$ vectors $v_1, \dots, v_{2n}$ that are linearly independent over $\mathbb{R}$. Corollary 1.17 shows that V/Λ is an n-dimensional complex Lie group, called a ***complex torus***. When $n = 0$, it is just a single point. When $n > 0$, the real-linear isomorphism $A : \mathbb{R}^{2n} \to V$ given by $A\left(x^1, \dots, x^{2n}\right) = x^j v_j$ descends to a diffeomorphism from $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ to V/Λ; since $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is diffeomorphic to the $2n$-torus $\mathbb{S}^1 \times \dots \times \mathbb{S}^1$, so is V/Λ. Thus the complex tori defined by different lattices are all diffeomorphic to each other. They are typically not biholomorphic, however; see Problem 1-4 for an example. //

Example 1.19 (Hopf Manifolds). Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an ordered n-tuple of real numbers with $0 < \lambda_j < 1$, and define an action of $\mathbb{Z}$ on $\mathbb{C}^n \smallsetminus \{0\}$ by $k \cdot z = ((\lambda_1)^k z^1, \dots, (\lambda_n)^k z^n)$. This action is holomorphic, free, and proper, so the quotient $H_\lambda = (\mathbb{C}^n \smallsetminus \{0\})/\mathbb{Z}$ is an n-dimensional complex manifold called a ***Hopf manifold***. Regarding $\mathbb{S}^{2n-1}$ as the set of unit vectors in $\mathbb{C}^n$, we define a smooth map $A : \mathbb{S}^{2n-1} \times \mathbb{R} \to \mathbb{C}^n \smallsetminus \{0\}$ by $A(z, t) = ((\lambda_1)^t z^1, \dots, (\lambda_n)^t z^n)$; if $\pi : \mathbb{C}^n \smallsetminus \{0\} \to H_\lambda$ is the quotient map, one can check that $\pi \circ A$ makes the same identifications as the quotient map from $\mathbb{S}^{2n-1} \times \mathbb{R}$ to $\mathbb{S}^{2n-1} \times (\mathbb{R}/\mathbb{Z}) \approx \mathbb{S}^{2n-1} \times \mathbb{S}^1$, so all Hopf manifolds are diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. //

Example 1.20 (Iwasawa Manifolds). Consider the subgroup $G \subseteq \mathrm{GL}(3, \mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix}$$

for $z^1, z^2, z^3 \in \mathbb{C}$. It is a complex Lie group, biholomorphic to $\mathbb{C}^3$, with multiplication given by

$$(z^1, z^2, z^3) \cdot (w^1, w^2, w^3) = (z^1 + w^1, z^2 + w^2, z^3 + w^3 + z^1 w^2).$$

For a discrete subgroup $\Gamma \subseteq G$, the left coset space G/Γ is a complex 3-manifold by Corollary 1.17. An ***Iwasawa manifold*** is a left coset space of the form G/Γ for a discrete subgroup Γ that is ***cocompact***, meaning that G/Γ is compact. (Some authors use quotients by left Γ-actions in their definitions, corresponding to right coset spaces; group inversion in G induces a biholomorphism between the left and right coset spaces, so there is no real difference.) The simplest example is the ***standard Iwasawa manifold***, obtained by taking Γ to be the subgroup consisting of matrices in which z^1, z^2, z^3 are ***Gaussian integers***, that is, complex numbers of the form $m + ni$ for $m, n \in \mathbb{Z}$. It is cocompact by the result of Problem 1-1. //

Some Complex Analysis

In this book, we assume you are familiar with basic undergraduate-level complex analysis in one variable; if your complex analysis is rusty, this would be a good time to review. (Some suggested texts are listed in the Preface.)

Recall the definition of a holomorphic function of one complex variable: if $W \subseteq \mathbb{C}$ is an open subset and $f : W \to \mathbb{C}$ is a function, then f is said to be ***holomorphic*** if it has a complex derivative at each point $p \in W$, defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Holomorphic functions are sometimes called ***complex-analytic***, or just ***analytic*** if there can be no confusion with real-analytic functions.

For convenience, let us recall some basic facts from the one-variable theory. In these statements, W represents an arbitrary open subset of $\mathbb{C}$ and $f : W \to \mathbb{C}$ is an arbitrary holomorphic function. We write the standard coordinate on $\mathbb{C}$ as $z = x + iy$.

- Cauchy Integral Formula: If $a \in W$ and $r > 0$ is chosen so that the closed disk $\overline{D}_r(a)$ is contained in W, then the following formula holds for all z in the open disk $D_r(a)$:

$$h(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{h(\zeta)}{\zeta - z}\, d\zeta.$$

- Cauchy–Riemann Equations: The real and imaginary parts u and v of f satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Power Series Expansion: For each $a \in W$, if the disk $D_r(a)$ is contained in W, then $f|_D$ is equal to a convergent series in powers of $(z-a)$. It has complex derivatives of all orders, which may be computed by differentiating the series term-by-term.
- Zeros Are Isolated and Have Finite Order: If $f(a) = 0$ for some $a \in W$ and f is not identically zero, then there is a disk $D_r(a) \subseteq W$ such that $f(z) \neq 0$ for $z \in D_r(a) \smallsetminus \{a\}$; and there is a positive integer m (called the ***order*** or ***multiplicity*** of the zero), such that $f(z) = (z-a)^m h(z)$ for some holomorphic function h that does not vanish at a. The order of a zero is equal to the smallest integer m such that $f^{(m)}(a) \neq 0$. A zero of order 1 is called a ***simple zero***.
- Maximum Principle: If W is connected and $|f(z)|$ attains a maximum at a point $z \in W$, then f is constant.
- Liouville's Theorem: If $W = \mathbb{C}$ and f is bounded, then it is constant.
- Riemann's Removable Singularity Theorem: If $W = \widetilde{W} \smallsetminus \{a\}$ for some open set $\widetilde{W}$ and some point $a \in \widetilde{W}$, and f is bounded, then f extends to a holomorphic function on all of $\widetilde{W}$.

For our study of complex manifolds, we need to extend some of the results of the one-variable theory to functions of several complex variables. Many of these results will look familiar, but some properties of holomorphic functions are decidedly different in higher dimensions.

We begin with the official definition of holomorphic functions of several variables. Suppose $U \subseteq \mathbb{C}^n$ is an open subset and $f : U \to \mathbb{C}$. For $p = (p^1, \dots, p^n) \in U$ and $j \in \{1, \dots, n\}$, we say f has a ***complex partial derivative at p with respect to z^j*** if the following limit exists:

$$\frac{\partial f}{\partial z^j}(p) = \lim_{h \to 0} \frac{f(p^1, \dots, p^j + h, \dots, p^n) - f(p^1, \dots, p^n)}{h}, \tag{1.2}$$

where the limit is taken over all h in some punctured disk centered at the origin in $\mathbb{C}$. Such a function is said to be ***holomorphic*** if it is continuous and has a complex partial derivative with respect to each variable $z^1, \dots, z^n$ at each point of U. More generally, a vector-valued function $F : U \to \mathbb{C}^k$ is said to be holomorphic if each of its component functions is holomorphic.

Our definition of holomorphic functions is essentially the same as the one-variable definition, except in that case the assumption of continuity is not needed because a simple argument shows that continuity follows from the existence of a complex derivative. It is worth noting, in fact, that the continuity assumption is actually not needed in higher dimensions either: the German mathematician Friedrich

Hartogs proved in 1906 [**Har06**] that a function that has complex partial derivatives at every point of an open subset of $\mathbb{C}^n$ is automatically continuous. That proof (which can be found in [**Kra01**, Section 2.4]) is difficult, though, so it is much more convenient simply to assume continuity as part of our definition.

In one complex variable, there are several equivalent ways to characterize holomorphic functions: having a complex derivative everywhere, or having continuous partial derivatives that satisfy the Cauchy–Riemann equations, or being the sum of a convergent power series in a neighborhood of each point. There are similar equivalent characterizations for holomorphic functions of several variables.

Theorem 1.21. *Let $U \subseteq \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$. The following are equivalent.*

(a) *f is holomorphic (i.e., it is continuous and has a complex partial derivative with respect to each variable at each point of U).*

(b) *f is smooth and satisfies the following Cauchy–Riemann equations:*

$$\frac{\partial u}{\partial x^j} = \frac{\partial v}{\partial y^j}, \qquad \frac{\partial u}{\partial y^j} = -\frac{\partial v}{\partial x^j}, \qquad j = 1, \dots, n,$$

where $z^j = x^j + i y^j$ and $f(z) = u(z) + iv(z)$.

(c) *For each $p = (p^1, \dots, p^n) \in U$, there exists a neighborhood of p in U on which f is equal to the sum of an absolutely convergent power series of the form*

$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} a_{k_1\dots k_n}(z^1 - p^1)^{k_1} \cdots (z^n - p^n)^{k_n}. \tag{1.3}$$

Remarks.

- In the decomposition $f(z) = u(z) + iv(z)$ in part (b), it is understood that $u(z)$ and $v(z)$ are real. The same applies to $z^j = x^j + i y^j$ and everywhere in the book when we write such a decomposition, unless otherwise specified.
- In (c), the reason we insist on absolute convergence is that a sum over multiple indices can be ordered in various ways, and absolute convergence ensures that the ordering of terms does not matter.

Proof. We will prove (a) ⇔ (b) and (a) ⇔ (c).

Suppose f satisfies (a). Because f is holomorphic in each variable separately, the one-variable theory shows that it satisfies the Cauchy–Riemann equations with respect to each variable. To show that it is smooth, given $p \in U$, choose $r > 0$ such that the closed polydisk $\overline{D}_r^n(p)$ is contained in U. Because f is holomorphic in each variable separately, we can apply the single-variable version of the Cauchy integral formula repeatedly to obtain the following for all $z \in D^n_r(p)$:

$$\begin{aligned}
&f(z^1,\dots,z^n)\\
&= \frac{1}{2\pi i}\int\limits_{|\zeta^n-p^n|=r}\frac{f(z^1,\dots,z^{n-1},\zeta^n)}{\zeta^n-z^n}\,d\zeta^n\\
&= \frac{1}{(2\pi i)^2}\int\limits_{|\zeta^n-p^n|=r}\ \int\limits_{|\zeta^{n-1}-p^{n-1}|=r}\frac{f(z^1,\dots,\zeta^{n-1},\zeta^n)}{(\zeta^n-z^n)(\zeta^{n-1}-z^{n-1})}\,d\zeta^{n-1}d\zeta^n\\
&\ \vdots\\
&= \frac{1}{(2\pi i)^n}\int\limits_{|\zeta^n-p^n|=r}\cdots\int\limits_{|\zeta^1-p^1|=r}\frac{f(\zeta^1,\dots,\zeta^n)}{(\zeta^n-z^n)\cdots(\zeta^1-z^1)}\,d\zeta^1\cdots d\zeta^n.
\end{aligned} \tag{1.4}$$

Since the domain of integration is compact and the integrand is continuous in all variables and smooth as a function of (the real and imaginary parts of) $z^1,\dots,z^n$, we can differentiate under the integral sign as often as we like with respect to x^j and y^j to conclude that f is smooth. This proves (b).

To prove that f also satisfies (c), note that

$$\frac{1}{\zeta^j-z^j}=\frac{1}{(\zeta^j-p^j)-(z^j-p^j)}=\frac{1}{\zeta^j-p^j}\,\frac{1}{1-\left(\dfrac{z^j-p^j}{\zeta^j-p^j}\right)},$$

and since $|z^j-p^j|/|\zeta^j-p^j|<1$ on the domain of integration in (1.4), we can expand the last fraction on the right in a power series to obtain

$$\frac{1}{\zeta^j-z^j}=\frac{1}{\zeta^j-p^j}\sum_{k=0}^{\infty}\left(\frac{z^j-p^j}{\zeta^j-p^j}\right)^k,$$

which converges uniformly and absolutely for z^j in any closed disk $\overline{D}_{r'}(p^j)$ with $0<r'<r$ by comparison with the geometric series $\sum_k (r'/r)^k$. Inserting this formula for each variable into (1.4), we conclude that f satisfies (1.3) with coefficients

$$a_{k_1\dots k_n}=\int\limits_{|\zeta^n-p^n|=r}\cdots\int\limits_{|\zeta^1-p^1|=r}\frac{f(\zeta^1,\dots,\zeta^n)}{(\zeta^n-p^n)^{k_n+1}\cdots(\zeta^1-p^1)^{k_1+1}}\,d\zeta^1\cdots d\zeta^n.$$

This completes the proof that (a) ⇒ (b) and (a) ⇒ (c).

Conversely, if f satisfies (b), then it is certainly continuous, and the one-variable theory implies that it has a complex derivative with respect to each variable, so it also satisfies (a).

Finally, assume f satisfies (c), and let $p \in U$ be arbitrary. There is some closed polydisk $\overline{D}^n_r(p)$ contained in U and centered at p on which the series converges absolutely. Because the series converges at $z_0=(p^1+r,\dots,p^n+r)$, the terms in the series for $f(z_0)$ are all uniformly bounded, which means there is a constant C such that

$$|a_{k_1\ldots k_n}| r^{k_1} \cdots r^{k_n} \le C.$$

On a polydisk $D^n_{r'}(p)$ for any $0 < r' < r$, the terms of the series satisfy the following bound:

$$\begin{aligned}(1.5)\qquad \sum_{k_1,\ldots,k_n=0}^{\infty} \left| a_{k_1\ldots k_n}(z^1 - p^1)^{k_1} \cdots (z^n - p^n)^{k_n} \right| \\ \le \sum_{k_1,\ldots,k_n=0}^{\infty} C \left(\frac{r'}{r}\right)^{k_1} \cdots \left(\frac{r'}{r}\right)^{k_n},\end{aligned}$$

and the series on the right is an iterated convergent geometric series. Therefore, the series for f converges uniformly and absolutely on $D^n_{r'}(p)$ by the Weierstrass M-test, so f is continuous there, and in particular at p. If we fix all the variables but z^j, we obtain a convergent power series in z^j, which is therefore holomorphic in z^j by the one-variable theory, thus proving (a). □

Next we enumerate the basic properties of holomorphic functions that we will use throughout the book.

Proposition 1.22 (Compositions of Holomorphic Functions are Holomorphic). *Suppose $Z \subseteq \mathbb{C}^m$ and $W \subseteq \mathbb{C}^n$ are open subsets and $f : Z \to W$ and $g : W \to \mathbb{C}^k$ are holomorphic functions. Then $g \circ f : Z \to \mathbb{C}^k$ is holomorphic.*

Proof. Certainly $g \circ f$ is smooth, so we just need to check that it satisfies the Cauchy–Riemann equations. Let us write the variables in Z as $z^j = x^j + iy^j$, those in W as $w^j = u^j + iv^j$, and the component functions of f and g as $f^k(z) = U^k(z) + iV^k(z)$, $g^l(w) = A^l(w) + iB^l(w)$. Applying the real-variable chain rule (and using the summation convention), we find

$$\frac{\partial(A^l \circ f)}{\partial x^j} - \frac{\partial(B^l \circ f)}{\partial y^j} = \frac{\partial A^l}{\partial u^k}\frac{\partial U^k}{\partial x^j} + \frac{\partial A^l}{\partial v^k}\frac{\partial V^k}{\partial x^j} - \frac{\partial B^l}{\partial u^k}\frac{\partial U^k}{\partial y^j} - \frac{\partial B^l}{\partial v^k}\frac{\partial V^k}{\partial y^j}.$$

Using the Cauchy–Riemann equations for g to replace $\partial A^l/\partial u^k$ by $\partial B^l/\partial v^k$ and $\partial A^l/\partial v^k$ by $-\partial B^l/\partial u^k$ and then applying the Cauchy–Riemann equations for f, we see that this expression is identically zero. A similar computation shows that the composition also satisfies the other set of Cauchy–Riemann equations. □

Proposition 1.23. *Suppose $f, g : U \to \mathbb{C}$ are holomorphic functions on an open subset $U \subseteq \mathbb{C}^n$. Then $f + g$, $f - g$, and fg are holomorphic on U, and f/g is holomorphic on $U \smallsetminus g^{-1}(0)$.*

► **Exercise 1.24.** Prove this proposition.

It follows easily from the two preceding propositions, for example, that all polynomial functions of $z^1, \dots, z^n$ are holomorphic on $\mathbb{C}^n$, and all rational functions (quotients of polynomials) are holomorphic wherever their denominators are nonzero.

Our next proposition relates partial derivatives with respect to complex variables to those with respect to real variables. If $f = u + iv$ is a complex-valued function, the notation $\partial f/\partial x^j$ denotes the complex-valued function $\partial u/\partial x^j + i\partial v/\partial x^j$, and similarly with y^j derivatives.

Proposition 1.25. *Suppose U is an open subset of $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ is holomorphic. Writing $z^j = x^j + iy^j$ for $j = 1, \dots, n$, we have*

$$\frac{\partial f}{\partial z^j} = \frac{\partial f}{\partial x^j} = \frac{1}{i}\frac{\partial f}{\partial y^j}. \tag{1.6}$$

Proof. Note that the existence of the limit in (1.2) as h approaches zero through all complex values implies that we obtain the same limit if we restrict h to approach zero through real values only or imaginary values only. Thus for any $p \in U$,

$$\frac{\partial f}{\partial z^j}(p) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(p^1, \dots, p^j + h, \dots, p^n) - f(p^1, \dots, p^n)}{h} = \frac{\partial f}{\partial x^j}(p),$$

$$\frac{\partial f}{\partial z^j}(p) = \lim_{\substack{k\to 0\\ k\in\mathbb{R}}} \frac{f(p^1, \dots, p^j + ik, \dots, p^n) - f(p^1, \dots, p^n)}{ik} = \frac{1}{i}\frac{\partial f}{\partial y^j}(p). \qquad \square$$

Next we establish some important properties of multivariable power series.

Proposition 1.26. *Suppose f is a holomorphic function given by an absolutely convergent power series of the form* (1.3) *on a polydisk $D^n_r(p) \subseteq \mathbb{C}^n$. The complex partial derivatives of f of all orders exist and are given by absolutely convergent power series on the same polydisk, which can be computed by differentiating the series term by term.*

Proof. For any $0 < r' < r_1 < r$, the series converges absolutely on $\overline{D}^n_{r_1}(p)$, and thus the proof of Theorem 1.21 shows that it converges uniformly and absolutely on $D^n_{r'}(p)$. Note that the complex derivative $\partial f/\partial z^j$ is equal to the real partial derivative $\partial f/\partial x^j$ by Proposition 1.25. A standard result in real analysis [**Rud76**, Thm. 7.17] shows that we can differentiate the power series term by term with respect to x^j on $D^n_{r'}(p)$ provided the differentiated series converges uniformly there.

With notation as in (1.5), the differentiated series satisfies

$$\begin{aligned}
&\sum_{k_1,\dots,k_n=0}^{\infty} \left| \frac{\partial}{\partial x^j} \left(a_{k_1\dots k_n} (z^1 - p^1)^{k_1} \cdots (z^n - p^n)^{k_n} \right) \right| \\
&\qquad = \sum_{k_1,\dots,k_n=0}^{\infty} \left| a_{k_1\dots k_n} (z^1 - p^1)^{k_1} \cdots k_j (z^j - p^j)^{k_j - 1} \cdots (z^n - p^n)^{k_n} \right| \\
&\qquad \le \sum_{k_1,\dots,k_n=0}^{\infty} C \left(\frac{r'}{r_1} \right)^{k_1} \cdots k_j \left(\frac{r'}{r_1} \right)^{k_j - 1} \cdots \left(\frac{r'}{r_1} \right)^{k_n}.
\end{aligned}$$

The last expression is an iterated sum in which $n - 1$ of the sums are convergent geometric series, while the jth one is the series $\sum_k kx^{k-1}$, which converges absolutely for $|x| < 1$ by the ratio test. Thus we may apply the Weierstrass M-test again to conclude that the differentiated series converges uniformly and absolutely on $D^n_{r'}(p)$, and therefore is equal to the derivative of f there. Since every point in $D^n_r(p)$ lies in $D^n_{r'}(p)$ for some $0 < r' < r_1 < r$, it follows that $\partial f / \partial z^j$ is equal to the sum of the differentiated series on all of $D^n_r(p)$. It then follows by induction that the same is true of all higher complex derivatives. □

Corollary 1.27. *If f is a holomorphic function given by an absolutely convergent power series of the form* (1.3) *on a polydisk $D^n_r(p) \subseteq \mathbb{C}^n$, then the power series is given explicitly by the following formula, called the* ***Taylor series of f centered at p****:*

$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} \frac{1}{k_1! \cdots k_n!} \frac{\partial^{k_1+\dots+k_n} f(p)}{(\partial z^1)^{k_1} \cdots (\partial z^n)^{k_n}} (z^1 - p^1)^{k_1} \cdots (z^n - p^n)^{k_n}.$$

Proof. Just differentiate (1.3) repeatedly term-by-term and evaluate at $z = p$ to determine the coefficients $a_{k_1\dots k_n}$. □

Proposition 1.28 (Identity Theorem). *Suppose $W \subseteq \mathbb{C}^n$ is a connected open subset, and $f, g : W \to \mathbb{C}$ are holomorphic functions that agree on a nonempty open subset of W. Then $f \equiv g$ on W.*

Proof. Set $h = f - g$, so $h \equiv 0$ on a nonempty open subset $U_0 \subseteq W$. Let

$$U = \{p \in W : h \text{ and its complex partial derivatives of all orders vanish at } p\}.$$

Then U is nonempty because $U_0 \subseteq U$. We will show that it is open and closed in W, which implies by connectivity that it is all of W.

Suppose $p \in U$. Then h is equal to a convergent power series in a neighborhood of p, and Corollary 1.27 shows that every term in the series is zero. Thus U is open in W.

Now suppose $p \in W$ is a limit point of U. There is a sequence of points $p_j \in U$ converging to p, and the hypothesis implies that all partial derivatives of h vanish at each p_j. Thus by continuity, they also vanish at p, showing that $p \in U$. Thus U is closed in W. □

Corollary 1.29 (Identity Theorem for Manifolds). *Suppose M and N are complex manifolds with M connected, and $f, g : M \to N$ are holomorphic maps that agree on a nonempty open subset of M. Then $f \equiv g$ on M.*

Proof. Proposition 1.28 applied to local coordinate representations of f and g shows that the set of points where f and g agree along with their partial derivatives of all orders is both open and closed in M, hence all of M. □

Proposition 1.30 (Liouville's Theorem). *Every holomorphic function that is defined on all of $\mathbb{C}^n$ and bounded is constant.*

Proof. Suppose $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and bounded. Given any point $z \in \mathbb{C}^n$, the function $g(\zeta) = f(\zeta z)$ is a bounded holomorphic function defined on all of $\mathbb{C}$, so it is constant by the one-variable version of Liouville's theorem. In particular, this means $f(z) = f(0)$. Since z is arbitrary, this shows f is constant. □

Liouville's theorem allows us to give our first example of two complex manifolds that are diffeomorphic but not biholomorphic.

Example 1.31 (The Unit Ball is Not Biholomorphic to $\mathbb{C}^n$). We know that $\mathbb{B}^{2n}$ and $\mathbb{C}^n$ are diffeomorphic (see [**LeeSM**, Example 2.14]). But if $F : \mathbb{C}^n \to \mathbb{B}^{2n}$ is any holomorphic map, each of its coefficient functions is a bounded holomorphic function on $\mathbb{C}^n$ and therefore constant. Thus there is no biholomorphism between $\mathbb{B}^{2n}$ and $\mathbb{C}^n$. //

Proposition 1.32 (The Maximum Principle). *Suppose $f : U \to \mathbb{C}$ is a holomorphic function on a connected open set $U \subseteq \mathbb{C}^n$. If $|f(z)|$ attains a maximum value at some point in U, then f is constant.*

Proof. Suppose $|f(z)|$ attains a maximum value at $z_0 \in U$. Let $c = f(z_0)$, and set $W = \{z \in U : f(z) = c\}$. Then W is nonempty because $z_0 \in W$, and it is closed in U by continuity. Given $z_1 \in W$, choose $\varepsilon > 0$ such that the ball $B_\varepsilon(z_1)$ is contained in U. For each $w \in \mathbb{C}^n$ with $|w| = 1$, the function $g(\zeta) = f(z_1 + \zeta w)$ is holomorphic on the disk $D_\varepsilon(0) \subseteq \mathbb{C}$ and achieves its maximum modulus at $\zeta = 0$. By the one-variable maximum principle, therefore, g is constant. Since w is arbitrary, this shows f is constant on $B_\varepsilon(z_1)$. Thus W is open, and by connectivity it is all of U. □

This result too has an immediate, and somewhat surprising, application to complex manifolds.

Corollary 1.33. *Let M be a connected compact complex manifold. Then every globally defined holomorphic function from M to $\mathbb{C}$ is constant.*

Proof. Suppose $f \in \mathcal{O}(M)$. By compactness, the continuous function $|f|$ attains a maximum value at a point $z_0 \in M$. In a holomorphic coordinate ball centered at z_0, the coordinate representation of f is a holomorphic function on an open ball in $\mathbb{C}^n$ that attains its maximum modulus at the origin, so it is constant on the entire coordinate domain. Thus by the identity theorem, it is constant on all of M. □

One of the most striking features of holomorphic functions is described in the next proposition, which shows in particular that uniform limits of holomorphic functions are holomorphic. It is worth noting that the analogous result for smooth functions, or even real-analytic functions, is not true.

Proposition 1.34. *Suppose $U \subseteq \mathbb{C}^n$ is open and $f_k : U \to \mathbb{C}$ is a sequence of holomorphic functions that converge uniformly on compact subsets of U to a function $f : U \to \mathbb{C}$. Then f is holomorphic.*

Proof. Given $p \in U$, choose $r > 0$ such that $\overline{D}^n_r(p) \subseteq U$. For all $z \in D^n_r(p)$, we can apply the Cauchy integral formula to f_k, and uniform convergence guarantees that

$$\begin{aligned} f(z) &= \lim_{k\to\infty} \frac{1}{(2\pi i)^n} \int\limits_{|\zeta^n - p^n| = r} \cdots \int\limits_{|\zeta^1 - p^1| = r} \frac{f_k(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n)\cdots(\zeta^1 - z^1)} \, d\zeta^1 \cdots d\zeta^n \\ &= \frac{1}{(2\pi i)^n} \int\limits_{|\zeta^n - p^n| = r} \cdots \int\limits_{|\zeta^1 - p^1| = r} \frac{f(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n)\cdots(\zeta^1 - z^1)} \, d\zeta^1 \cdots d\zeta^n. \end{aligned}$$

The integrand in the last expression is continuous in all variables and smooth in $z^1, \dots, z^n$, so we can differentiate under the integral sign with respect to x^j and y^j as many times as we like to conclude that f is smooth. In particular, since the integrand is holomorphic in $z^1, \dots, z^n$, we see that f satisfies the Cauchy–Riemann equations on $D^n_r(p)$. □

Our next result is a little less elementary, so its one-variable analogue is not always covered in undergraduate complex analysis texts. We will use it only once, when we study sections of holomorphic vector bundles (Thm. 3.13).

Proposition 1.35 (Montel's Theorem). *Suppose $U \subseteq \mathbb{C}^n$ is open and $f_k : U \to \mathbb{C}$ is a sequence of holomorphic functions that are uniformly bounded, meaning there is some $C > 0$ such that $|f_k(z)| < C$ for all $k \geq 1$ and all $z \in U$. Then there is a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ that converges uniformly on compact subsets of U to a holomorphic function defined on all of U.*

Proof. For any closed polydisk $\overline{D}^n_r(p) \subseteq U$, we can use Cauchy's formula to write each f_k on $D^n_r(p)$ in the form

$$f_k(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta^n - p^n| = r} \cdots \int\limits_{|\zeta^1 - p^1| = r} \frac{f_k(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n) \cdots (\zeta^1 - z^1)} \, d\zeta^1 \cdots d\zeta^n.$$

Differentiating under the integral sign, we obtain

$$\frac{\partial f_k(z)}{\partial x^j} = \frac{1}{(2\pi i)^n} \int \cdots \int \frac{f_k(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n) \cdots (\zeta^j - z^j)^2 \cdots (\zeta^1 - z^1)} \, d\zeta^1 \cdots d\zeta^n.$$

A simple computation shows that a contour integral over a circle c of radius r satisfies $\left|\int_c h(\zeta) d\zeta\right| \leq 2\pi r \sup_c |h|$. Applying this in turn to each contour integral in the above formula gives

$$\left|\frac{\partial f_k(z)}{\partial x^j}\right| \leq \frac{C}{r}.$$

This shows that the partial derivatives of f_k with respect to $x^1, \dots, x^n$ are uniformly bounded on $D^n_r(p)$, and then the Cauchy–Riemann equations show the same is true of the derivatives with respect to $y^1, \dots, y^n$. Therefore, each f_k satisfies a Lipschitz estimate of the form $|f_k(z_1) - f_k(z_2)| \leq (C'/r)|z_1 - z_2|$ there. By continuity, the same bound holds on the closed polydisk $\overline{D}^n_r(p)$. Thus the functions f_k are uniformly bounded and uniformly equicontinuous on $\overline{D}^n_r(p)$, so the Arzelà–Ascoli theorem [**Rud76**, Thm. 7.25] guarantees that a subsequence $\{f_{k_j}\}_{j=1}^\infty$ converges uniformly there.

Every $p \in U$ is contained in some polydisk $D^n_r(p)$ such that $\overline{D}^n_r(p) \subseteq U$. The set of all such polydisks is an open cover of U, and thus U is covered by countably many such polydisks. Let $\{V_m\}_{m=1}^\infty$ be such a countable cover. By the above argument, we may choose a subsequence $\{f_{1,j}\}_{j=1}^\infty$ of the original sequence that converges uniformly on $\overline{V}_1$. From that subsequence, we may choose a further subsequence $\{f_{2,j}\}_{j=1}^\infty$ that also converges uniformly on $\overline{V}_2$. Continuing by induction, for each m we get a subsequence $\{f_{m,j}\}_{j=1}^\infty$ converging uniformly on $\overline{V}_1 \cup \cdots \cup \overline{V}_m$, such that the mth sequence is a subsequence of the $(m-1)$st one. Finally, let $\{f_{k_j}\}_{j=1}^\infty$ be the diagonal subsequence $f_{k_j} = f_{j,j}$. If $K \subseteq U$ is any compact set, there is some m such that $K \subseteq V_1 \cup \cdots \cup V_m$. Since $\{f_{k_j}\}$ is a subsequence of $\{f_{i,j}\}$ for each i, it converges uniformly on K. By Proposition 1.34, the limit function is holomorphic. □

So far, all these facts about holomorphic functions of several variables have been straightforward generalizations of standard facts about holomorphic functions of one variable. The next result, however, is radically different from anything in the one-variable theory. It was proved by Friedrich Hartogs in 1906 [**Har06**].

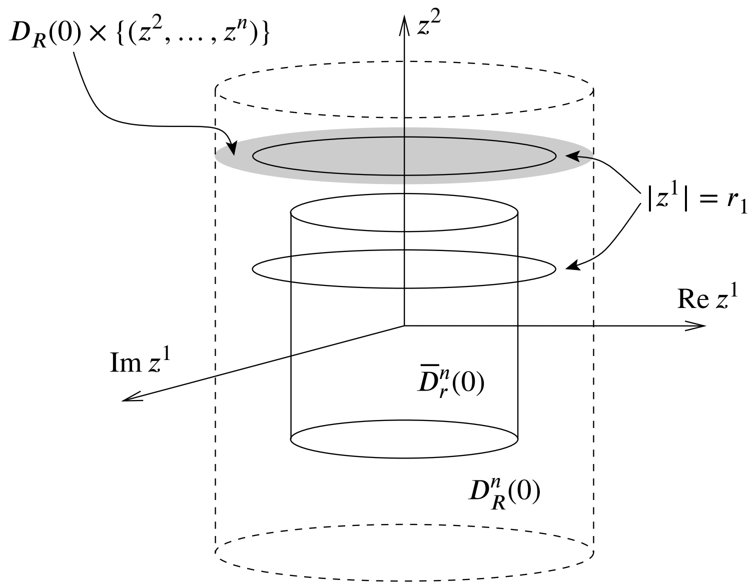

Figure 1.1. Proof of Hartogs's extension theorem

Theorem 1.36 (Hartogs's Extension Theorem). *Let $n \geq 2$, and let $\Omega \subseteq \mathbb{C}^n$ be an open set of the form $D^n_R(p) \smallsetminus \overline{D}^n_r(p)$ for some $p \in \mathbb{C}^n$ and $0 < r < R$. Every holomorphic function $f : \Omega \to \mathbb{C}$ has a unique extension to a holomorphic function on all of $D^n_R(p)$.*

Proof. After a translation, we may assume that $p = 0$. Choose any r_1 such that $r < r_1 < R$. As long as $r < |z^2| < R$, the function $z^1 \mapsto f(z^1, \dots, z^n)$ is holomorphic on the entire disk $D_R(0) \subseteq \mathbb{C}$ (see Fig. 1.1), so Cauchy's formula shows that

$$f(z^1, \dots, z^n) = \frac{1}{2\pi i} \int_{|\zeta|=r_1} \frac{f(\zeta, z^2, \dots, z^n)}{\zeta - z^1}\, d\zeta.$$

But this formula actually makes sense for all $(z^1, \dots, z^n) \in D^n_{r_1}(0)$ because the integration contour is contained in Ω in that case, and it defines a holomorphic function f_1 there by differentiation under the integral sign. Because f_1 agrees with f on the open subset of $D^n_{r_1}(0)$ where $r < |z^2| < r_1$, the identity theorem shows that it agrees on the entire connected set $D^n_{r_1}(0) \smallsetminus \overline{D}^n_r(0)$. Thus we can define a holomorphic function on all of $D^n_R(0)$ by letting it be equal to f on Ω and to f_1 on $D^n_{r_1}(0)$. Uniqueness follows immediately from the identity theorem. □

This theorem is false in the case $n = 1$, because there are many holomorphic functions with isolated singularities, such as $1/z$ or $e^{1/z}$, which are holomorphic on annuli centered at a singular point but have no holomorphic extensions across that point. Hartogs's theorem implies that singularities of holomorphic functions in two or more variables are never isolated. Moreover, it says something important about zeros of holomorphic functions as well. In one complex variable, zeros of holomorphic functions of one variable are always isolated. But if a holomorphic function f had an isolated zero at $p \in \mathbb{C}^n$ with $n \geq 2$, then $1/f$ would have an isolated singularity, which is impossible. Thus zeros of holomorphic functions of more than one variable are never isolated either.

The Complexified Tangent and Cotangent Bundles

Now we introduce some extensions to the theory of smooth manifolds that we will need for working with complex-valued functions. Writing such a function as $f = u + iv$, we would like to express its differential as $df = du + i dv$. But this is not an ordinary 1-form in the sense that the term is used in smooth manifold theory: sections of a real vector bundle like the cotangent bundle can be multiplied by real numbers, but not by complex ones.

To make sense of this, we make the following definition. If V is a real vector space, we define the ***complexification of V***, denoted by $V_{\mathbb{C}}$, to be the vector space $V \oplus V$ with multiplication by complex numbers defined as follows:

$$(a + ib)(u, v) = (au - bv, av + bu) \quad \text{for } a + ib \in \mathbb{C}.$$

Together with the usual addition in $V \oplus V$, it turns $V_{\mathbb{C}}$ into a vector space over $\mathbb{C}$. The map $u \mapsto (u, 0)$ is a real-linear isomorphism from V onto the (real) subspace $V \oplus \{0\} \subseteq V_{\mathbb{C}}$, and we typically identify V with its image under this map, thus considering V itself to be a real-linear subspace of $V_{\mathbb{C}}$. With this identification, we can write $(u, v) = u + iv$, and we can think of $V_{\mathbb{C}}$ as consisting of the set of all linear combinations of elements of V with complex coefficients.

If $(b_1, \dots, b_n)$ is any basis for V (over $\mathbb{R}$), then $\big((b_1, 0), \dots, (b_n, 0)\big)$ is a basis for $V_{\mathbb{C}}$ over $\mathbb{C}$, which under our identification we can just write as $(b_1, \dots, b_n)$. It follows that the complex dimension of $V_{\mathbb{C}}$ is the same as the real dimension of V.

For example, the complexification of $\mathbb{R}^n$ can be naturally identified with $\mathbb{C}^n$.

If $L: V \to W$ is a linear map between real vector spaces, it extends canonically to a complex-linear map $L_{\mathbb{C}}: V_{\mathbb{C}} \to W_{\mathbb{C}}$, called the ***complexification of L***, satisfying $L_{\mathbb{C}}(u + iv) = L(u) + iL(v)$. In cases where it will not cause confusion, we will often denote the complexification of a linear map L by the same symbol L.

► **Exercise 1.37.** Show that the assignment $V \mapsto V_{\mathbb{C}}$, $L \mapsto L_{\mathbb{C}}$ defines a covariant functor from the category of real vector spaces to the category of complex ones.

The next exercise describes an alternative definition of the complexification.

► **Exercise 1.38.** Let V be a real vector space. Give the space $V \otimes_{\mathbb{R}} \mathbb{C}$ (the abstract tensor product of V and $\mathbb{C}$, considered as real vector spaces), the structure of a complex vector space with the usual addition and with scalar multiplication defined by

$$\alpha\Big(\sum_{j=1}^{k} v_j \otimes \beta_j\Big) = \sum_{j=1}^{k} v_j \otimes (\alpha\beta_j),$$

for $v_j \in V$ and $\alpha, \beta_j \in \mathbb{C}$. Show that this turns $V \otimes_{\mathbb{R}} \mathbb{C}$ into a complex vector space, which is canonically isomorphic to $V_{\mathbb{C}}$ via the map $(u, v) \mapsto u \otimes 1 + v \otimes i$.

► **Exercise 1.39.** Suppose V is a real vector space.

(a) Given $w = (u, v) \in V_{\mathbb{C}}$, define the ***conjugate of w*** by $\overline{w} = (u, -v)$. Show that the map $w \mapsto \overline{w}$ is a bijective conjugate-linear map from $V_{\mathbb{C}}$ to itself satisfying $\overline{\overline{w}} = w$ for all $w \in V_{\mathbb{C}}$. (A map $F : V \to W$ between complex vector spaces is said to be ***conjugate-linear*** if it is linear over $\mathbb{R}$ and satisfies $F(\alpha v) = \overline{\alpha} F(v)$ for all $\alpha \in \mathbb{C}$ and $v \in V$.)

(b) An element $w \in V_{\mathbb{C}}$ is said to be ***real*** if $\overline{w} = w$. Show that w is real if and only if it lies in the real subspace $V \subseteq V_{\mathbb{C}}$ defined above.

(c) For $w \in V_{\mathbb{C}}$, define $\operatorname{Re} w = \frac{1}{2}(w + \overline{w})$ and $\operatorname{Im} w = \frac{1}{2i}(w - \overline{w})$. Show that $\operatorname{Re} w$ and $\operatorname{Im} w$ are real, and $w = \operatorname{Re} w + i \operatorname{Im} w$.

The complexification functor can be adapted easily to vector bundles. First we establish some definitions.

Suppose M is a topological space. A ***complex vector bundle of rank k over M*** is defined analogously to a real vector bundle (e.g., as in [**LeeSM**, Chap. 10]): it is a topological space E together with a continuous surjective map $\pi : E \to M$ such that each fiber $E_p = \pi^{-1}(p)$ is given the structure of a k-dimensional complex vector space, and each $p \in M$ has a neighborhood U over which there exists a ***local trivialization***, which is a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ that restricts to a complex-linear isomorphism from E_q to $\{q\} \times \mathbb{C}^k$ for each $q \in U$. This means, in particular, that the following diagram commutes, where $\pi_1 : U \times \mathbb{C}^k \to U$ is the projection on the first factor:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\Phi\quad} & U \times \mathbb{C}^k \\ & {\scriptstyle \pi|_{\pi^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U. & \end{array}$$

If M and E are smooth manifolds, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, it is a ***smooth complex vector bundle***; and if M and E are complex manifolds, π is holomorphic, and the local trivializations can be chosen to be biholomorphisms, it is a ***holomorphic vector bundle***. Any open cover of M such that E admits a trivialization over each of the open sets of the cover is called a ***trivializing cover for E***. If there is a global trivialization (that is, a local trivialization over all of M), the bundle is said to be a ***trivial bundle***. A ***line bundle*** is a (real or complex) vector bundle of rank 1.

If $\pi : E \to M$ and $\pi' : E' \to M$ are complex vector bundles over M, a map $F : E \to E'$ is called a ***bundle homomorphism*** if $\pi' \circ F = \pi$ and for each $p \in M$, the map $F|_{E_p} : E_p \to E'_p$ is a complex-linear map. A bundle homomorphism that is also a homeomorphism between E and E' is called a ***bundle isomorphism***, and the bundles E and E' are said to be ***isomorphic***, denoted by $E \cong E'$, if there is a bundle isomorphism between them. If the bundles are smooth and F is a diffeomorphism, it is called a ***smooth isomorphism***, and if the bundles are holomorphic and F is a biholomorphism, it is a ***holomorphic isomorphism***. In each of these cases, it is easy to check that the inverse map is also a bundle isomorphism. (For some purposes, it is useful to introduce a more general notion of vector bundle homomorphisms between bundles over different manifolds, and the kind we have defined here is identified as a ***bundle homomorphism over M***; see [**LeeSM**, Chap. 10] for details. Since we will not have any need for that extra generality, we always understand bundle homomorphisms to be the type we have defined here.)

Most of the standard constructions used for real vector bundles, such as Whitney sums [**LeeSM**, Example 10.7] and smooth subbundles [**LeeSM**, pp. 264–266], carry over in obvious ways to smooth complex bundles.

We will have much more to say about holomorphic vector bundles in Chapter 3; for now we focus attention on smooth bundles.

If $\pi : E \to M$ is a smooth (real or complex) vector bundle, a ***(global) section of E*** is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. For any open subset $U \subseteq M$, a ***local section of E over U*** is a continuous map $\sigma : U \to E$ satisfying $\sigma \circ \pi = \mathrm{Id}_U$. Every smooth vector bundle has a smooth ***zero section*** ζ, for which $\zeta(p)$ is the zero element of E_p for each $p \in M$. Any section that is not equal to the zero section will be called a ***nontrivial section***. A ***rough (local or global) section of E*** is a map $\sigma : U \to E$ ***rough (local or global) section of E*** is a map $\sigma : U \to E$ satisfying $\sigma \circ \pi = \mathrm{Id}_U$, but not assumed to be smooth or even continuous. We denote the space of smooth global sections of E by $\Gamma(E)$. A ***local frame for E*** is an ordered k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ over an open set $U \subseteq M$ whose values at each $p \in U$ form a basis for the fiber E_p.

If $\pi : E \to M$ is a smooth rank-k real vector bundle over a smooth manifold M, we define the ***complexification of E*** to be the set $E_{\mathbb{C}} = \bigcup_{p \in M} (E_p)_{\mathbb{C}}$ together with the obvious projection $\pi_{\mathbb{C}} : E_{\mathbb{C}} \to M$. For each smooth local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, we define a local trivialization $\Phi_{\mathbb{C}} : \pi_{\mathbb{C}}^{-1}(U) \to U \times \mathbb{C}^k$ by

$$\Phi_{\mathbb{C}}(\xi) = \big(\pi_{\mathbb{C}}(\xi),\ \big(\Phi|_{E_{\pi_{\mathbb{C}}(\xi)}}\big)_{\mathbb{C}}(\xi)\big).$$

Wherever two such trivializations (U, Φ) and (V, Ψ) overlap, [**LeeSM**, Lemma 10.15] shows that we can write $\Psi \circ \Phi^{-1}(p, v) = (p, \tau(p)v)$ for some smooth transition function $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$, and it is straightforward to check that the transition function from $\Phi_{\mathbb{C}}$ to $\Psi_{\mathbb{C}}$ is the same: $\Psi_{\mathbb{C}} \circ \Phi_{\mathbb{C}}^{-1}(p, v) = (p, \tau(p)v)$, where now we are considering τ as a map into $\mathrm{GL}(k, \mathbb{C})$. It follows from the vector bundle chart lemma [**LeeSM**, Lemma 10.6] (adapted in the obvious way for complex vector bundles) that $\pi_{\mathbb{C}} : E_{\mathbb{C}} \to M$ has a unique structure as a smooth rank-k complex vector bundle, with the maps constructed above as smooth local trivializations.

What this really amounts to in practice is that, given any smooth local frame $(b_1, \dots, b_k)$ for E, we can write a section of $E_{\mathbb{C}}$ locally as a sum $f^j b_j$, where now the coefficient functions f^j are allowed to be complex-valued.

► **Exercise 1.40.** Let $E \to M$ be a smooth real vector bundle. Show that every smooth (local or global) section of $E_{\mathbb{C}}$ can be written uniquely as a sum $\alpha + i\beta$, where α and β are smooth local or global sections of E.

The result of Exercise 1.39 shows that for any real vector bundle $E \to M$, conjugation defines a smooth conjugate-linear bundle homomorphism from $E_{\mathbb{C}}$ to itself, and the set of real elements (those satisfying $\overline{w} = w$) forms a real-linear subbundle canonically isomorphic to the original bundle E. It is important to note that the existence of such a conjugation operator is a special feature of complexifications: in fact, as Problem 1-6 shows, a complex vector bundle admits such a conjugation operator if and only if it is isomorphic to the complexification of a real bundle.

When we apply this construction to the tangent and cotangent bundles of a smooth manifold M, we obtain the ***complexified tangent bundle*** $T_{\mathbb{C}}M$ and the ***complexified cotangent bundle*** $T_{\mathbb{C}}^*M$, respectively. A section of $T_{\mathbb{C}}M$, called a ***complex vector field***, can be written locally as a linear combination of coordinate vector fields with complex-valued coefficient functions, or as a sum of a real vector field plus i times another real vector field. A complex vector field $Z = X + iY$ acts on a smooth real-valued function f by $Zf = Xf + iYf$, and on a complex-valued function $f = u + iv$ by the same formula, where we interpret Xf to mean $Xu + iXv$ and similarly for Y. The Lie bracket operation can be extended to pairs of smooth complex vector fields by complex bilinearity: $[X_1 + iY_1, X_2 + iY_2] = \big([X_1, X_2] - [Y_1, Y_2]\big) + i\big([X_1, Y_2] + [Y_2, X_1]\big)$. It is straightforward to check that the formula $[fV, gW] = fg[V, W] + f(Vg)W - g(Wf)V$ holds equally well when the vector fields V, W and the functions f, g are allowed to be complex.

Similarly, a section of $T_{\mathbb{C}}^*M$ is called a ***complex* 1*-form*** or a ***complex covector field***, and can be written locally as a linear combination of coordinate 1-forms with complex coefficients, or as a sum of a real 1-form plus i times another real 1-form. With this construction, we are now justified in writing $df = du + i\,dv$ whenever $f = u + iv$ is a complex-valued smooth function.

► **Exercise 1.41.** Prove that there is a canonical smooth bundle isomorphism between $T^*_{\mathbb{C}}M$ and the bundle $\operatorname{Hom}_{\mathbb{C}}(T_{\mathbb{C}}M, \mathbb{C})$ whose fiber at a point $p \in M$ is the space of complex-linear maps from $(T_pM)_{\mathbb{C}}$ to $\mathbb{C}$.

Let us specialize to the case of $\mathbb{C}^n$, with its standard holomorphic coordinates $z^j = x^j + iy^j$. Considering $\mathbb{C}^n$ as a smooth manifold of (real) dimension $2n$, we can use (x^j, y^j) as smooth global coordinates. We have a smooth global coframe $\{dx^j, dy^j\}$ for $T^*\mathbb{C}^n$, which is therefore also a coframe for $T^*_{\mathbb{C}}\mathbb{C}^n$. Consider the $2n$ complex 1-forms $dz^j = dx^j + i\,dy^j$ and $d\bar{z}^j = dx^j - i\,dy^j$. Because we can solve for $dx^j = \frac{1}{2}(dz^j + d\bar{z}^j)$ and $dy^j = \frac{1}{2i}(dz^j - d\bar{z}^j)$, it follows that $\{dz^j, d\bar{z}^j\}$ is also a smooth coframe for $T^*_{\mathbb{C}}\mathbb{C}^n$, and arbitrary complex 1-forms can also be expressed in terms of this coframe. In particular, if $f : U \to \mathbb{C}$ is a smooth function on an open subset $U \subseteq \mathbb{C}^n$, we can write

$$df = \frac{\partial f}{\partial x^j}\,dx^j + \frac{\partial f}{\partial y^j}\,dy^j = A_j\,dz^j + B_j\,d\bar{z}^j$$

for some coefficient functions A_j and B_j. (When using the summation convention, the understanding is that an upper index "in the denominator" is to be treated as a lower index.) To see what these coefficients are, just substitute the formulas for dx^j and dy^j in terms of dz^j, $d\bar{z}^j$ and collect terms:

$$\begin{aligned} df &= \frac{\partial f}{\partial x^j}\left(\frac{dz^j + d\bar{z}^j}{2}\right) + \frac{\partial f}{\partial x^j}\left(\frac{dz^j - d\bar{z}^j}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x^j} - i\frac{\partial f}{\partial y^j}\right)dz^j + \frac{1}{2}\left(\frac{\partial f}{\partial x^j} + i\frac{\partial f}{\partial y^j}\right)d\bar{z}^j. \end{aligned} \tag{1.7}$$

Motivated by this calculation, we define $2n$ smooth complex vector fields $\partial/\partial z^j$ and $\partial/\partial \bar{z}^j$ on $\mathbb{C}^n$ by

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right). \tag{1.8}$$

(Be sure to notice that the negative sign appears in the formula for $\partial/\partial z^j$, not $\partial/\partial \bar{z}^j$; this is not a typo!) A simple computation shows that $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ is the smooth global frame for $T_{\mathbb{C}}\mathbb{C}^n$ dual to $\{dz^j, d\bar{z}^j\}$. For a smooth complex-valued function f defined on an open subset $U \subseteq \mathbb{C}^n$, formula (1.7) can be rewritten in terms of this frame as

$$df = \frac{\partial f}{\partial z^j}\,dz^j + \frac{\partial f}{\partial \bar{z}^j}\,d\bar{z}^j. \tag{1.9}$$

In the special case in which f is a holomorphic function on an open subset of $\mathbb{C}^n$, you will notice that we had already defined the expression $\partial f/\partial z^j$ by equation (1.2); now we seem to have introduced a different meaning for the same expression. The next proposition ensures that the two definitions are equivalent for holomorphic functions.

Proposition 1.42. *Suppose $U \subseteq \mathbb{C}^n$ is open. Let $f : U \to \mathbb{C}$ be any smooth function, and let $\partial/\partial z^j$ and $\partial/\partial \bar{z}^j$ be the complex vector fields on U defined by* (1.8).

(a) *f is holomorphic if and only if $\partial f/\partial \bar{z}^j = 0$ for $j = 1, \dots, n$.*

(b) *If f is holomorphic, then for each j, the expression $\partial f/\partial z^j$ obtained by applying the complex vector field $\partial/\partial z^j$ to f is equal to the complex partial derivative defined by* (1.2).

Proof. After we substitute $f = u + iv$ into the equation $\partial f/\partial \bar{z}^j = 0$ and separate its real and imaginary parts, it becomes the jth pair of Cauchy–Riemann equations for f, thus proving (a). Then (b) follows from Proposition 1.25. □

One must be careful not to read too much into the expressions $\partial f/\partial z^j$ and $\partial f/\partial \bar{z}^j$ when f is merely smooth: despite the notation, they are not partial derivatives in the ordinary sense, because, for example, it does not make sense to take a derivative of a function with respect to z^1 while holding $z^2, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n$ fixed. If you fix $\bar{z}^1$, then z^1 remains fixed as well. However, there is a sense in which these operators behave like partial derivatives, which we now explain.

Suppose p is any (not necessarily holomorphic) complex-valued polynomial function of the real variables $\{x^j, y^j\}$:

$$p(x, y) = \sum_{\substack{l_1,\dots,l_n \\ m_1,\dots,m_n}} a_{l_1,\dots,l_n,m_1,\dots,m_n} (x^1)^{l_1} \cdots (x^n)^{l_n} (y^1)^{m_1} \cdots (y^n)^{m_n}.$$

Substituting $x^j = \frac{1}{2}(z^j + \bar{z}^j)$ and $y^j = \frac{1}{2i}(z^j - \bar{z}^j)$ and collecting like terms, we can express p as a polynomial expression in $z^j, \bar{z}^j$, which we denote by $\tilde{p}$:

$$\begin{aligned} \tilde{p}(z) &= p\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \\ &= \sum_{\substack{l_1,\dots,l_n \\ m_1,\dots,m_n}} \tilde{a}_{l_1,\dots,l_n,m_1,\dots,m_n} (z^1)^{l_1} \cdots (z^n)^{l_n} (\bar{z}^1)^{m_1} \cdots (\bar{z}^n)^{m_n}. \end{aligned}$$

To separate the dependence on z^j and $\bar{z}^j$, we can introduce new independent variables w^j in place of $\bar{z}^j$. Let $q : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the polynomial function

$$q(z, w) = p\left(\frac{z + w}{2}, \frac{z - w}{2i}\right),$$

so that $\tilde{p}(z) = q(z, \bar{z})$. Now it makes sense to ask whether q is independent of $w^1, \dots, w^n$.

► **Exercise 1.43.** Prove that the original polynomial p defines a holomorphic function if and only if $\partial q/\partial w^j = 0$ for each j.

So for a polynomial function p, in this sense we can say p is holomorphic if and only if it depends only on $z^1, \dots, z^n$ with no occurrences of $\bar{z}^1, \dots, \bar{z}^n$. Exactly the same argument can be made when p is a real-analytic function, except then the finite sums above become absolutely convergent infinite series; the absolute convergence ensures that the convergence is not affected by rearranging the terms. In that case as well, a real-analytic function f is holomorphic if and only if it can be written as a power series in $z^1, \dots, z^n$, with no occurrences of $\bar{z}^1, \dots, \bar{z}^n$.

For a function that is merely smooth, these computations do not make sense, because you cannot plug complex numbers into a function that is defined only for real values of $(x^1, \dots, x^n, y^1, \dots, y^n)$. But motivated by the computations above, it is sometimes helpful to think about a holomorphic function intuitively as a "smooth function that is independent of $\bar{z}^1, \dots, \bar{z}^n$."

Complex Coordinate Frames

Now suppose M is a complex manifold and $(z^1, \dots, z^n)$ are local holomorphic coordinates on an open subset $U \subseteq M$. The coordinate map $\varphi : U \to \mathbb{C}^n$ can also be thought of as a smooth coordinate map from U to $\mathbb{R}^{2n}$, with smooth coordinate functions $(x^1, y^1, \dots, x^n, y^n)$ where $z^j = x^j + i y^j$. These coordinates yield smooth coordinate vector fields $(\partial/\partial x^1, \partial/\partial y^1, \dots, \partial/\partial x^n, \partial/\partial y^n)$, which act on a smooth function $f : U \to \mathbb{C}$ by

$$\text{(1.10)} \qquad \left.\frac{\partial}{\partial x^j}\right|_p f = \left.\frac{\partial}{\partial x^j}\right|_{\varphi(p)} (f \circ \varphi^{-1}), \qquad \left.\frac{\partial}{\partial y^j}\right|_p f = \left.\frac{\partial}{\partial y^j}\right|_{\varphi(p)} (f \circ \varphi^{-1}),$$

where the expressions on the right-hand sides are ordinary partial derivatives on $\mathbb{R}^{2n}$ (see [**LeeSM**, p. 60]). We define a smooth local complex frame $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ for $T_{\mathbb{C}}M$ by (1.8), where now $\partial/\partial x^j$ and $\partial/\partial y^j$ are interpreted as smooth vector fields on $U \subseteq M$. These vector fields are called ***complex coordinate vector fields***, and the corresponding local frame is called a ***complex coordinate frame***.

Lemma 1.44. *Suppose M is a complex manifold and $f : M \to \mathbb{C}$ is a smooth function. If $(z^1, \dots, z^n)$ are holomorphic coordinates on a subset $U \subseteq M$ and $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ are the corresponding complex coordinate vector fields, then f is holomorphic on U if and only if $\partial f/\partial \bar{z}^j \equiv 0$ on U for $j = 1, \dots, n$.*

Proof. Let $\varphi : U \to \mathbb{C}^n$ be the holomorphic coordinate map, and let $\widehat{U} = \varphi(U) \subseteq \mathbb{C}^n$. It follows from (1.10) together with the definition of $\partial/\partial \bar{z}^j$ that for all $p \in U$,

$$\frac{\partial f}{\partial \bar{z}^j}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial \bar{z}^j}(\varphi(p)).$$

The lemma then follows from the fact that $f : U \to \mathbb{C}$ is holomorphic by definition if and only if $f \circ \varphi^{-1} : \widehat{U} \to \mathbb{C}$ is holomorphic. □

When M and N are complex manifolds, the ***total derivative*** or ***differential*** of a smooth map $F : M \to N$ at a point $p \in M$ is a real-linear map from T_pM to $T_{F(p)}N$, and its complexification is a complex-linear map from $(T_pM)_{\mathbb{C}}$

to $(T_{F(p)}N)_{\mathbb{C}}$. For smooth manifolds, the differential is often denoted by dF_p, but for reasons that will be explained shortly, in this book we will denote the differential at p (or its complexification) by $DF(p)$, and the associated bundle homomorphism, called the ***global differential of F***, by $DF: T_{\mathbb{C}}M \to T_{\mathbb{C}}N$. The next proposition shows how to compute it in terms of holomorphic coordinates.

Proposition 1.45 (The Total Derivative in Holomorphic Coordinates). *Let M and N be complex manifolds and $F: M \to N$ be a smooth map. Given $p \in M$, let $z^j = x^j + iy^j$ be local holomorphic coordinates for M in a neighborhood of p, and $w^j = u^j + iv^j$ for N in a neighborhood of $F(p)$. In terms of the complex local frames $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ for M and $\{\partial/\partial w^j, \partial/\partial \overline{w}^j\}$ for N, the total derivative of F at p has the following coordinate representation:*

$$DF(p)\Big(\frac{\partial}{\partial z^j}\Big|_p\Big) = \frac{\partial F^k}{\partial z^j}(p)\,\frac{\partial}{\partial w^k}\Big|_{F(p)} + \frac{\partial \overline{F}^k}{\partial z^j}(p)\,\frac{\partial}{\partial \overline{w}^k}\Big|_{F(p)}, \tag{1.11}$$

$$DF(p)\Big(\frac{\partial}{\partial \bar{z}^j}\Big|_p\Big) = \frac{\partial F^k}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial w^k}\Big|_{F(p)} + \frac{\partial \overline{F}^k}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial \overline{w}^k}\Big|_{F(p)}. \tag{1.12}$$

Proof. Write the real and imaginary parts of (the coordinate representation of) F as $F = U + iV$. Considering M and N as smooth manifolds, we have the usual coordinate formula for $DF(p)$:

$$\begin{aligned} DF(p)\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) &= \frac{\partial U^k}{\partial x^j}(p)\,\frac{\partial}{\partial u^k}\Big|_{F(p)} + \frac{\partial V^k}{\partial x^j}(p)\,\frac{\partial}{\partial v^k}\Big|_{F(p)}, \\ DF(p)\Big(\frac{\partial}{\partial y^j}\Big|_p\Big) &= \frac{\partial U^k}{\partial y^j}(p)\,\frac{\partial}{\partial u^k}\Big|_{F(p)} + \frac{\partial V^k}{\partial y^j}(p)\,\frac{\partial}{\partial v^k}\Big|_{F(p)}. \end{aligned} \tag{1.13}$$

To transform this to holomorphic coordinates, begin with the definitions of $\partial/\partial z^j$ and $\partial/\partial \bar{z}^j$, and use (1.13) together with the complex linearity of $DF(p)$ to obtain

$$\begin{aligned} DF(p)\Big(\frac{\partial}{\partial z^j}\Big|_p\Big) &= \frac{\partial U^k}{\partial z^j}(p)\,\frac{\partial}{\partial u^k}\Big|_{F(p)} + \frac{\partial V^k}{\partial z^j}(p)\,\frac{\partial}{\partial v^k}\Big|_{F(p)}, \\ DF(p)\Big(\frac{\partial}{\partial \bar{z}^j}\Big|_p\Big) &= \frac{\partial U^k}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial u^k}\Big|_{F(p)} + \frac{\partial V^k}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial v^k}\Big|_{F(p)}. \end{aligned}$$

Now substitute $\partial/\partial u^k = \partial/\partial w^k + \partial/\partial \overline{w}^k$ and $\partial/\partial v^k = i(\partial/\partial w^k - \partial/\partial \overline{w}^k)$ and collect terms:

$$\begin{aligned} DF(p)\Big(\frac{\partial}{\partial z^j}\Big|_p\Big) = \Big(\frac{\partial U^k}{\partial z^j}(p) + i\frac{\partial V^k}{\partial z^j}(p)\Big)\,\frac{\partial}{\partial w^k}\Big|_{F(p)} \\ + \Big(\frac{\partial U^k}{\partial z^j}(p) - i\frac{\partial V^k}{\partial z^j}(p)\Big)\,\frac{\partial}{\partial \overline{w}^k}\Big|_{F(p)}. \end{aligned}$$

This is (1.11), and a similar computation proves (1.12). □

Corollary 1.46. *In addition to the hypotheses of* 1.45*, suppose F is holomorphic. Then in terms of the local frames $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ and $\{\partial/\partial w^j, \partial/\partial \overline{w}^j\}$, $DF(p)$ is represented by the block-diagonal matrix*

$$\begin{pmatrix} D'F(p) & 0 \\ 0 & \overline{D'F(p)} \end{pmatrix}, \tag{1.14}$$

*where $D'F$ denotes the $n \times n$ complex matrix-valued function $\left(\partial F^k/\partial z^j\right)$, called the **holomorphic Jacobian of F**. Thus the linear map $DF(p)$ is invertible if and only if the holomorphic Jacobian of F is invertible at p.*

Proof. The fact that F is holomorphic means that each component function of its coordinate representation is holomorphic. Thus $\partial F^k/\partial \bar{z}^j$ vanishes identically, and by conjugation so does $\partial \overline{F}^k/\partial z^j$. Therefore, $DF(p)$ has the given matrix representation by Proposition 1.45. The last statement then follows from the fact that $\det DF(p) = |\det D'F(p)|^2$. □

Proposition 1.47 (Chain Rule for Smooth Functions). *Suppose M and N are complex manifolds, $F : M \to N$ is a smooth map, and $h : N \to \mathbb{C}$ is a smooth function. In terms of local holomorphic coordinates (z^j) for M and (ζ^k) for N,*

$$\frac{\partial(h \circ F)}{\partial z^j} = \frac{\partial h}{\partial \zeta^k}\frac{\partial F^k}{\partial z^j} + \frac{\partial h}{\partial \bar{\zeta}^k}\frac{\partial \overline{F}^k}{\partial z^j},$$

$$\frac{\partial(h \circ F)}{\partial \bar{z}^j} = \frac{\partial h}{\partial \zeta^k}\frac{\partial F^k}{\partial \bar{z}^j} + \frac{\partial h}{\partial \bar{\zeta}^k}\frac{\partial \overline{F}^k}{\partial \bar{z}^j}.$$

Proof. Proposition 1.45 shows that the value of $\partial(h \circ F)/\partial z^j$ at $p \in M$ is equal to the $\partial/\partial w$ component of $D(h \circ F)(p)\left(\partial/\partial z^j|_p\right)$ (where w denotes the standard holomorphic coordinate of $\mathbb{C}$). By smooth manifold theory, $D(h \circ F)(p) = Dh(F(p)) \circ DF(p)$, which can be computed by applying the formula of Proposition 1.45 to h and to F and composing the two linear maps. A similar argument applies to the $\bar{z}^j$ derivative. □

Corollary 1.48 (Chain Rule for Holomorphic Functions). *Under the hypotheses of Proposition* 1.47*, suppose in addition that F and h are holomorphic. Then*

$$d(h \circ F) = \frac{\partial h}{\partial w^k}\frac{\partial F^k}{\partial z^j} dz^j. \qquad \square$$

In the theory of smooth (real) manifolds, the differential of a smooth real-valued function f at a point $p \in M$ can be considered either as a linear map from T_pM to $\mathbb{R}$ (a covector) or as a linear map from T_pM to $T_{f(p)}\mathbb{R}$; in view of the canonical identification between $T_{f(p)}\mathbb{R}$ and $\mathbb{R}$, these are the same map, so it makes sense to use the same notation df_p to denote both of them. But in complex manifold theory, something different happens. Suppose $f = u + iv : M \to \mathbb{C}$ is a complex-valued smooth function on a complex manifold M. On the one hand, df_p denotes the value at p of the complex-valued 1-form $df = du + i\,dv$, an element of $(T_p^*M)_{\mathbb{C}}$, which can also be viewed as a complex-linear map from $(T_pM)_{\mathbb{C}}$ to $\mathbb{C}$ (by Exercise 1.41). Using the coordinate formula (1.9), we find, for example, that

$$df_p\Big(\frac{\partial}{\partial \bar z^j}\Big|_p\Big) = \frac{\partial f}{\partial \bar z^j}(p) \in \mathbb{C}.$$

On the other hand, $Df(p)$ is a complex-linear map from $(T_pM)_{\mathbb{C}}$ to $(T_{f(p)}\mathbb{C})_{\mathbb{C}}$, and Proposition 1.45 shows that

$$Df(p)\Big(\frac{\partial}{\partial \bar z^j}\Big|_p\Big) = \frac{\partial f}{\partial \bar z^j}(p)\,\frac{\partial}{\partial w}\Big|_{f(p)} + \frac{\partial \overline{f}}{\partial \bar z^j}(p)\,\frac{\partial}{\partial \overline{w}}\Big|_{f(p)} \in (T_{f(p)}\mathbb{C})_{\mathbb{C}}.$$

These are distinctly different objects—for example, if f is holomorphic, then $df(\partial/\partial \bar z^j)$ vanishes identically, but $Df(\partial/\partial \bar z^j)$ does not. This is why we use different notations for the two kinds of derivatives, and prefer the term "total derivative" for $Df(p)$.

Orientations

The computations we just did lead to another important property of complex manifolds: they all have canonical orientations. (Just to be clear: when we speak of an orientation of a complex manifold, it means an orientation of its underlying smooth real manifold.)

Proposition 1.49. *Every complex manifold has a canonical orientation, uniquely determined by the following two properties:*

(i) *The canonical orientation of $\mathbb{C}^n$ is the one determined by the $2n$-form*

$$\omega_n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{1.15}$$

(ii) *Every local biholomorphism is orientation-preserving.*

Proof. Let us begin by expressing the real $2n$-form ω_n in terms of the complex coordinates $(z^1, \dots, z^n)$. Observe that for each j, we have $dz^j \wedge d\bar z^j = (dx^j + i\,dy^j) \wedge (dx^j - i\,dy^j) = -2i dx^j \wedge dy^j$. Therefore

$$\omega_n = \Big(\frac{i}{2}\Big)^n dz^1 \wedge d\bar z^1 \wedge \cdots \wedge dz^n \wedge d\bar z^n. \tag{1.16}$$

Let $U \subseteq \mathbb{C}^n$ be an open subset and $F : U \to \mathbb{C}^n$ be a local biholomorphism. The Jacobian matrix of F has the form (1.14) when expressed in terms of the ordered frame $\big(\partial/\partial z^1, \dots, \partial/\partial z^n, \partial/\partial \bar z^1, \dots, \partial/\partial \bar z^n\big)$. Note that this order is not the same as the one used in formula (1.16)—they differ by a permutation whose sign is $(-1)^{(n-1)n/2}$, as you can check, and therefore

$$\omega_n = (-1)^{(n-1)n/2}\Big(\frac{i}{2}\Big)^n dz^1 \wedge \cdots \wedge dz^n \wedge d\bar z^1 \wedge \cdots \wedge d\bar z^n.$$

The formula for the pullback of a top-degree form (see [**LeeSM**, Prop. 14.9], which works equally well for complex-valued forms) gives

$$\begin{aligned}
F^*\big(dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n\big) \\
&= \det\begin{pmatrix} D'F & 0 \\ 0 & \overline{D'F} \end{pmatrix} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \\
&= |\det D'F|^2 dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n,
\end{aligned}$$

and multiplying both sides by $(-1)^{(n-1)n/2}(i/2)^n$ implies

$$F^*\omega_n = |\det D'F|^2\omega_n.$$

This shows that every biholomorphism between open subsets of $\mathbb{C}^n$ is orientation-preserving.

Now let M be an n-dimensional complex manifold. Because every holomorphic coordinate chart is a local biholomorphism, if there is to be an orientation of M satisfying (i) and (ii), it must be determined in the domain of each holomorphic chart by the pullback of ω_n under the coordinate map, and it is uniquely determined by this property. We just need to verify that the orientations determined by different holomorphic charts agree.

Suppose two holomorphic charts (U,φ) and (V,ψ) overlap. The transition function $\psi\circ\varphi^{-1}$ is a biholomorphism between open subsets of $\mathbb{C}^n$, so the above computation shows that $(\psi\circ\varphi^{-1})^*\omega_n = u\,\omega_n$, where u is the positive smooth function $|\det D'(\psi\circ\varphi^{-1})|^2$. Thus on $U\cap V$ we have

$$\begin{aligned}
\psi^*\omega_n &= \varphi^*(\varphi^{-1})^*\psi^*\omega_n \\
&= \varphi^*\big((\psi\circ\varphi^{-1})^*\omega_n\big) \\
&= \varphi^*\big(u\,\omega_n\big) \\
&= (u\circ\varphi)\varphi^*\omega_n.
\end{aligned}$$

Thus the n-forms determined by φ and ψ are positive multiples of each other, so they determine the same orientation on $U\cap V$.

Finally, we need to show that every local biholomorphism between complex manifolds is orientation-preserving. Suppose $F\colon M\to N$ is a local biholomorphism. Let $p\in M$, and choose holomorphic charts (U,φ) for M and (V,ψ) for N such that $p\in U$, $F(U)\subseteq V$, and $F|_U$ is a biholomorphism onto its image. Then on U,

$$F = \big(\psi^{-1}\big)\circ\big(\psi\circ F\circ\varphi^{-1}\big)\circ\big(\varphi\big).$$

The three maps in parentheses above are all orientation-preserving: the first and third by the way we have defined the orientations on N and M, and the second because it is a biholomorphism between open subsets of $\mathbb{C}^n$. □

Almost Complex Structures

To delve further into the interaction between a holomorphic structure and its underlying smooth structure, we introduce the following linear-algebraic construction. Let V be an n-dimensional complex vector space, and let $V_{\mathbb{R}}$ be its underlying real vector space—the same set as V, but considered only as a vector space over $\mathbb{R}$. Then $V_{\mathbb{R}}$ is a $2n$-dimensional real vector space. The fact that V is a complex vector space is encoded in the rule for multiplying vectors by i, which is the map $J: V \to V$ sending each vector v to iv. By ignoring the complex vector space structure, we can also think of J as a real-linear map $J: V_{\mathbb{R}} \to V_{\mathbb{R}}$ satisfying $J \circ J = -\operatorname{Id}$.

Now suppose V is any vector space over $\mathbb{R}$. A ***complex structure on V*** is a real-linear endomorphism $J: V \to V$ satisfying $J \circ J = -\operatorname{Id}$.

Lemma 1.50. *Suppose V is a real vector space and J is a complex structure on V. Then the multiplication by complex scalars defined by $(a + bi)v = av + bJv$, together with the given vector addition operation, turns the set V into a complex vector space.*

► **Exercise 1.51.** Prove this lemma by showing that complex multiplication is associative and distributive.

To understand a complex structure J on a vector space V more deeply, we need to look at its eigenvalues. The fact that $J \circ J = -\operatorname{Id}$ means that every eigenvalue λ must satisfy $\lambda^2 = -1$. Thus J has no real eigenvalues, and the only possible complex eigenvalues are $\pm i$. To find eigenspaces, therefore, we must complexify V and J. Let $V_{\mathbb{C}}$ be the complexification of V, and denote the complexification of J by $J: V_{\mathbb{C}} \to V_{\mathbb{C}}$. It still satisfies $J \circ J = -\operatorname{Id}$.

Proposition 1.52. *If J is a complex structure on the real vector space V, then $V_{\mathbb{C}}$ has a complete eigenspace decomposition of the form*

$$V_{\mathbb{C}} = V' \oplus V'',$$

where $V' \subseteq V_{\mathbb{C}}$ is the i-eigenspace of J and V'' is the $(-i)$-eigenspace. The eigenspace decomposition of $w \in V_{\mathbb{C}}$ is given by $w = w' + w''$, where

$$w' = \tfrac{1}{2}(w - iJw), \qquad w'' = \tfrac{1}{2}(w + iJw). \tag{1.17}$$

If V is finite-dimensional, then V' and V'' have the same complex dimension.

Proof. Given $w \in V_{\mathbb{C}}$, define $w', w'' \in V_{\mathbb{C}}$ by (1.17). Simple computations show that $Jw' = iw'$ and $Jw'' = -iw''$. Because $w = w' + w''$, this shows that $V_{\mathbb{C}} = V' + V''$. On the other hand, a nonzero vector cannot be an eigenvector with two different eigenvalues, so $V' \cap V'' = \{0\}$, which shows that the sum is direct.

To see that the eigenspaces have the same dimension, note that conjugation (Exercise 1.39) is a bijective real-linear map from $V_{\mathbb{C}}$ to itself, and it interchanges

V' and V''. Thus the underlying real spaces of V' and V'' have the same real dimension, and because the complex dimension is half the real dimension, V' and V'' have the same complex dimension. □

Corollary 1.53. *If a finite-dimensional real vector space admits a complex structure, then it is even-dimensional.*

Proof. If V admits a complex structure, the preceding proposition shows that $V_{\mathbb{C}}$ is even-dimensional. The result follows from the fact that the complex dimension of $V_{\mathbb{C}}$ is equal to the real dimension of V. □

Let us apply this construction to $\mathbb{C}^n$ with its standard complex structure. Let $(X_1, \dots, X_n)$ denote the standard basis for $\mathbb{C}^n$ as a complex vector space, where $X_j = (0, \dots, 1, \dots, 0)$ with a 1 in the jth place. Let $Y_j = JX_j = (0, \dots, i, \dots, 0)$. Then $(X_1, Y_1, \dots, X_n, Y_n)$ is a basis over $\mathbb{R}$ for the underlying real vector space $(\mathbb{C}^n)_{\mathbb{R}}$, and J satisfies $JX_j = Y_j$, $JY_j = -X_j$. From Proposition 1.52, we see that the i-eigenspace $(\mathbb{C}^n)'$ is spanned by $(Z_1, \dots, Z_n)$, where $Z_j = \frac{1}{2}(X_j - iY_j)$, and $(\mathbb{C}^n)''$ is spanned by $(\overline{Z}_1, \dots, \overline{Z}_n)$.

All of these constructions can be applied to vector bundles. If $E \to M$ is a smooth real vector bundle, a ***complex structure on E*** is a smooth bundle endomorphism $J : E \to E$ satisfying $J \circ J = -\operatorname{Id}$.

Consider the case of $\mathbb{C}^n$ as a smooth manifold. For each point $p \in \mathbb{C}^n$, using the standard identification of $T_p\mathbb{C}^n$ with $(\mathbb{C}^n)_{\mathbb{R}}$, we have the following correspondences:

$$\left.\frac{\partial}{\partial x^j}\right|_p \leftrightarrow X_j, \qquad \left.\frac{\partial}{\partial y^j}\right|_p \leftrightarrow Y_j, \qquad \left.\frac{\partial}{\partial z^j}\right|_p \leftrightarrow Z_j.$$

Thus the bundle $T\mathbb{C}^n$ has a canonical complex structure $J_{\mathbb{C}^n}$, which satisfies

$$J_{\mathbb{C}^n}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \qquad J_{\mathbb{C}^n}\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}.$$

The complexified tangent bundle $T_{\mathbb{C}}\mathbb{C}^n$ splits as $T_{\mathbb{C}}\mathbb{C}^n = T'\mathbb{C}^n \oplus T''\mathbb{C}^n$, with $T'\mathbb{C}^n$ spanned by the complex vector fields $\partial/\partial z^1, \dots, \partial/\partial z^n$, and $T''\mathbb{C}^n$ spanned by $\partial/\partial \bar{z}^1, \dots, \partial/\partial \bar{z}^n$.

Lemma 1.54. *For an open subset $U \subseteq \mathbb{C}^n$, a smooth function $F : U \to \mathbb{C}^m$ is holomorphic if and only if the following relation holds for all $p \in U$:*

$$DF(p) \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ DF(p). \tag{1.18}$$

Proof. First suppose that (1.18) holds for all $p \in U$. After both sides are extended by complex linearity to act on complex vectors, the two expressions yield the same result when applied to the elements of the complex coordinate frame

$\{\partial/\partial z^j,\ \partial/\partial \bar{z}^j\}$. Using (1.12), we obtain

$$\begin{aligned}
0 &= DF\Big(J_{\mathbb{C}^n}\frac{\partial}{\partial \bar{z}^j}\Big) - J_{\mathbb{C}^m}\Big(DF\frac{\partial}{\partial \bar{z}^j}\Big)\\
&= DF\Big(-i\frac{\partial}{\partial \bar{z}^j}\Big) - J_{\mathbb{C}^m}\Big(DF\frac{\partial}{\partial \bar{z}^j}\Big)\\
&= -i\frac{\partial F^k}{\partial \bar{z}^j}\frac{\partial}{\partial w^k} - i\frac{\partial \overline{F}^k}{\partial \bar{z}^j}\frac{\partial}{\partial \overline{w}^k} - J_{\mathbb{C}^m}\frac{\partial F^k}{\partial \bar{z}^j}\frac{\partial}{\partial w^k} - J_{\mathbb{C}^m}\frac{\partial \overline{F}^k}{\partial \bar{z}^j}\frac{\partial}{\partial \overline{w}^k}\\
&= -2i\frac{\partial F^k}{\partial \bar{z}^j}\frac{\partial}{\partial w^k}.
\end{aligned}$$

This shows $\partial F^k/\partial \bar{z}^j \equiv 0$ for all j, k, so F is holomorphic.

Conversely, if F is holomorphic, the computation above shows that both sides of (1.18) yield the same result when applied to $\partial/\partial \bar{z}^j$, and conjugation shows that the same is true when applied to $\partial/\partial z^j$, using the fact that $\partial \overline{F}^k/\partial z^j = \overline{\partial F^k/\partial \bar{z}^j} = 0$. Since both sides are linear over $C^\infty(M;\mathbb{C})$, this shows the equation holds when applied to arbitrary vector fields. □

Lemma 1.54 enables us to define a canonical complex structure on the tangent bundle of every complex manifold.

Proposition 1.55. *For every complex manifold M, there is a canonical complex structure on TM, denoted by $J_M : TM \to TM$. If N is another complex manifold and $F : M \to N$ is a smooth map, then F is holomorphic if and only if*

(1.19) $$DF \circ J_M = J_N \circ DF.$$

Proof. Let n be the complex dimension of M. We define J_M as follows: given $p \in M$, choose a holomorphic coordinate chart (U, φ) on a neighborhood of p, and define $J_M : TM|_U \to TM|_U$ by

(1.20) $$J_M = D\varphi^{-1} \circ J_{\mathbb{C}^n} \circ D\varphi.$$

Wherever two holomorphic charts (U, φ) and (V, ψ) overlap, the transition map $\psi \circ \varphi^{-1}$ is a holomorphic map between open subsets of $\mathbb{C}^n$, so its differential commutes with $J_{\mathbb{C}^n}$ by Lemma 1.54. Therefore,

$$\begin{aligned}
D\psi^{-1} \circ J_{\mathbb{C}^n} \circ D\psi &= D\psi^{-1} \circ J_{\mathbb{C}^n} \circ \big(D\psi \circ D\varphi^{-1}\big) \circ D\varphi\\
&= D\psi^{-1} \circ \big(D\psi \circ D\varphi^{-1}\big) \circ J_{\mathbb{C}^n} \circ D\varphi\\
&= D\varphi^{-1} \circ J_{\mathbb{C}^n} \circ D\varphi,
\end{aligned}$$

so J_M is well defined. The fact that it satisfies $J_M \circ J_M = -\operatorname{Id}$ follows from the corresponding fact for $J_{\mathbb{C}^n}$.

Now let N be a complex m-manifold and $F: M \to N$ be a smooth map. Because (1.19) is a local statement, it suffices to choose arbitrary local holomorphic charts (U, φ) for M and (V, ψ) for N such that $F(U) \subseteq V$, and prove that the restriction of F to U is holomorphic if and only if it satisfies (1.19) there. By definition, F is holomorphic on U if and only if its coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is holomorphic, which in turn is true if and only if $D\widehat{F} \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ D\widehat{F}$ by Lemma 1.54. Using (1.20) for both M and N, we compute

$$\begin{aligned} D\widehat{F} \circ J_{\mathbb{C}^n} - J_{\mathbb{C}^m} \circ D\widehat{F} &= D\psi \circ DF \circ D\varphi^{-1} \circ J_{\mathbb{C}^n} - J_{\mathbb{C}^m} \circ D\psi \circ DF \circ D\varphi^{-1} \\ &= D\psi \circ DF \circ J_M \circ D\varphi^{-1} - D\psi \circ J_N \circ DF \circ D\varphi^{-1} \\ &= D\psi \circ (DF \circ J_M - J_N \circ DF) \circ D\varphi^{-1}. \end{aligned}$$

Since $D\psi$ and $D\varphi^{-1}$ are bundle isomorphisms, this last expression is zero if and only if (1.18) holds, thus completing the proof. □

Proposition 1.56. *Let M be a complex manifold and let $J_M : TM \to TM$ be the associated complex structure on TM. There are smooth subbundles $T'M$, $T''M \subseteq T_{\mathbb{C}}M$ whose fibers at each point are the i-eigenspace and $(-i)$-eigenspace of (the complexification of) J_M, respectively. The complexified tangent bundle decomposes as a Whitney sum: $T_{\mathbb{C}}M = T'M \oplus T''M$. In terms of any local holomorphic coordinates $z^j = x^j + iy^j$, the complex vector fields $\partial/\partial z^j$ defined by* (1.8) *form a local frame for $T'M$; and the vector fields $\partial/\partial \bar{z}^j$ form a local frame for $T''M$.*

Proof. For each $p \in M$, the space $(T_pM)_{\mathbb{C}}$ has such a decomposition by Proposition 1.52. Suppose $z^j = x^j + iy^j$ are holomorphic local coordinates on M. Because the endomorphism J_M is defined by using the coordinate map to transport $J_{\mathbb{C}^n}$ to the manifold, it follows that the vector fields $\partial/\partial z^j$ provide a local frame for $T'M$, as do $\partial/\partial \bar{z}^j$ for $T''M$. Because both subbundles are spanned locally by smooth vector fields, they are smooth. □

We call the bundles $T'M$ and $T''M$ the ***holomorphic tangent bundle*** and ***antiholomorphic tangent bundle of M***, respectively. The fibers T'_pM and T''_pM at a point $p \in M$ are called the ***holomorphic tangent space*** and ***antiholomorphic tangent space at p***, respectively.

The decomposition of $T_{\mathbb{C}}M$ into holomorphic and antiholomorphic tangent bundles allows us to give a coordinate-free interpretation to the holomorphic Jacobian of a holomorphic map. It follows from Proposition 1.55 that if $F: M \to N$ is holomorphic, then $DF(T'M) \subseteq T'N$. In local holomorphic coordinates (z^j) for M and (w^k) for N, Corollary 1.46 shows that the restriction of $DF(p)$ to T'_pM is represented by the holomorphic Jacobian matrix $(\partial F^k(p)/\partial z^j)$. Henceforth, we will use the notation $D'F(p)$ and the term ***holomorphic Jacobian*** to refer either to this complex-linear map from $T'_p M$ to $T'_{F(p)}N$ or to its matrix representation in local holomorphic coordinates.

For a finite-dimensional real vector space with its natural smooth structure, the tangent space at each point is canonically identified with the vector space itself [**LeeSM**, Prop. 3.13]. The following proposition shows that there is a corresponding identification for complex vector spaces.

Proposition 1.57 (Holomorphic Tangent Space to a Complex Vector Space). *Suppose V is a finite-dimensional complex vector space with its standard holomorphic structure. For each $a \in V$, there is a canonical (basis-independent) complex-linear isomorphism $\Phi_a : V \cong T'_a V$. It is natural in the following sense: if $L : V \to W$ is a complex-linear map between finite-dimensional complex vector spaces, then the following diagram commutes for each $a \in V$:*

$$\begin{array}{ccc} V & \xrightarrow{\Phi_a} & T'_a V \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle D'L(a)} \\ W & \xrightarrow[\Phi_{L(a)}]{} & T'_{L(a)}W. \end{array} \tag{1.21}$$

Proof. Given $a, w \in V$, let $\lambda_{a,w} : \mathbb{C} \to V$ be the holomorphic map $\lambda_{a,w}(\tau) = a + \tau w$. We define $\Phi_a : V \to T'_a V$ by

$$\Phi_a(w) = D'(\lambda_{a,w})(0)\left(\left.\frac{\partial}{\partial \tau}\right|_0 \right).$$

The definition shows that this is independent of any choice of basis for V. To see that it satisfies the required conditions, choose any basis for V and let $(z^1, \dots, z^n)$ be the corresponding linear coordinates. Then a simple computation based on Corollary 1.46 shows that Φ_a has the coordinate representation

$$\Phi_a(w^1, \dots, w^n) = w^j \left.\frac{\partial}{\partial z^j}\right|_a,$$

which shows that it is a complex-linear isomorphism. If W is another finite-dimensional complex vector space and $L : V \to W$ is a complex-linear map, then in terms of any linear coordinates $(\zeta^1, \dots, \zeta^m)$ for W, we see that

$$D'L(a)(\Phi_a(w^1, \dots, w^n)) = L^j_k w^k \left.\frac{\partial}{\partial \zeta^j}\right|_{L(a)} = \Phi_{L(a)}(L(w^1, \dots, w^n)),$$

which proves (1.21). □

For a complex manifold M, we have now introduced several different varieties of tangent bundles: TM, $T_{\mathbb{C}}M$, $T'M$, and $T''M$. In case you are not confused enough already, we now define one more: $T_J M$ is the complex vector bundle with the same total space as the ordinary tangent bundle TM, but endowed with the complex vector space structure on fibers determined by J_M as in Lemma 1.50.

Proposition 1.58. *Let M be a complex n-manifold. Then $T_J M$ is a smooth rank-n complex vector bundle over M. The complex vector bundles $T_J M$ and $T'M$ are isomorphic via the map $\xi : T_J M \to T'M$ given by $\xi(v) = v - iJv$.*

Proof. Problem 1-7. □

For easy reference, here is a summary of all of these bundles. Suppose M is a complex n-manifold.

- $\boldsymbol{TM}$: The ordinary tangent bundle of the smooth manifold M. It is a real vector bundle of rank $2n$.
- $\boldsymbol{T_{\mathbb{C}}M}$: The complexified tangent bundle, a complex vector bundle of rank $2n$.
- $\boldsymbol{T'M}$: The holomorphic tangent bundle, a complex rank-n vector subbundle of $T_{\mathbb{C}}M$. Its fiber at each point is the i-eigenspace of J_M.
- $\boldsymbol{T''M}$: The antiholomorphic tangent bundle, another complex rank-n vector subbundle of $T_{\mathbb{C}}M$, whose fibers are $(-i)$-eigenspaces of J_M.
- $\boldsymbol{T_J M}$: The ordinary tangent bundle of M equipped with the complex structure J_M, which turns it into a complex vector bundle of rank n.

Now suppose M is an arbitrary smooth manifold. It makes sense to ask whether there is a complex structure on TM, that is, a smooth bundle endomorphism $J : TM \to TM$ satisfying $J \circ J = -\mathrm{Id}$. The existence of such an endomorphism is a necessary condition for the existence of a holomorphic structure on M, but it is not sufficient, as we will see below. For this reason, a manifold whose tangent bundle is endowed with such a complex structure J is called an ***almost complex manifold***, and J is called an ***almost complex structure on M***. Note the potentially confusing shift in terminology: a *complex structure on TM* is called an *almost complex structure on M* (to distinguish it from the traditional use of "complex structure on M" to denote what we are calling a holomorphic structure).

Proposition 1.55 shows that a holomorphic structure on a manifold M determines an almost complex structure on M (that is, a complex structure on TM). The question naturally arises whether the reverse is true: Given an almost complex structure J on a smooth manifold M, is there a holomorphic structure for which J is the canonical almost complex structure as described in Proposition 1.55? In general, the answer is no, because there is a nontrivial necessary condition, as a consequence of the next proposition.

Proposition 1.59. *Suppose M is a complex manifold and $V, W \in \Gamma(T'M)$. Then $[V, W] \in \Gamma(T'M)$.*

Proof. In local holomorphic coordinates, we can write

$$V = V^j \frac{\partial}{\partial z^j}, \qquad W = W^k \frac{\partial}{\partial z^k},$$

and therefore,

$$[V, W] = V^j \left(\frac{\partial W^k}{\partial z^j} \right) \frac{\partial}{\partial z^k} - W^k \left(\frac{\partial V^j}{\partial z^k} \right) \frac{\partial}{\partial z^j}.$$

This last expression takes its values in $T'M$. □

For almost complex structures, it makes sense to ask if the same result holds, by virtue of the following lemma.

Lemma 1.60. *Suppose M is a smooth $2n$-manifold endowed with an almost complex structure J. Then there are smooth rank-n complex subbundles $T'M, T''M \subseteq T_{\mathbb{C}}M$ whose fibers are the i-eigenspaces and $(-i)$-eigenspaces of J, respectively, such that $T_{\mathbb{C}}M = T'M \oplus T''M$.*

► **Exercise 1.61.** Prove this lemma.

An almost complex structure on a smooth manifold M is said to be ***integrable*** if whenever V, W are smooth sections of $T'M$, then $[V, W]$ is also a section of $T'M$. Every almost complex structure on a 2-dimensional real manifold is integrable (see Problem 1-8), but in higher dimensions integrability is a nontrivial condition, as Problems 1-11 and 1-13 illustrate.

The integrability condition looks formally similar to the condition of involutivity for a distribution (subbundle of the tangent bundle) on a smooth manifold, which is a necessary and sufficient condition for the distribution to be tangent to a foliation (see [**LeeSM**, Chap. 19]). But there is no foliation associated with $T'M$ because it is not a subbundle of the (real) tangent bundle of M.

The following corollary is an immediate consequence of Proposition 1.59.

Corollary 1.62. *If J is an almost complex structure on a smooth manifold, a necessary condition for J to be the canonical almost complex structure associated with a holomorphic structure is that J be integrable.* □

The following important converse was proved in 1957 by August Newlander and Louis Nirenberg, showing that integrability is also sufficient.

Theorem 1.63 (Newlander–Nirenberg). *If an almost complex structure on a smooth manifold is integrable, then it arises from a holomorphic structure.*

We will neither prove nor use this theorem (except in Example 1.64 and Problem 7-10 below, which are not essential to our main story). There are several known proofs, all based on deep results from the theory of partial differential equations. Two different proofs can be found in [**Nir73**] and [**Hör90**].

Not every smooth manifold admits an almost complex structure. Two simple requirements are that the manifold must be even-dimensional (Cor. 1.53) and orientable (Problem 1-9). In two real dimensions, these conditions are sufficient, as

Example 1.64 below will show. But in higher dimensions, there are other topological obstructions that are not so easily described. Two spheres that admit almost complex structures are $\mathbb{S}^2$ (by Example 1.64 below) and $\mathbb{S}^6$ (by Problem 1-13). The structure on $\mathbb{S}^2$ is integrable, and turns it into a complex manifold biholomorphic to $\mathbb{CP}^1$ (see Problem 2-4). The known structure on $\mathbb{S}^6$ is not integrable, and it is not known whether $\mathbb{S}^6$ carries a holomorphic structure. It was proved by Armand Borel and Jean-Pierre Serre in 1953 [**BS53**] that $\mathbb{S}^2$ and $\mathbb{S}^6$ are the only spheres that carry almost complex structures; so no other spheres can be made into complex manifolds. A modern proof of this fact can be found in [**May99**, p. 208]. Since our main concern is to study complex manifolds, which already come equipped with canonical almost complex structures, we do not pursue the general question of existence of almost complex structures any further.

Example 1.64 (Holomorphic Structures on 2-Manifolds). Suppose M is an orientable real 2-manifold. We can always endow M with a Riemannian metric and an orientation. With that data, we can define an almost complex structure on M by letting J be "counterclockwise rotation by 90°." More precisely, for each nonzero $v \in T_pM$, we let Jv be the unique vector w such that $\langle v, w\rangle = 0$, $|w| = |v|$, and (v, w) is an oriented basis for T_pM. If (b_1, b_2) is any smooth oriented orthonormal local frame, then we have $Jb_1 = b_2$ and $Jb_2 = -b_1$, which shows that J is smooth. This almost complex structure is integrable by the result of Problem 1-8, so it arises from a holomorphic structure by the Newlander–Nirenberg theorem. Thus every orientable smooth real 2-manifold can be given a holomorphic structure. A real 2-manifold endowed with a particular holomorphic structure is called a ***Riemann surface***. Be careful about the terminology: a Riemann surface is a *complex curve* (1-dimensional complex manifold), while a *complex surface* is a 2-dimensional complex manifold. (It is possible for the same real 2-manifold to have different holomorphic structures that are not biholomorphic to each other, however; see Problem 1-4.) //

Problems

1-1. With $G \subseteq \mathrm{GL}(3, \mathbb{C})$ as in Example 1.20, let $\Gamma \subseteq G$ be the subgroup consisting of matrices whose entries are Gaussian integers. Prove that Γ is cocompact by showing that every coset in G/Γ has at least one representative lying in the unit cube $[0, 1]^6 \subseteq \mathbb{C}^3$.

1-2. Suppose $U \subseteq \mathbb{C}^n$ is open and $f : U \to \mathbb{C}$ is a holomorphic function that is nonzero on $U \smallsetminus S$, where $S \subseteq \mathbb{C}^n$ is a complex-linear subspace of codimension at least 2. Show that f is nonzero everywhere in U.

1-3. Prove that every 1-dimensional Hopf manifold is biholomorphic to a complex torus $\mathbb{C}/\Lambda$, and determine an explicit lattice Λ.

1-4. For any two vectors $v, w \in \mathbb{C}$ that are linearly independent over $\mathbb{R}$, let $T_{v,w} = \mathbb{C}/\Lambda(v,w)$ denote the 1-dimensional complex torus obtained as a quotient of $\mathbb{C}$ by the lattice $\Lambda(v,w)$ generated by v and w.
   (a) For any such v, w, show that there exists $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ such that $T_{v,w}$ is biholomorphic to $T_{1,\tau}$.
   (b) Let $\mathrm{SL}(2,\mathbb{Z})$ denote the group of integer matrices with determinant 1. Suppose $\tau, \tau' \in \mathbb{C}$ satisfy $\operatorname{Im}\tau > 0$ and $\operatorname{Im}\tau' > 0$. Show that $T_{1,\tau}$ is biholomorphic to $T_{1,\tau'}$ if and only if there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ such that $\tau' = (a\tau + b)/(c\tau + d)$. [Hint: Show that any biholomorphism $T_{1,\tau} \to T_{1,\tau'}$ lifts to an automorphism of $\mathbb{C}$.]

1-5. Show that the Lie group $\mathrm{U}(n)$ acts continuously and transitively on the Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$ by $A \cdot S = A(S)$ for $A \in \mathrm{U}(n)$ and $S \subseteq \mathbb{C}^n$ a subspace of dimension k. Use this to show that $\mathrm{G}_k(\mathbb{C}^n)$ is compact for every k and n.

1-6. Suppose $E \to M$ is a complex vector bundle. Show that there exists a conjugation operator, that is, a conjugate-linear bundle homomorphism $c\colon E \to E$ satisfying $c \circ c = \mathrm{Id}$, if and only if E is isomorphic (over $\mathbb{C}$) to the complexification of a real bundle.

1-7. Prove Proposition 1.58 ($T_J M$ is a smooth complex vector bundle isomorphic to $T'M$).

1-8. Prove that every almost complex structure on a real 2-manifold is integrable.

1-9. Suppose M is a smooth manifold that admits an almost complex structure. Prove that M is orientable.

1-10. Let M be a smooth manifold and J be an almost complex structure on M. Define a map $N\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ by

$$N(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y].$$

   (a) Show that N is bilinear over $C^\infty(M)$, and therefore defines a $(1,2)$-tensor field on M, called the ***Nijenhuis tensor of J***.
   (b) Show that J is integrable if and only if $N \equiv 0$. [Hint: Extend N to act on complex vector fields, and take X and Y to be smooth sections of $T'M$ or $T''M$.]

1-11. For $n \geq 2$, define an almost complex structure on $\mathbb{C}^n$ as follows:

$$J\frac{\partial}{\partial x^1} = (1 + (x^2)^2)\frac{\partial}{\partial y^1}, \qquad J\frac{\partial}{\partial y^1} = -\frac{1}{(1+(x^2)^2)}\frac{\partial}{\partial x^1},$$
$$J\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}, \qquad J\frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k}, \qquad k = 2, \dots, n.$$

Show that J is not integrable.

1-12. Let M be a $2n$-dimensional smooth manifold. Suppose ζ is a smooth closed complex n-form on M that is ***locally decomposable*** (i.e., can locally be written as a wedge product of complex 1-forms), and satisfies $\zeta \wedge \bar{\zeta} \neq 0$ everywhere on M. Show that there is a unique integrable almost complex structure on M for which $T'M = \{v \in T_{\mathbb{C}}M : v \lrcorner \bar{\zeta} = 0\}$ (where $\lrcorner$ denotes ***interior multiplication*** with a vector field, defined by $(v \lrcorner \bar{\zeta})(\dots) = \bar{\zeta}(v, \dots)$; see [**LeeSM**, p. 358]).

1-13. AN ALMOST COMPLEX STRUCTURE ON $\mathbb{S}^6$: Let $\mathbb{O}$ denote the algebra of ***octonions***, which is an 8-dimensional nonassociative algebra over $\mathbb{R}$ defined as follows. Start with the ***quaternions***, the 4-dimensional associative algebra $\mathbb{H}$ over $\mathbb{R}$ with basis $(\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k})$ and bilinear multiplication defined by

$$\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = -\mathbb{1}, \quad \mathbb{1}q = q\mathbb{1} = q \text{ for all } q \in \mathbb{H},$$
$$\mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i} = \mathbb{k}, \quad \mathbb{j}\mathbb{k} = -\mathbb{k}\mathbb{j} = \mathbb{i}, \quad \mathbb{k}\mathbb{i} = -\mathbb{i}\mathbb{k} = \mathbb{j}.$$

Then define $\mathbb{O} = \mathbb{H} \times \mathbb{H}$, with the bilinear product defined by

$$(p,q)(r,s) = (pr - sq^*,\ p^*s + rq),$$

where the conjugate of a quaternion is

$$(w\mathbb{1} + x\mathbb{i} + y\mathbb{j} + z\mathbb{k})^* = w\mathbb{1} - x\mathbb{i} - y\mathbb{j} - z\mathbb{k}.$$

Define the conjugate of an octonion $P = (p,q)$ by $P^* = (p^*, -q)$. Let $\mathbb{R} = \{P \in \mathbb{O} : P^* = P\}$ denote the set of real octonions, identified with the real numbers in the natural way, and $\mathbb{E} = \{P \in \mathbb{O} : P^* = -P\}$ the set of imaginary octonions. Define an inner product on $\mathbb{O}$ by $\langle P, Q\rangle = \frac{1}{2}(P^*Q + Q^*P) \in \mathbb{R}$. Let $\mathbb{S} = \{P \in \mathbb{E} : |P| = 1\}$ be the unit sphere in $\mathbb{E}$, and for each $P \in \mathbb{S}$, define a map $J_P : T_P\mathbb{S} \to \mathbb{O}$ by $J_P(Q) = QP$, where we identify $T_P\mathbb{S}$ with the real-linear subspace $P^\perp \cap \mathbb{E} \subseteq \mathbb{O}$. Although the multiplication in $\mathbb{O}$ is not associative, it is the case that $(PQ)^* = Q^*P^*$ and $(PQ)P = P(QP)$ for all $P, Q \in \mathbb{O}$, and you may use these facts without proof. (See also Problem 8-7 in [**LeeSM**].)

(a) Show that J_P maps $T_P\mathbb{S}$ to itself, and defines an almost complex structure on $\mathbb{S}$.

(b) Show that this almost complex structure is not integrable.

[Remark: It is still unknown whether $\mathbb{S}^6$ admits an integrable almost complex structure. Many well-known and respected mathematicians have written papers purporting to answer this question one way or the other, but all the proofs have been found to be wrong or incomplete.]

1-14. Let (M, g) and (N, h) be Riemannian manifolds of the same dimension. A smooth map $F : M \to N$ is said to be ***conformal*** if $F^*h = \lambda g$ for some smooth, positive function λ on M.

(a) Suppose (M, g) and (N, h) are oriented Riemannian 2-manifolds, and give M and N the holomorphic structures described in Example 1.64. Suppose $F : M \to N$ is a local diffeomorphism. Show that F is holomorphic if and only if it is conformal and orientation-preserving.

(b) Give examples of diffeomorphisms $F, G : \mathbb{C}^2 \to \mathbb{C}^2$ such that F is holomorphic but not conformal, and G is conformal and orientation-preserving but not holomorphic.

10.1090/gsm/244/02

Chapter 2

Complex Submanifolds

In this chapter, we introduce tools for studying complex submanifolds, which will yield a rich new source of examples of complex manifolds.

Variations on the Inverse Function Theorem

In smooth manifold theory, the primary technical tools for constructing new smooth manifolds out of old ones are the inverse function theorem and its friends, the implicit function theorem and the rank theorem (see [**LeeSM**, Chaps. 4 and 5]). All of these generalize straightforwardly to the holomorphic category.

Theorem 2.1 (Holomorphic Inverse Function Theorem). *Suppose M and N are complex n-manifolds, $F : M \to N$ is holomorphic, and the holomorphic Jacobian $D'F(p)$ is nonsingular for some $p \in M$. Then there exist connected neighborhoods U_0 of p and V_0 of $F(p)$ such that F restricts to a biholomorphism from U_0 to V_0.*

Proof. The hypothesis implies that $DF(p)$ is nonsingular, so by the ordinary inverse function theorem there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism. We already know F is holomorphic, so it remains only to show that F^{-1} is also.

By choosing local holomorphic coordinates on U_0 and V_0 (after shrinking both neighborhoods if necessary) and replacing F by its coordinate representation, we can reduce the problem to the case in which F is a holomorphic diffeomorphism between open subsets of $\mathbb{C}^n$, and we can use coordinates $(z^1, \dots, z^n)$ for both the domain and codomain. Let $G = F^{-1}$, and consider the lth coordinate function G^l. It satisfies $G^l \circ F(z) = z^l$. From the chain rule (Prop. 1.47), we see that

$$0 = \frac{\partial z^l}{\partial \bar{z}^j} = \frac{\partial (G^l \circ F)}{\partial \bar{z}^j} = \frac{\partial G^l}{\partial z^k}\frac{\partial F^k}{\partial \bar{z}^j} + \frac{\partial G^l}{\partial \bar{z}^k}\frac{\partial \overline{F}^k}{\partial \bar{z}^j}.$$

The first term on the right-hand side vanishes because F is holomorphic. Because the matrix $(\partial\overline{F}^k \partial\bar{z}^k)$ is invertible (it is the conjugate of the holomorphic Jacobian of F), this implies that $\partial G^l/\partial\bar{z}^k \equiv 0$. Since this is true for all l and k, it follows that G is holomorphic. □

Theorem 2.2 (Holomorphic Implicit Function Theorem). *Let $U \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be an open subset, and denote the standard holomorphic coordinates there by $(z, w) = (z^1, \dots, z^n, w^1, \dots, w^m)$. Suppose $\Phi : U \to \mathbb{C}^m$ is a holomorphic map, and the $m \times m$ matrix $(\partial\Phi^j/\partial w^k)$ is nonsingular at some $(a, b) \in U$. Let $c = \Phi(a, b)$. Then there exist neighborhoods V_0 of a in $\mathbb{C}^n$ and W_0 of b in $\mathbb{C}^m$ and a holomorphic function $F : V_0 \to W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F:*

$$\Phi^{-1}(c) \cap (V_0 \times W_0) = \{(z, w) \in V_0 \times W_0 : w = F(z)\}.$$

Proof. By shrinking U if necessary, we may assume that the matrix $(\partial\Phi^j/\partial w^k)$ is nonsingular on all of U. The hypothesis implies that the holomorphic Jacobian of the map $w \mapsto \Phi(a, w)$ is nonsingular at $w = b$, so Corollary 1.46 shows that the total derivative of this map is also nonsingular. The smooth version of the implicit function theorem then implies the existence of neighborhoods V_0 and W_0 and a smooth function $F : V_0 \to W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F. It remains only to show that F is holomorphic.

The fact that $\Phi(z, F(z)) \equiv c$ for $z \in V_0$, together with the chain rule, implies that for each l and j,

$$0 = \frac{\partial}{\partial\bar{z}^j}\Phi^l(z, F(z)) = \frac{\partial\Phi^l}{\partial\bar{z}^j} + \frac{\partial\Phi^l}{\partial w^k}\frac{\partial F^k}{\partial\bar{z}^j} + \frac{\partial\Phi^l}{\partial\bar{w}^k}\frac{\partial\overline{F}^k}{\partial\bar{z}^j}.$$

The first term and the last term on the right-hand side are zero because Φ is holomorphic. As in the previous proof, the invertibility of the matrix $(\partial\Phi^l/\partial w^k)$ on U ensures that $\partial F^k/\partial\bar{z}^j \equiv 0$. □

Theorem 2.3 (Holomorphic Rank Theorem). *Suppose M and N are complex manifolds of dimensions m and n, respectively, and $F : M \to N$ is a holomorphic map whose holomorphic Jacobian has constant rank r. For each $p \in M$ there exist holomorphic charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form*

$$\widehat{F}(z^1, \dots, z^r, z^{r+1}, \dots, z^m) = (z^1, \dots, z^r, 0, \dots, 0).$$

► **Exercise 2.4.** Prove this theorem by verifying that the proof of the ordinary rank theorem [**LeeSM**, Thm. 4.12] goes through essentially unchanged with the holomorphic inverse function theorem substituted for its smooth counterpart.

A holomorphic map between complex manifolds is called a ***submersion***, ***immersion***, or ***embedding*** if it has the corresponding property when considered as a smooth map between smooth manifolds: a submersion has surjective total derivative everywhere, an immersion has injective total derivative everywhere, and an embedding is an immersion that is also a homeomorphism onto its image in the subspace topology. Because the rank of the total derivative of a holomorphic map is twice that of its holomorphic Jacobian, these properties can also be characterized in terms of the latter; in particular, a holomorphic map $F: M \to N$ between complex manifolds is a submersion if and only if $D'F$ has rank equal to the complex dimension of N everywhere, and is an immersion if and only if $D'F$ has rank equal to the complex dimension of M everywhere.

For any continuous map $F: M \to N$ between topological spaces, recall that a ***local section of F*** is a continuous map $\sigma: U \to M$ defined on an open subset $U \subseteq N$ and satisfying $F \circ \sigma = \mathrm{Id}_U$.

Corollary 2.5 (Holomorphic Local Section Theorem). *Suppose $\pi: M \to N$ is a holomorphic submersion. Then every point of M is in the image of a holomorphic local section of π.*

Proof. For any $p \in M$, by the rank theorem we can choose holomorphic coordinates centered at p and $\pi(p)$ in which the coordinate representation of π is $\widehat{\pi}(z^1, \dots, z^n) = (z^1, \dots, z^r)$. The map $\sigma(z^1, \dots, z^r) = (z^1, \dots, z^r, 0, \dots, 0)$ is a holomorphic local section of π sending $\pi(p)$ to p. □

Corollary 2.6 (Characteristic Property of Surjective Holomorphic Submersions). *Suppose M, N, and P are complex manifolds. If $\pi: M \to N$ is a surjective holomorphic submersion, then a map $F: N \to P$ is holomorphic if and only if $F \circ \pi$ is holomorphic.*

Proof. The "only if" claim is just the fact that compositions of holomorphic maps are holomorphic. For the "if" part, suppose $F \circ \pi$ is holomorphic. Given $q \in N$, we can find a local holomorphic section σ of π defined on a neighborhood U of q, and then $F|_U = (F \circ \pi) \circ \sigma$, which is a composition of holomorphic maps. □

Corollary 2.7 (Passing to a Holomorphic Quotient). *Suppose M, N, and P are complex manifolds and $\pi: M \to N$ is a surjective holomorphic submersion. If $F: M \to P$ is a holomorphic map that is constant on the fibers of π, then there is a unique holomorphic map $\widetilde{F}: N \to P$ such that $\widetilde{F} \circ \pi = F$.*

► **Exercise 2.8.** Prove this corollary.

Example 2.9 (Projective Transformations of $\mathbb{CP}^n$). Suppose $A: \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is an invertible complex-linear map. Because $A(\lambda z) = \lambda A(z)$ for $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^{n+1}$, Corollary 2.7 applied to $\pi \circ A: \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$ shows that A descends to a holomorphic map $\widetilde{A}: \mathbb{CP}^n \to \mathbb{CP}^n$ defined by $\widetilde{A}([z]) = [Az]$. Since $\widetilde{A^{-1}}$ is an inverse for $\widetilde{A}$, it follows that $\widetilde{A}$ is an automorphism, called a ***projective transformation***. In fact, as we will see eventually, every automorphism of $\mathbb{CP}^n$ is a projective transformation. (See Problem 2-9 for the case $n = 1$, and Problem 9-9 for higher dimensions.) //

► **Exercise 2.10.** Let V be an n-dimensional complex vector space and let $\mathbb{P}(V)$ be its projectivization (see Example 1.11). Show that the bijections $\mathbb{CP}^n \to \mathbb{P}(V)$ determined by any two different bases of V differ by a projective transformation of $\mathbb{CP}^n$, and therefore the complex manifold structure of $\mathbb{P}(V)$ is independent of the choice of basis.

Complex Submanifolds

If M is a complex manifold, an ***(embedded) complex submanifold of M*** is a subset $S \subseteq M$ that is a topological manifold in the subspace topology, and is endowed with a holomorphic structure such that the inclusion $S \hookrightarrow M$ is a holomorphic embedding. The ***(complex) codimension of S*** is $\dim_{\mathbb{C}} M - \dim_{\mathbb{C}} S$. A ***complex hypersurface*** is a complex submanifold of complex codimension 1. (Note that algebraic geometers, such as [**Har77**, **GH94**], typically define hypersurfaces more generally to include subsets with certain kinds of singularities; but for us a hypersurface will always mean a complex submanifold.) It is also possible to define *immersed complex submanifolds* analogously to immersed smooth ones; but we will not be making use of the immersed case, so we assume henceforth without further comment that all complex submanifolds are embedded.

For a smooth submanifold of a smooth manifold, we can always canonically identify the tangent space to the submanifold with a subspace of the ambient tangent space. There is an analogous identification for holomorphic tangent spaces of complex submanifolds. Suppose M is a complex manifold and $S \subseteq M$ is an embedded complex submanifold, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. Since ι is a holomorphic embedding, for each $p \in S$ the holomorphic Jacobian $D'\iota(p)\colon T'_pS \to T'_pM$ is an injective complex-linear map. Using this map, we can identify T'_pS as a complex-linear subspace of T'_pM.

Suppose (U, φ) is a holomorphic coordinate chart for M and $K \subseteq \mathbb{C}^n$ is a complex-linear subspace of dimension k. The subset $\varphi^{-1}(K)$ is called a ***holomorphic k-slice of U***.

Proposition 2.11 (Holomorphic Slice Criterion). *Suppose M is a complex n-manifold and S is a subset of M with the subspace topology. Then S has a unique complex manifold structure that makes it into an embedded complex submanifold if and only if for some fixed k, each $p \in S$ is contained in the domain of a holomorphic coordinate chart (U, φ) in which $S \cap U$ is a holomorphic k-slice (called a* ***holomorphic slice chart for S****).*

► **Exercise 2.12.** Prove this proposition by verifying that the analogous proof for smooth manifolds [**LeeSM**, Thms. 5.8 and 5.31] carries through with the holomorphic rank theorem in place of its smooth analogue.

Corollary 2.13. *Suppose M is a complex n-manifold and $S \subseteq M$. Then S is a complex k-submanifold if and only if each $p \in S$ has a neighborhood U in M such that $S \cap U$ is the zero set of a holomorphic submersion $F : U \to \mathbb{C}^{n-k}$ (called a* ***local defining function for S****).*

► **Exercise 2.14.** Prove this corollary. [Hint: Start by showing that the subspace K in the definition of holomorphic slice charts can always be taken to be the subspace defined by $z^{k+1} = \cdots = z^n = 0$.]

Proposition 2.15 (Local Characterization of Submanifolds). *Suppose M is a complex manifold. A subset $S \subseteq M$ is an embedded complex manifold if and only if each point of S has a neighborhood U in M such that $U \cap S$ is an embedded complex submanifold of U.*

► **Exercise 2.16.** Prove this proposition.

A complex submanifold of codimension d is also a real submanifold of codimension $2d$. When d is positive, this results in the following important property.

Proposition 2.17. *Suppose M is a connected complex manifold and $S \subseteq M$ is a closed complex submanifold of positive codimension. Then $M \smallsetminus S$ is path-connected.*

Proof. Let n be the complex dimension of M and k the complex dimension of S, so that $k < n$. Let p and q be distinct points in $M \smallsetminus S$. Since connected manifolds are path-connected, there is a continuous path $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. For each $t \in [0,1]$, we choose a neighborhood U_t of $\gamma(t)$ in M as follows: If $\gamma(t) \notin S$, then because S is closed we can choose U_t such that $U_t \cap S = \varnothing$ and U_t is path-connected. If $\gamma(t) \in S$, let U_t be the domain of a holomorphic local slice chart, so $U_t \cap S$ is a holomorphic k-slice of U. In the latter case, we may shrink U_t so that it is biholomorphic to a product $\mathbb{B}^{2k} \times \mathbb{B}^{2n-2k}$, under a biholomorphism that sends $U_t \cap S$ to $\mathbb{B}^{2k} \times \{0\}$. Thus $U_t \smallsetminus S$ is biholomorphic to $\mathbb{B}^{2k} \times \left(\mathbb{B}^{2n-2k} \smallsetminus \{0\}\right)$. Because $2n - 2k \geq 2$, the set $\mathbb{B}^{2n-2k} \smallsetminus \{0\}$ is path-connected, and thus so is $U_t \smallsetminus S$.

By the Lebesgue number lemma [**LeeTM**, Lemma 7.18], there are points $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\gamma([t_{i-1}, t_i])$ is contained in one of the sets U_t described above for each i; call this set U_i. For each $i = 1, \ldots, m-1$, the open set $U_i \cap U_{i+1}$ is nonempty because it contains $\gamma(t_i)$; and because S contains no nonempty open subset of M, we can choose a point $x_i \in U_i \cap U_{i+1} \smallsetminus S$. Let $x_0 = p$ and $x_m = q$. Then because $U_i \smallsetminus S$ is path connected, for each $i = 1, \ldots, m$ there is a path σ_i in $U_i \smallsetminus S$ from x_{i-1} to x_i; and then the concatenation of these paths is a path in $M \smallsetminus S$ from p to q. □

Proposition 2.18 (Restricting Domains or Codomains of Holomorphic Maps). *Suppose M and N are complex manifolds and $F : M \to N$ is a holomorphic map.*

(a) *If $S \subseteq M$ is an embedded complex submanifold, then $F|_S : S \to N$ is holomorphic.*

(b) *If $T \subseteq N$ is an embedded complex submanifold such that $F(M) \subseteq T$, then F is holomorphic as a map from M to T.*

Proof. Part (a) follows because $F|_S$ is the composition of the inclusion $S \hookrightarrow M$ followed by F. For Part (b), it follows from smooth manifold theory that F is smooth as a map into T, and then holomorphicity can be checked by using holomorphic slice coordinates. □

Examples of Complex Submanifolds

The notion of complex submanifolds gives us some tools for defining many more examples of complex manifolds.

Example 2.19 (Open Submanifolds). If M is a complex manifold, every open subset of M is an embedded complex submanifold of codimension 0. //

Example 2.20 (Images of Holomorphic Embeddings). If $F : N \to M$ is a holomorphic embedding, then the image $F(N)$ is an embedded complex submanifold, directly from the definition. //

Example 2.21 (Graphs of Holomorphic Maps). Suppose $f : M \to N$ is a holomorphic map. The ***graph of f*** is the set $\Gamma(f) = \{(p, f(p)) \subseteq M \times N : p \in M\}$. This is a complex submanifold of $M \times N$, biholomorphic to M, because the map from M to $M \times N$ given by $p \mapsto (p, f(p))$ is a holomorphic embedding. //

Example 2.22 (Regular Level Sets). Suppose M and N are complex manifolds, $\Phi : M \to N$ is a holomorphic map, and $c \in N$ is a ***regular value of Φ*** (meaning that $D\Phi(p)$ is surjective for each $p \in \Phi^{-1}(c)$). Then the level set $\Phi^{-1}(c)$ is a complex submanifold of M whose codimension is equal to the dimension of N. The proof is a simple application of the holomorphic rank theorem and the holomorphic slice criterion. //

Example 2.23 (Transverse Preimages). Suppose M and N are complex manifolds and $S \subseteq N$ is a complex submanifold. A holomorphic map $F : M \to N$ is said to be ***transverse to S*** if for every $x \in F^{-1}(S)$, the subspaces $DF(x)(T_x M)$ and $T_{F(x)}S$ together span $T_{F(x)}N$. If this is the case, then $F^{-1}(S)$ is a complex submanifold whose codimension in M is equal to the codimension of S in N. //

► **Exercise 2.24.** Prove the above claim: the preimage of a complex submanifold under a holomorphic map that is transverse to the submanifold is a complex submanifold of the same codimension. (See Theorem 6.30 of [**LeeSM**] for the smooth version; essentially the same proof goes through in the holomorphic case.)

Example 2.25 (Complex Submanifolds of $\mathbb{C}^n$). It is important to observe, to begin with, that there are no compact complex submanifolds of $\mathbb{C}^n$ except 0-dimensional ones. To verify this, note that if $M \subseteq \mathbb{C}^n$ is a compact complex submanifold, then each of the coordinate functions z^j restricts to a global holomorphic function on M, which must therefore be constant on each connected component of M by Corollary 1.33.

On the other hand, many important examples of noncompact complex manifolds arise as complex submanifolds of $\mathbb{C}^n$. Some obvious ones are graphs of holomorphic functions from $\mathbb{C}^k$ to $\mathbb{C}^{n-k}$, and level sets of holomorphic submersions from $\mathbb{C}^n$ to $\mathbb{C}^{n-k}$.

A particularly important class of examples is defined as follows: a subset $V \subseteq \mathbb{C}^n$ is called an ***affine algebraic variety*** if it is the common zero set of a finite collection $p_1, \dots, p_k$ of holomorphic polynomials. (Some authors require that a variety be ***irreducible***, meaning that it is not the union of two or more nonempty varieties, but we do not require this.) A point $z_0 \in V$ is a ***regular point of V*** if it has a neighborhood $U \subseteq \mathbb{C}^n$ such that $V \cap U$ is the common zero set of some of the polynomials $p_{i_1}, \dots, p_{i_r}$ with the property that the differentials $dp_{i_1}, \dots, dp_{i_r}$ are linearly independent at every point of $V \cap U$; points that are not regular are called ***singular points of V***. An affine algebraic variety is said to be ***nonsingular*** or ***smooth*** if it has no singular points; in this case V is a complex submanifold of $\mathbb{C}^n$ by Corollary 2.13. //

Example 2.26 (Complex Lie Subgroups). Suppose G is a complex Lie group. An (embedded) complex submanifold of G that is also a subgroup is called a ***complex Lie subgroup***; it inherits a complex Lie group structure of its own because the group operations are holomorphic by restriction. Some specific examples:

- The group $\mathrm{SL}(n,\mathbb{C})$ of complex matrices with determinant 1 is a complex Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$ because it is a level set of the holomorphic submersion $\det\colon \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}$.
- $\mathrm{O}(n,\mathbb{C})$ denotes the group of $n \times n$ complex matrices A satisfying $A^T A = \mathrm{Id}$. It is a regular level set of the holomorphic polynomial function $A \mapsto A^T A$, so it is a complex Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$.
- $\mathrm{SO}(n,\mathbb{C})$ is the subgroup of $\mathrm{O}(n,\mathbb{C})$ consisting of matrices of determinant 1; because it is a regular level set of $\det\colon \mathrm{O}(n,\mathbb{C}) \to \mathbb{C}$, it is a complex Lie subgroup.

It should be noted that the unitary group $\mathrm{U}(n) \subseteq \mathrm{GL}(n,\mathbb{C})$ is *not* a complex Lie group, because it does not have a holomorphic defining function, even locally. (See Problem 2-1.) //

Complex Submanifolds of Projective Spaces

A complex manifold M is called a ***projective manifold*** if it is biholomorphic to a closed complex submanifold of $\mathbb{CP}^n$. Because projective manifolds will be among our main objects of study, it is worthwhile to devote a little more time to understanding the structure of submanifolds of $\mathbb{CP}^n$.

The most important examples (in fact, thanks to Chow's theorem discussed below, the only examples) are algebraic ones, which we will define shortly. But since there are no nonconstant holomorphic functions from $\mathbb{CP}^n$ to $\mathbb{C}$, polynomial or otherwise, we have to be a little more careful about what we mean.

Suppose $p\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic polynomial function (that is, a polynomial function of the variables $w^0, \dots, w^n$). It is said to be ***homogeneous of degree d*** for a nonnegative integer d if $p(\lambda w) = \lambda^d w$ for all $\lambda \in \mathbb{C}$ and $w \in \mathbb{C}^{n+1}$. This just means that p can be written as a sum of monomial terms each of which has the same total degree d. Although such a polynomial does not produce a well-defined function from $\mathbb{CP}^n$ to $\mathbb{C}$, nonetheless its zero set is a well-defined subset of $\mathbb{CP}^n$, because if $p(w) = 0$, then $p(\lambda w) = 0$ for all $\lambda \in \mathbb{C} \smallsetminus \{0\}$.

A ***projective algebraic variety*** is a subset $V \subseteq \mathbb{CP}^n$ of the form

$$V = \{[w] \in \mathbb{CP}^n : p_1(w) = \cdots = p_k(w) = 0\},$$

where $p_1, \dots, p_k\colon \mathbb{C}^{n+1} \to \mathbb{C}$ are homogeneous holomorphic polynomials of various degrees. A projective variety is said to be ***nonsingular*** or ***smooth*** if the affine variety $\widetilde{V} \subseteq \mathbb{C}^{n+1}$ defined by $p_1, \dots, p_k$ has no singular points other than perhaps the origin. It follows from the next lemma that every nonsingular projective algebraic variety is actually a complex submanifold of $\mathbb{CP}^n$. Projective algebraic varieties are the main objects of study in complex algebraic geometry.

Lemma 2.27. *Suppose $M \subseteq \mathbb{CP}^n$ is a subset and $\widetilde{M} = \pi^{-1}(M) \subseteq \mathbb{C}^{n+1} \smallsetminus \{0\}$, where $\pi\colon \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$ is the canonical projection. Then M is a complex k-submanifold of $\mathbb{CP}^n$ if and only if $\widetilde{M}$ is a complex $(k+1)$-dimensional submanifold of $\mathbb{C}^{n+1} \smallsetminus \{0\}$.*

Proof. Problem 2-2. □

Lemma 2.28. *Every projective algebraic variety in $\mathbb{CP}^n$ is compact and therefore closed in $\mathbb{CP}^n$.*

Proof. Let $V \subseteq \mathbb{CP}^n$ be the variety determined by homogeneous polynomials $p_1, \dots, p_k\colon \mathbb{C}^{n+1} \to \mathbb{C}$, and let $\widetilde{V} = \pi^{-1}(V) \subseteq \mathbb{C}^{n+1} \smallsetminus \{0\}$, where $\pi\colon \mathbb{C}^{n+1} \smallsetminus \{0\}$ is the canonical projection. By homogeneity, every point $[w] \in V$ has a preimage $w' = w/|w|$ that lies in $\widetilde{V} \cap \mathbb{S}^{2n+1}$. Thus V is the image under π of the compact set $\widetilde{V} \cap \mathbb{S}^{2n+1}$, so it is compact. Since $\mathbb{CP}^n$ is a compact Hausdorff space, all compact subsets of it are closed. □

The algebraic submanifolds of projective space might seem like a restrictive special class, but in fact they are quite general, thanks to the following remarkable theorem proved in 1949 by Wei-Liang Chow [**Cho49**].

Theorem 2.29 (Chow's Theorem). *Every closed complex submanifold of* $\mathbb{CP}^n$ *is algebraic.*

Later, we will give a proof of Chow's theorem for the special case of hypersurfaces (see Cor. 9.53). We will neither prove nor use the general case, but you can find a proof in [**GH94**, p. 167]. (Actually, Chow proved something stronger—if $V \subseteq \mathbb{CP}^n$ is a closed ***analytic variety***, meaning that each point of V has a neighborhood U such that $V \cap U$ is the common zero set of finitely many holomorphic functions defined on U, then V is actually a projective algebraic variety. In 1956, Jean-Pierre Serre introduced a vast generalization of Chow's theorem in a famous paper called "Géométrie algébrique et géométrie analytique" [**Ser55b**]: roughly speaking, it showed that virtually any geometric structure that can be defined holomorphically in complex projective space can actually be defined algebraically. That general fact is now known as the *GAGA principle*, after the French title of Serre's paper.)

We note in passing that in algebraic geometry, there is another topology that is usually used for affine or projective algebraic varieties. The ***Zariski topology*** on $\mathbb{CP}^n$ is defined by declaring a set to be open if and only if its complement is an algebraic variety; the Zariski topology on $\mathbb{C}^n$ is defined similarly. The Zariski topology on a projective or affine algebraic variety is then defined as the induced subspace topology. From an algebraic point of view, this topology is advantageous because algebraic properties of subsets are directly reflected in topological properties, and because the topology makes sense for algebraic varieties defined over arbitrary fields. But the Zariski topology has properties quite different from those of the usual manifold topologies on $\mathbb{C}^n$ and $\mathbb{CP}^n$, most notably the fact that is never Hausdorff except in the 0-dimensional case. In this book, we will use only the standard topologies on $\mathbb{C}^n$ and $\mathbb{CP}^n$ and their subsets.

Algebraic varieties in $\mathbb{CP}^1$ are easy to describe.

Proposition 2.30. *A subset* $V \subseteq \mathbb{CP}^1$ *is an algebraic variety if and only if it is either finite or all of* $\mathbb{CP}^1$.

Proof. Suppose $V = \{a_1, \dots, a_k\}$ is a finite subset of $\mathbb{CP}^1$. If $V = \varnothing$, then it is the variety determined by the polynomial $f \equiv 1$. Otherwise, for each j, we can write $a_j = [a_j^0, a_j^1]$ for some point $(a_j^0, a_j^1) \in \mathbb{C}^2$, so $\{a_j\}$ is the variety determined by the polynomial $f_j(w^0, w^1) = a_j^1 w^0 - a_j^0 w^1$. It follows that V is the variety determined by the product $f_1 \cdot \dots \cdot f_k$. On the other hand, $\mathbb{CP}^1$ itself is the variety determined by the zero polynomial.

Conversely, if V is the variety determined by the homogeneous polynomials $f_1, \dots, f_k$, and V is not equal to $\mathbb{CP}^1$, then at least one of the polynomials, say f_j, is not identically zero. The points of V in $\mathbb{CP}^1 \smallsetminus \{[0,1]\}$ can all be written in the form $[1,z]$ where $f_j(1,z) = 0$. The polynomial function $z \mapsto f_j(1,z)$ has only finitely many zeros by the fundamental theorem of algebra. □

Our definition of a projective algebraic variety $V \subseteq \mathbb{CP}^n$ of codimension k allows for the possibility that more than k homogeneous polynomials are required to define it. It turns out that for nonsingular projective *hypersurfaces*, a single polynomial always suffices. We will not have the tools to prove this until Chapter 9, but we state it here for convenience because it will make some results easier to state. For the proof, see Corollary 9.52.

Proposition 2.31 (Algebraic Hypersurface Is Cut Out by a Single Polynomial). *If $V \subseteq \mathbb{CP}^n$ is a codimension-1 nonsingular algebraic variety, then V is the variety defined by a single homogeneous polynomial $p\colon \mathbb{C}^{n+1} \to \mathbb{C}$.*

To understand submanifolds of $\mathbb{CP}^n$ more deeply, we will use the fact that $\mathbb{CP}^n$ has a lot of symmetry. Because $\mathrm{GL}(n+1,\mathbb{C})$ acts transitively on 1-dimensional subspaces of $\mathbb{C}^{n+1}$, it follows that the group of projective transformations acts transitively on $\mathbb{CP}^n$. Two subsets $S_1, S_2 \subseteq \mathbb{CP}^n$ are said to be ***projectively equivalent*** if there is a projective transformation taking S_1 to S_2.

Let us explore some particular complex submanifolds of $\mathbb{CP}^n$. First of all, if $p_1, \dots, p_{n-k}$ are linearly independent holomorphic *linear* functions on $\mathbb{C}^{n+1}$, the variety V they determine is a k-dimensional complex submanifold of $\mathbb{CP}^n$ called a ***projective linear subspace***. If we choose a basis $b_0, \dots, b_k$ for the common kernel of $p_1, \dots, p_{n-k}$ in $\mathbb{C}^{n+1}$, then the map $[w^0, \dots, w^k] \mapsto [w^0 b_0 + \dots + w^k b_k]$ is a holomorphic embedding of $\mathbb{CP}^k$ into $\mathbb{CP}^n$, whose image is exactly the variety V. Thus every k-dimensional projective linear subspace of $\mathbb{CP}^n$ is biholomorphic to $\mathbb{CP}^k$. In particular, a 1-dimensional projective linear subspace is called a ***projective line***; a 2-dimensional one is called a ***projective plane***; and an $(n-1)$-dimensional one in $\mathbb{CP}^n$ is called a ***projective hyperplane***. Since any two $(k+1)$-dimensional subspaces of $\mathbb{C}^{n+1}$ are related by a holomorphic linear isomorphism, it follows that any two k-dimensional projective linear subspaces of $\mathbb{CP}^n$ are projectively equivalent.

Example 2.32 (Dual Projective Spaces). For any positive integer n, the set of projective hyperplanes in $\mathbb{CP}^n$ is called the ***dual projective space*** to $\mathbb{CP}^n$, and denoted by $(\mathbb{CP}^n)^*$. Since each projective hyperplane is the variety determined by a nonzero complex-linear functional $f \in (\mathbb{C}^{n+1})^*$, and two such functionals determine the same variety if and only if they differ by a nonzero complex multiple, $(\mathbb{CP}^n)^*$ is canonically identified with the projectivization $\mathbb{P}((\mathbb{C}^{n+1})^*)$, and therefore $(\mathbb{CP}^n)^*$ is biholomorphic to $\mathbb{CP}^n$. One such biholomorphism is obtained by sending $[a^0, \dots, a^n] \in \mathbb{CP}^n$ to the variety determined by the linear functional

$f_a(w) = a^0 w^0 + \cdots + a^n w^n$. (But note that this biholomorphism is heavily dependent on working with the standard basis of $\mathbb{C}^{n+1}$; another choice of basis will result in a different isomorphism between $\mathbb{CP}^n$ and $(\mathbb{CP}^n)^*$.) //

Now consider the map $I : \mathbb{C}^n \to \mathbb{CP}^n$ given by $I(z^1, \dots, z^n) = [1, z^1, \dots, z^n]$. It is the inverse of one of the holomorphic coordinate charts we defined in Example 1.9, and thus it is a biholomorphism onto its image. That image is exactly the complement of the projective hyperplane Π defined by $z^0 = 0$, so as a set, $\mathbb{CP}^n$ is the disjoint union of $I(\mathbb{C}^n)$ and Π. We can think of $\mathbb{CP}^n$ as the union of a copy of $\mathbb{C}^n$ together with the (projective) "hyperplane at infinity."

By following the embedding I with a suitable projective transformation, we can obtain a holomorphic embedding of $\mathbb{C}^n$ into $\mathbb{CP}^n$ whose image is the complement of any arbitrary projective hyperplane. Any such embedding is called an ***affine embedding of*** $\mathbb{C}^n$; the embedding I described above is called the ***standard affine embedding***. Thus any nontrivial complex-linear functional p on $\mathbb{C}^{n+1}$ determines a decomposition of $\mathbb{CP}^n$ into a projective hyperplane (determined by the kernel of p) and its complement, which is a dense, open subspace that is the image of an affine embedding.

Proposition 1.57 showed that the holomorphic tangent space to a point of $\mathbb{C}^n$ is canonically identified with $\mathbb{C}^n$ itself, from which it follows that the holomorphic tangent space to a complex submanifold of $\mathbb{C}^n$ can be identified with a linear subspace of $\mathbb{C}^n$. The next proposition gives a projective version of this identification.

Proposition 2.33 (The Projective Tangent Space). *Suppose $M \subseteq \mathbb{CP}^n$ is a k-dimensional complex submanifold. For each $p \in M$, there is a unique k-dimensional projective linear subspace $\Pi \subseteq \mathbb{CP}^n$ containing p with the property that $T'_p M = T'_p \Pi$. It is called the* ***projective tangent space to M at p***.

Proof. First we prove existence. Let $\widetilde{M} = \pi^{-1}(M) \subseteq \mathbb{C}^{n+1} \setminus \{0\}$. Lemma 2.27 shows that $\widetilde{M}$ is a $(k+1)$-dimensional complex submanifold. Given $p \in M$, let $\tilde{p}$ be any point in $\pi^{-1}(p)$. Since π maps $\widetilde{M}$ to M, the linear map $D'\pi(\tilde{p})$ maps $T'_{\tilde{p}}\widetilde{M}$ into $T'_p M$. The kernel of $D'\pi(\tilde{p})$ is spanned by the vector $\tilde{p}^j \partial/\partial w^j\big|_{\tilde{p}}$. This span is contained in $T'_{\tilde{p}}\widetilde{M}$, so when $D'\pi(\tilde{p})$ is restricted to the $(k+1)$-dimensional subspace $T'_{\tilde{p}}\widetilde{M} \subseteq T'_{\tilde{p}}\mathbb{C}^{n+1}$, its rank is exactly k. Thus it is surjective onto $T'_p M$ for dimensional reasons: $T'_p M = D'\pi(\tilde{p})\big(T'_{\tilde{p}}\widetilde{M}\big)$.

Under the canonical identification of $T'_{\tilde{p}}\mathbb{C}^{n+1}$ with $\mathbb{C}^{n+1}$ itself (see Prop. 1.57), we may consider $T'_{\tilde{p}}\widetilde{M}$ as a $(k+1)$-dimensional linear subspace $\widetilde{\Pi} \subseteq \mathbb{C}^{n+1}$. Because $\tilde{p}^j \partial/\partial w^j\big|_{\tilde{p}} \in T'_{\tilde{p}}\widetilde{M}$, it follows that $\tilde{p} \in \widetilde{\Pi}$. The image of $\widetilde{\Pi}$ in $\mathbb{CP}^n$ is a k-dimensional projective linear subspace Π containing p, and the argument above with $\widetilde{\Pi}$ in place of $\widetilde{M}$ shows that

$$T'_p\Pi = D'\pi(\tilde{p})\big(T'_{\tilde{p}}\widetilde{\Pi}\big) = D'\pi(\tilde{p})\big(T'_{\tilde{p}}\widetilde{M}\big) = T'_p M.$$

Thus Π is the projective subspace we seek.

If Π' is another such subspace and $\widetilde{\Pi}' = \pi^{-1}(\Pi')$, then $T'_{\tilde{p}}\widetilde{\Pi}'$ contains the vector $\tilde{p}^j \partial/\partial z^j\big|_{\tilde{p}}$ and is mapped onto $T'_p M$ by $D'\pi(\tilde{p})$. It follows that $T'_{\tilde{p}}\widetilde{\Pi}' = T'_{\tilde{p}}\widetilde{\Pi}$, from which it follows that $\widetilde{\Pi}' = \widetilde{\Pi}$ and therefore $\Pi' = \Pi$. □

The case of $\mathbb{CP}^1$ is especially notable. In this case, a projective hyperplane is just a single point, so $\mathbb{CP}^1$ is the union of a copy of $\mathbb{C}$ (which we may identify with the set of points of the form $[1, z]$ under the standard affine embedding) together with the "point at infinity" (the point $[0, 1]$, which we denote by ∞ in this context). Topologically, it is homeomorphic to the one-point compactification of $\mathbb{C}$, also called the ***Riemann sphere***. In fact, Problem 2-4 shows that $\mathbb{CP}^1$ is *diffeomorphic* to $\mathbb{S}^2$.

Thanks to the result of Problem 2-9, every automorphism of $\mathbb{CP}^1$ is a map of the form $m([w, z]) = [cz + dw, az + bw]$ for some complex numbers a, b, c, d with $ad - bc \neq 0$ (to ensure that the corresponding linear map on $\mathbb{C}^2$ is invertible). Assuming $c \neq 0$, we can write this map in affine coordinates (defined by $w = 1$) as

$$m(z) = \begin{cases} \dfrac{az + b}{cz + d}, & z \neq \infty,\ -d/c, \\ \dfrac{a}{c}, & z = \infty, \\ \infty, & z = -d/c. \end{cases}$$

In the remaining case $c = 0$, it just maps $z \in \mathbb{C}$ to $(az + b)/d$ and ∞ to ∞. Any such automorphism is called a ***Möbius transformation***.

The higher-dimensional projective spaces are not diffeomorphic to spheres. But we do have the following.

Proposition 2.34. *For every $n \geq 1$, $\mathbb{CP}^n$ is simply connected.*

Proof. We will prove this by induction on n. For $n = 1$, it follows from the fact that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$, which is simply connected. So suppose $n \geq 1$ and we have shown that $\mathbb{CP}^n$ is simply connected. We can write $\mathbb{CP}^{n+1} = U \cup V$, where

$$U = \left\{[w^0, \dots, w^{n+1}] \in \mathbb{CP}^{n+1} : w^0 \neq 0\right\},$$
$$V = \mathbb{CP}^{n+1} \smallsetminus \{[1, 0, \dots, 0]\}.$$

Then U is the image of the standard affine embedding and thus biholomorphic to $\mathbb{C}^{n+1}$, so it is simply connected. We will show that V is homotopy equivalent to $\mathbb{CP}^n$. Consider the map $H : V \times [0, 1] \to V$ given by

$$H([w^0, \dots, w^{n+1}], t) = [tw^0, w^1, \dots, w^{n+1}].$$

If we let Π denote the projective hyperplane defined by $w^0 = 0$, then Π is biholomorphic to $\mathbb{CP}^n$, and H is a strong deformation retraction of V onto Π. It follows from the induction hypothesis that V is also simply connected, so the Seifert–Van Kampen theorem (specifically, [**LeeTM**, Cor. 10.5]) shows that $\mathbb{CP}^{n+1}$ is simply connected. □

Next we explore some examples of algebraic submanifolds of $\mathbb{CP}^n$ of degree higher than 1. Suppose $S \subseteq \mathbb{CP}^n$ is a projective algebraic variety defined by a single homogeneous polynomial of degree d; in this case we say d is the ***degree*** of the variety. Projective varieties cut out by single polynomials of degree 2, 3, 4, and 5 are called ***projective quadrics***, ***cubics***, ***quartics***, and ***quintics***, respectively.

To study projective quadrics, we begin with some general considerations about homogeneous quadratic polynomials.

Lemma 2.35. *Suppose p is a homogeneous quadratic holomorphic polynomial function on an n-dimensional complex vector space V. It is always possible to find a basis for V in which p has the coordinate representation*

$$p(w^1, \dots, w^n) = (w^1)^2 + \cdots + (w^r)^2 \tag{2.1}$$

for some $0 \le r \le n$. The integer r, called the ***rank of p****, is independent of the choice of basis.*

Proof. The proof is essentially a complex-linear version of the Gram-Schmidt algorithm. Begin by choosing a basis for V and writing p in the form $p(z) = \sum_{j,k} p_{ij} z^i z^j$. We can assume that the matrix p_{ij} is symmetric—if not, just replace p_{ij} with $\frac{1}{2}(p_{ij} + p_{ji})$, which does not change the values of the polynomial p. Define $B\colon V \times V \to \mathbb{C}$ by $B(z, w) = \sum_{j,k} p_{ij} z^i w^j$, so that B is a symmetric complex-bilinear form satisfying $B(z, z) = p(z)$. Note that B can also be written in the form $B(z, w) = \frac{1}{4}\big(p(z + w) - p(z - w)\big)$, so it is independent of the choice of basis.

We will prove by induction on $n = \dim V$ that V has a basis in which B has the form $B(z, w) = z^1 w^1 + \cdots + z^r w^r$ for some r, which implies the result. For $n = 0$, there is nothing to prove.

Assume the result is true for spaces of dimension $n - 1$ and suppose V has dimension n. If $B \equiv 0$, any basis will do. Otherwise, there is some $b_1 \in V$ such that $B(b_1, b_1) \ne 0$. After multiplying b_1 by an appropriate complex scalar, we can arrange that $B(b_1, b_1) = 1$. Let $S \subseteq V$ be the set $\{z \in V : B(z, b_1) = 0\}$. Because $z \mapsto B(z, b_1)$ is a nonzero linear functional, S is a complex-linear subspace of dimension $n - 1$. The induction hypothesis shows that S has a basis $(b_2, \dots, b_n)$ in which the restriction of B to S has the coordinate formula $B(z, w) = z^2 w^2 + \cdots + z^r w^r$, and then a simple computation shows that the basis $(b_1, \dots, b_n)$ for V has the desired property.

To prove that the rank is independent of the choice of basis, just note that $n - r$ is equal to the largest dimension of a complex-linear subspace $W \subseteq V$ such that $B(w, z) = 0$ for all $w \in W$ and $z \in V$. □

Proposition 2.36. *Suppose p is a holomorphic homogeneous quadratic polynomial on $\mathbb{C}^{n+1}$. The variety $V \subseteq \mathbb{CP}^n$ defined by p is nonsingular if and only if the rank of p is equal to $n + 1$.*

Proof. After a complex-linear change of basis, we may assume that p has the form $p(w) = (w^0)^2 + \cdots + (w^{r-1})^2$, where r is the rank of p. If $r = n + 1$, then we can write the differential of p on $\mathbb{C}^{n+1}$ as

$$dp = 2\big(w^0\, dw^0 + \cdots + w^n\, dw^n\big).$$

This never vanishes except at the origin, showing that V is nonsingular.

Conversely, suppose $r < n + 1$. In the affine coordinate chart $(z^1, \dots, z^n) \leftrightarrow [z^1, \dots, z^n, 1]$, V is the set of points where $(z^1)^2 + \cdots + (z^r)^2 = 0$. All of the following smooth curves starting at the origin lie in V:

$$\begin{aligned} &t \mapsto (t, 0, \dots, 0, it, 0, \dots, 0), \qquad &&t \mapsto (it, 0, \dots, 0, t, 0, \dots, 0), \qquad j = 2, \dots, n\\ &t \mapsto (t, -it, 0, \dots, 0), &&t \mapsto (it, -t, 0, \dots, 0) \end{aligned}$$

(where the nonzero entries in the first line are in positions 1 and j). The initial velocity vectors of these curves span the entire (real) tangent space to $\mathbb{C}^n$ at the origin. Since elsewhere V is a complex codimension-1 submanifold and thus a smooth submanifold of real codimension 2, this shows that V is not a smooth submanifold in a neighborhood of the origin in this affine chart. □

Proposition 2.37. *All nonsingular projective quadrics in $\mathbb{CP}^n$ are projectively equivalent.*

Proof. If $V \subseteq \mathbb{CP}^n$ is a nonsingular projective quadric defined by a quadratic polynomial p, Proposition 2.36 shows that p has rank $n+1$, and then Lemma 2.35 shows that a change of basis in $\mathbb{C}^{n+1}$ (which induces a projective transformation on $\mathbb{CP}^n$), puts p into the standard form $(w^0)^2 + \cdots + (w^n)^2$. □

Problems 2-5 and 2-6 show that nonsingular quadrics in $\mathbb{CP}^2$ are biholomorphic to $\mathbb{CP}^1$, and those in $\mathbb{CP}^3$ are biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. The higher-dimensional ones are diffeomorphic to certain Grassmann manifolds; see Problem 2-7.

Now let us look at some more general projective manifolds. In general, one can get an idea of the structure of an algebraic variety by expressing it in affine coordinates; computationally, this just amounts to setting one of the homogeneous coordinates to 1, which produces an affine algebraic variety that typically includes an open dense subset of the corresponding projective variety. That is what we did in the proof of Proposition 2.36, for example.

Conversely, given an affine algebraic hypersurface $V \subseteq \mathbb{C}^n$ determined by a holomorphic polynomial p, we can complete it to a projective variety in the following way. Suppose $p\colon \mathbb{C}^n \to \mathbb{C}$ is a holomorphic polynomial of degree m. By separating the monomial terms by degree, we can write $p = \sum_{d=0}^m p_{(d)}$, where $p_{(d)}$ is a homogenenous polynomial of degree d. Then define a homogeneous polynomial

$\tilde{p}\colon \mathbb{C}^{n+1} \to \mathbb{C}$, called the ***homogenization of p***, by

$$\tilde{p}(w^0, \dots, w^n) = \sum_{d=0}^{m} (w^0)^{m-d} p_{(d)}(w^1, \dots, w^n).$$

This homogeneous polynomial defines a projective hypersurface $\overline{V} \subseteq \mathbb{CP}^n$, called the ***projective completion of V***. The intersection of $\overline{V}$ with the standard embedded copy of $\mathbb{C}^n$ is the original affine algebraic variety. Even if the original variety V was smooth, though, $\overline{V}$ might have singularities on the hyperplane at infinity, which must be checked separately. (Projective completions can also be defined for affine algebraic varieties of higher codimension, but this requires more delicate algebraic considerations.)

Example 2.38. Consider the affine cubic $V = \{(z, w) : w = z^3\} \subseteq \mathbb{C}^2$. It is the graph of a holomorphic function and thus a nonsingular affine algebraic variety. Its projective completion is the variety $\overline{V} = \{[\zeta, z, w] \in \mathbb{CP}^2 : \zeta^2 w - z^3 = 0\}$. The projective hyperplane at infinity is the set of points of the form $[0, z, w]$, and $\overline{V}$ contains one point on that hyperplane, namely $[0, 0, 1]$. To analyze the structure of the variety near that point, we can switch to affine coordinates $(\alpha, \beta) \leftrightarrow [\alpha, \beta, 1]$. In these coordinates $\overline{V}$ has the equation $\alpha^2 - \beta^3 = 0$. This polynomial has a singular point at the origin, which can be verified as follows: suppose $\gamma(t) = (f(t), g(t))$ is a smooth curve lying in $\overline{V}$ with $\gamma(0) = 0$. Then $f(t)^2 \equiv g(t)^3$. Taking two derivatives and setting $t = 0$ shows that $f'(0) = 0$, and then taking another derivative shows that $g'(0) = 0$. Thus every tangent vector to $\overline{V}$ at the origin is zero. If $\overline{V}$ were a nonsingular variety, its tangent space at the origin would have to have real dimension 2. //

Example 2.39 (Fermat Hypersurfaces). The algebraic hypersurface in $\mathbb{CP}^n$ defined in homogeneous coordinates by the equation

$$(w^0)^d + \cdots + (w^n)^d = 0 \tag{2.2}$$

for an integer $d > 0$ is called a ***Fermat hypersurface*** of degree d. Problem 2-8 shows that each such hypersurface is nonsingular. In the special case $n = 2$, it is called a ***Fermat curve***. (The name reflects the fact that the equation $(w^0)^d + (w^1)^d + (w^2)^d = 0$ is projectively equivalent under the change of variables $w_0 \mapsto i w_0$ to the equation $(w^1)^d + (w^2)^d = (w^0)^d$, which is the basis of Fermat's last theorem.) //

For future use, we record the following important property of algebraic hypersurfaces in $\mathbb{CP}^n$. When $n \geq 2$, it is easy to see that two projective hyperplanes in $\mathbb{CP}^n$ must intersect, because two linear codimension-1 subspaces in $\mathbb{C}^{n+1}$ must have a nontrivial intersection by linear algebra. The next lemma generalizes this to (possibly singular) algebraic hypersurfaces of any degree.

Lemma 2.40. *Let $n \geq 2$. If $V, W \subseteq \mathbb{CP}^n$ are algebraic varieties in $\mathbb{CP}^n$, each defined by a single homogeneous polynomial, then $V \cap W \neq \varnothing$.*

Proof. We will prove this by induction on n. We begin with the case $n = 2$. Let $[w^0, w^1, w^2]$ denote the homogeneous coordinates on $\mathbb{CP}^2$. Suppose $V, W \subseteq \mathbb{CP}^2$ are the varieties determined by homogeneous polynomials $P, Q\colon \mathbb{C}^3 \to \mathbb{C}$ of degrees d and e, respectively. After a change of basis if necessary, we may assume that $P(0,0,1)$ and $Q(0,0,1)$ are both nonzero; this means that the coefficient of $(w^2)^d$ in P and that of $(w^2)^e$ in Q are nonzero. After dividing by suitable constants, we may assume that both these coefficients are equal to 1.

Define polynomials $p, q\colon \mathbb{C}^2 \to \mathbb{C}$ by

$$p(x,y) = P(1,x,y), \qquad q(x,y) = Q(1,x,y).$$

By collecting terms in y, we can write

$$p(x,y) = y^d + a_1(x)y^{d-1} + \cdots + a_{d-1}(x)y + a_d(x),$$

for some one-variable polynomials $a_1, \dots, a_d$. By the fundamental theorem of algebra, for each $x \in \mathbb{C}$ the polynomial $p(x,\cdot)$ has d roots $\lambda_1(x), \dots, \lambda_d(x)$ (listed in some order, possibly with some roots repeated), and we can write

$$p(x,y) = (y - \lambda_1(x)) \cdot \cdots \cdot (y - \lambda_d(x)).$$

Let $r\colon \mathbb{C} \to \mathbb{C}$ be the function

$$r(x) = q(x, \lambda_1(x)) \cdot \cdots \cdot q(x, \lambda_d(x)),$$

called the ***resultant of p and q***. Although the λ_j's are not polynomial functions of x, it turns out that r is a polynomial in x. To see why, note that the function $R\colon \mathbb{C}^{d+1} \to \mathbb{C}$ given by

$$R(x, k_1, \dots, k_d) = q(x, k_1) \cdot \cdots \cdot q(x, k_d)$$

is a ***symmetric polynomial*** in $k = (k_1, \dots, k_d)$ with coefficients given by polynomials in x, meaning it is unchanged by applying any permutation to the coordinates of k. A basic theorem in algebra called the ***fundamental theorem on symmetric polynomials*** says that any such polynomial can be expressed as a polynomial in the ***elementary symmetric polynomials*** $\sigma_1(k), \dots \sigma_d(k)$, where $\sigma_j(k)$ is defined as the coefficient of y^{d-j} in the expansion

$$(y - k_1) \cdot \cdots \cdot (y - k_d) = y^d + \sigma_1(k)y^{d-1} + \cdots + \sigma_{d-1}(k)y + \sigma_d(k),$$

so that $\sigma_j(k)$ is the sum of all possible j-fold products of distinct components of k. (See [**Lan02**, Thm. 6.1 on p. 191] for a proof of this theorem.) Since the coefficients $a_j(x)$ of p are exactly the elementary symmetric polynomials in the roots $\lambda_1(x), \dots, \lambda_d(x)$, it follows that $r(x) = R(x, \lambda_1(x), \dots, \lambda_d(x))$ is a polynomial function of $\big(x, a_1(x), \dots, a_d(x)\big)$, and therefore a polynomial function of x. By the fundamental theorem of algebra, it has a root $x_0 \in \mathbb{C}$. This implies that one of the factors $q(x_0, \lambda_j(x_0))$ in the definition of r must vanish. Since $p(x_0, \lambda_j(x_0)) = 0$ by definition, it follows that the point $[x_0, \lambda_j(x_0), 1]$ lies on both V and W. This completes the $n = 2$ case.

Now let $n \geq 2$ and assume the result is true for $\mathbb{CP}^n$, and let V, W be algebraic varieties in $\mathbb{CP}^{n+1}$ defined by homogeneous polynomials $\widetilde{P}, \widetilde{Q}$. Let $\Pi \subseteq \mathbb{CP}^{n+1}$ be the projective hyperplane determined by $w^{n+1} = 0$; it is biholomorphic to $\mathbb{CP}^n$. After a projective transformation, we may assume that Π is not contained in either V or W. Define $P, Q : \mathbb{C}^{n+1} \to \mathbb{C}$ by

$$P(w^0, \dots, w^n) = \widetilde{P}(w^0, \dots, w^n, 0), \quad Q(w^0, \dots, w^n) = \widetilde{Q}(w^0, \dots, w^n, 0).$$

Then P and Q are nontrivial homogeneous polynomials, and the varieties in $\Pi \approx \mathbb{CP}^n$ they determine are $V' = V \cap \Pi$ and $W' = W \cap \Pi$. By the inductive hypothesis, $V' \cap W'$ is nonempty, which shows that V and W intersect. □

With more work, the argument above in the case of $\mathbb{CP}^2$ can be adapted to prove the stronger result known as ***Bézout's theorem***: if $V, W \subseteq \mathbb{CP}^2$ are distinct projective algebraic curves defined by irreducible polynomials of degrees d and e, respectively, then $V \cap W$ contains exactly de points counted with multiplicity. See [**Har77**, Cor. 7.8] for a proof.

Although nonsingular projective quadrics in $\mathbb{CP}^n$ are all projectively equivalent, the analogous statement is not true for projective hypersurfaces of higher degree: as long as $n \geq 1$ and $d \geq 3$, there are nonsingular projective hypersurfaces of degree d in $\mathbb{CP}^n$ that are not biholomorphic to each other. However, it is a remarkable fact that as smooth manifolds, they are all diffeomorphic, as Theorem 2.43 will show below. We will not use this result anywhere in the book; but it provides an interesting insight into the nature of projective hypersurfaces. The proof will depend on two preliminary results: one from differential geometry that we will prove, and one from commutative algebra that we will accept without proof.

First, the result from differential geometry.

Lemma 2.41. *Suppose M and N are smooth manifolds and $\pi : M \to N$ is a proper smooth submersion. If N is connected, then all fibers of π are diffeomorphic to each other.*

Proof. For each smooth vector field Y on N, we will show that there exists a ***lift of Y***, that is, a smooth vector field X on M that is π-related to Y, meaning that $D\pi(q)\big(X_q\big) = Y_{\pi(q)}$ for each $q \in M$. To see this, note first that the rank theorem shows that for each $q \in M$ there are smooth coordinate charts $\big(U, (u^1, \dots, u^m)\big)$ for M containing q and $\big(V, (v^1, \dots, v^n)\big)$ for N containing $\pi(q)$ in which π has the local expression $\pi(u^1, \dots, u^n, u^{n+1}, \dots, u^m) = (u^1, \dots, u^n)$. If we write the restriction of Y to V as $Y = \sum_{j=1}^n Y^j(v) \partial/\partial v^j$, an easy computation shows that the vector field $\sum_{j=1}^n Y^j(u^1, \dots, u^n) \partial/\partial u^j$ on U is π-related to Y. Blending together all of these vector fields on M with a partition of unity yields a global vector field that is π-related to Y.

Now given $p \in N$, let $(x^1, \dots, x^n)$ be smooth coordinates centered at p on an open set $U \subseteq N$ whose image is the unit ball $B_1(0) \subseteq \mathbb{R}^n$. Given $q \in U$ with coordinate representation $(q^1, \dots, q^n)$, let W be the constant-coefficient vector field $q^j \partial/\partial x^j$ on U, and let Y be the vector field on N that is zero outside U and equal to ψW on U, where $\psi : N \to [0,1]$ is a smooth bump function that is compactly supported in U and equal to 1 on the line segment from p to q. Because Y is compactly supported, it is complete. Let $\theta : \mathbb{R} \times N \to N$ be its flow. The curve $\gamma(t) = (tq^1, \dots, tq^n)$ satisfies $\gamma'(t) = Y_{\gamma(t)}$ for $t \in [0,1]$, so it is a portion of the integral curve of Y starting at p. Thus the time-1 flow $\theta_1 : N \to N$ takes p to q.

Let X be a lift of Y to M. Because π is proper, X is supported in the compact set $\pi^{-1}(\operatorname{supp} Y)$, so it is also complete. Let $\Theta : \mathbb{R} \times M \to M$ be its flow. The fact that Y and X are π-related implies $\theta_1 \circ \pi = \pi \circ \Theta_1$ by [**LeeSM**, Prop. 9.13]. In particular, this means that for all $\tilde{p} \in \pi^{-1}(p)$, we have $\pi(\Theta_1(\tilde{p})) = \theta_1(\pi(\tilde{p})) = \theta_1(p) = q$, so Θ_1 maps $\pi^{-1}(p)$ to $\pi^{-1}(q)$. Its inverse Θ_{-1} maps $\pi^{-1}(q)$ to $\pi^{-1}(p)$, showing that the fibers $\pi^{-1}(p)$ and $\pi^{-1}(q)$ are diffeomorphic.

Now define an equivalence relation on N by $p_1 \sim p_2$ if and only if $\pi^{-1}(p_1)$ is diffeomorphic to $\pi^{-1}(p_2)$. The argument above shows that each equivalence class is open in N, and since N is connected, there is only one equivalence class. □

And here is the commutative algebra result. Proofs can be found in [**vdW50**, Section 82] or [**GKZ08**, Chapter 13].

Lemma 2.42. *Suppose $f_0, \dots, f_n$ are homogeneous polynomials in $n+1$ complex variables. There is a quantity $R(f_0, \dots, f_n) \in \mathbb{C}$, called the* ***resultant of $f_0, \dots, f_n$****, which is given by a homogeneous polynomial in the coefficients of $f_0, \dots, f_n$, and which is zero if and only if $f_0, \dots, f_n$ have a common zero other than the origin.*

Theorem 2.43. *Let n and m be positive integers, with $n \geq 2$. All nonsingular projective hypersurfaces of degree m in $\mathbb{CP}^n$ are diffeomorphic to each other.*

Proof. Let n and m be fixed. By Proposition 2.31, each nonsingular projective hypersurface $S \subseteq \mathbb{CP}^n$ of degree m is the variety determined by a homogeneous polynomial p of degree m, and nonsingularity means that dp_w does not vanish at any nonzero point such that $p(w) = 0$. Any two such polynomials that are nonzero constant multiples of each other determine the same hypersurface. Conversely, if p_1 and p_2 determine the same nonsingular hypersurface, then the ratio p_1/p_2 extends holomorphically across the zero set $p_2^{-1}(0)$ to define a global holomorphic function on $\mathbb{C}^{n+1} \smallsetminus \{0\}$, and thus to all of $\mathbb{C}^{n+1}$ by Hartogs's extension theorem. Because it is homogeneous of degree zero, it is constant.

Therefore we can parametrize the set of nonsingular projective hypersurfaces of degree m by a certain subset of the projective space $\mathbb{P}(H)$, where H is the complex vector space of homogeneous polynomials of degree m in $n+1$ complex variables. The variety determined by a nonzero element $p \in H$ is nonsingular if and only if

$p, \partial_0 p, \dots, \partial_n p$ have no common zeros other than the origin (where $\partial_j p$ is shorthand for $\partial p/\partial w^j$). Because p is homogeneous, it satisfies ***Euler's identity***:

$$w^j \frac{\partial p}{\partial w^j}(w) = mp(w),$$

which is proved by differentiating the equation $p(\lambda w) = \lambda^m p(w)$ with respect to λ and setting $\lambda = 1$. Thus if $a \in \mathbb{C}^{n+1} \smallsetminus \{0\}$ is a common zero of $\partial_0 p, \dots, \partial_n p$, it is also a zero of p.

This shows that the nonsingular hypersurfaces of degree m are exactly the varieties V_p for which the homogeneous polynomials $\partial_0 p, \dots, \partial_n p$ have no common zeros other than the origin. Let $W \subseteq \mathbb{P}(H)$ be the corresponding set. It follows from Lemma 2.42 that W is the complement of the algebraic variety V_R defined by a homogeneous polynomial R. If V_R were nonsingular, it would follow from Proposition 2.17 that W is path-connected; we can show that the same result holds even if V_R has singularities, as follows. Given distinct points $[p], [q] \in W$, let $F\colon \mathbb{CP}^1 \to \mathbb{P}(H)$ be the holomorphic embedding $F([w^0, w^1]) = [w^0 p + w^1 q]$, and let L denote the projective line $F(\mathbb{CP}^1) \subseteq \mathbb{P}(H)$. Then

$$F^{-1}(L \cap V_R) = \big\{[w^0, w^1] : R(w^0 p + w^1 q) = 0\big\},$$

which is an algebraic subvariety of $\mathbb{CP}^1$. It is not all of $\mathbb{CP}^1$ because it does not contain $[0,1]$ or $[1,0]$, so by Proposition 2.30 it is finite. Since $\mathbb{CP}^1$ is homeomorphic to $\mathbb{S}^2$, the complement of a finite set in $\mathbb{CP}^1$ is path-connected, so there is a path in $L \cap W$ from p to q.

Now define a subset $X \subseteq \mathbb{P}(H) \times \mathbb{CP}^n$ by

$$X = \big\{([p],[w]) : [p] \in \mathbb{P}(H) \text{ and } [w] \in V_p\big\},$$

and let $X_0 = X \cap \pi_1^{-1}(W)$, where $\pi_1\colon \mathbb{P}(H) \times \mathbb{CP}^n \to \mathbb{P}(H)$ is the projection. Let $\Pi\colon X_0 \to W$ be the restriction of π_1. For each $[p] \in W$, the preimage $\Pi^{-1}\{[p]\}$ is the set $\{[p]\} \times V_p$, which is biholomorphic to V_p; thus the fibers of Π are exactly the nonsingular degree-m hypersurfaces in $\mathbb{CP}^n$.

Note that X is the image of the set

$$\widetilde{X} = \big\{(p,w) : p(w) = 0\big\} \subseteq (H \smallsetminus \{0\}) \times (\mathbb{C}^{n+1} \smallsetminus \{0\})$$

under the projection $q\colon (H \smallsetminus \{0\}) \times (\mathbb{C}^{n+1} \smallsetminus \{0\}) \to \mathbb{P}(H) \times \mathbb{CP}^n$ given by $q(p,w) = ([p],[w])$. Since $\widetilde{X}$ is the zero set of the continuous function $f(p,w) = p(w)$, it is closed. Moreover, every point $([p],[w]) \in X$ has a representative $(p', w') = (p/|p|, w/|w|)$ in the compact set $K \times \mathbb{S}^{2n+1} \subseteq H \times \mathbb{C}^{n+1}$, where K is the unit sphere in H with respect to any choice of norm. Thus X is the image of the compact set $\widetilde{X} \cap (K \times \mathbb{S}^{2n+1})$, so it is compact, and therefore the restriction of π_1 to X is proper. It follows that $\Pi\colon X_0 \to W$ is also proper because it is the restriction of $\pi_1|_X$ to an open subset $X_0 \subseteq X$ that is ***saturated***, meaning it is the full preimage of a subset of $\mathbb{P}(H)$ (see [**LeeTM**, Prop. 4.93(f)]).

Now we show that X_0 is a codimension-1 complex submanifold of $\mathbb{P}(H)\times\mathbb{CP}^n$. Given any point $x_0 = ([p_0],[w_0]) \in X_0$, we can choose a neighborhood U on which we have affine coordinates $(q^1,\dots,q^N)$ for $\mathbb{P}(H)$ (where $N = \dim\mathbb{P}(H)$) and affine coordinates $(z^1,\dots,z^n)$ for $\mathbb{CP}^n$, and then the set $X_0\cap U$ is characterized by a polynomial equation $f(q,z) = 0$ which is the coordinate representation of $p(w) = 0$. The fact that $[p_0] \in W$ guarantees that some z^j-derivative of f does not vanish at x_0, so $X_0 \cap U$ is a regular level set of a single holomorphic function, showing that X_0 is a complex submanifold of dimension $N + n - 1$. Moreover, the fact that some z^j-derivative of f is nonzero means that the tangent space $T_{x_0}X_0$ does not contain all of the direct summand $T_{[z_0]}\mathbb{CP}^n \subseteq T_{x_0}(\mathbb{P}(H) \times \mathbb{CP}^n)$; thus the dimension of $\operatorname{Ker} d\Pi_{x_0} = T_{[z_0]}\mathbb{CP}^n \cap T_{x_0}X_0$ is less than n. It follows that the rank of $d\Pi_{x_0}$ is at least $(N + n - 1) - (n - 1) = N$, so Π is a submersion. Because W is connected, it follows from Lemma 2.41 that all of the fibers of Π are diffeomorphic to each other. □

The Holomorphic Embedding Problem

One of the fundamental results in smooth manifold theory is the Whitney embedding theorem, which says that every smooth manifold can be properly embedded in some Euclidean space; see [**LeeSM**, Thm. 6.15] for a proof. (Properness of the embedding is equivalent to the image being a closed subset; see [**LeeSM**, Prop. 5.5].)

In complex manifold theory, things are very different. In the first place, we cannot hope to find holomorphic embeddings of compact complex manifolds of positive dimension into $\mathbb{C}^n$ (see Example 2.25). It turns out that the appropriate place to look for embeddings of compact complex manifolds is in $\mathbb{CP}^n$, but not every such manifold admits such an embedding even in that case. The question of characterizing which compact manifolds admit embeddings into $\mathbb{CP}^n$ will occupy much of the rest of this book.

What about noncompact complex manifolds? Here again, the situation is not straightforward. There are nontrivial necessary conditions for a noncompact complex manifold M to be embeddable in $\mathbb{C}^n$. For example, the preceding observation implies that if M contains a positive-dimensional compact complex submanifold, then M cannot be embedded in $\mathbb{C}^n$. It turns out that the class of complex manifolds that can be properly embedded in $\mathbb{C}^n$ for some n has a nice intrinsic characterization. First, a few definitions.

Let M be a complex manifold and $\mathcal{O}(M)$ its ring of global holomorphic functions. We say $\mathcal{O}(M)$ ***separates points*** if for every pair of distinct points $p, q \in M$, there exists $f \in \mathcal{O}(M)$ that satisfies $f(p) = 0$ and $f(q) \neq 0$. We say $\mathcal{O}(M)$ ***separates directions*** if for every $p \in M$ and every nonzero $v \in T'_pM$, there exists $f \in \mathcal{O}(M)$ such that $vf \neq 0$. The significance of the latter condition is explained by the following lemma.

Lemma 2.44. *For a complex n-manifold M, $\mathcal{O}(M)$ separates directions if and only if for each $p \in M$ there exist global holomorphic functions $z^1, \dots, z^n \in \mathcal{O}(M)$ that restrict to local holomorphic coordinates in a neighborhood of p.*

Proof. On the one hand, if there exist such functions $z^1, \dots, z^n$, then for any nonzero vector $v = v^j \partial/\partial z^j|_p \in T'_p M$, one of the components v^j must be nonzero, and $v(z^j) = v^j \neq 0$.

Conversely, suppose $\mathcal{O}(M)$ separates directions, and let $p \in M$ be arbitrary. We will show by induction on k that for each $k = 1, \dots, n$, there exist $z^1, \dots, z^k \in \mathcal{O}(M)$ such that $dz^1|_p, \dots, dz^k|_p$ are linearly independent. If this is true for $k = n$, the holomorphic inverse function theorem shows that $(z^1, \dots, z^n)$ restrict to holomorphic coordinates in a neighborhood of p.

For $k = 1$, just choose any $z^1 \in \mathcal{O}(M)$ such that $v(z^1) \neq 0$ for some nonzero $v \in T'_p M$, which implies $dz^1|_p \neq 0$. Now suppose the claim is true for some $k < n$. The complex-linear map from $T'_p M$ to $\mathbb{C}^k$ given by $v \mapsto (dz^1|_p(v), \dots, dz^k|_p(v))$ has rank k, so it has a kernel of dimension $n - k$. Choose $v \neq 0$ in that kernel, and let $z^{k+1} \in \mathcal{O}(M)$ be a function such that $v(z^{k+1}) \neq 0$. It follows that $dz^{k+1}|_p$ is linearly independent of $dz^1|_p, \dots, dz^k|_p$, thus completing the induction. □

One last definition: for any subset $K \subseteq M$, define the ***holomorphic hull of K*** as the set

$$\widehat{K} = \left\{ z \in M \,:\, |f(z)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(M) \right\}.$$

(The terminology is motivated by analogy with the ***convex hull*** of a subset $K \subseteq \mathbb{R}^n$, which is the intersection of all convex subsets of $\mathbb{R}^n$ containing K. Problem 2-12 shows that the convex hull can be characterized in an analogous way using linear functions instead of holomorphic ones.) We say M is ***holomorphically convex*** if whenever K is a compact subset of M, $\widehat{K}$ is also compact.

A complex manifold M is called a ***Stein manifold*** if it satisfies the following three conditions:

(i) $\mathcal{O}(M)$ separates points.

(ii) $\mathcal{O}(M)$ separates directions.

(iii) M is holomorphically convex.

Proposition 2.45. *Suppose $M \subseteq \mathbb{C}^n$ is a properly embedded complex submanifold. Then M is Stein.*

Proof. Clearly $\mathcal{O}(M)$ separates points: given distinct points $p, q \in M$, some holomorphic coordinate function z^j takes on different values at those two points, and then the restriction to M of $f(z) = z^j - p^j$ is zero at p and nonzero at q.

Similarly, $\mathcal{O}(M)$ separates directions: given $p \in M$ and $v \neq 0 \in T'_pM$, we can consider v as an element of $T'_p\mathbb{C}^n$ and write $v = v^j \partial/\partial z^j|_p$. Some component v^j is nonzero, from which it follows that $v(z^j) = v^j \neq 0$.

To show that M is holomorphically convex, let $K \subseteq M$ be a compact subset and let $\widehat{K} \subseteq M$ be its holomorphic hull. First note that $\widehat{K}$ is closed in M: if $p \in M$ is a boundary point of $\widehat{K}$, there is a sequence of points p_j in K such that $p_j \to p$. Then for any $f \in \mathcal{O}(M)$, since $|f(p_j)| \leq \sup_K |f|$ for each j, it follows by continuity that $|f(p)| \leq \sup_K |f|$, so $p \in \widehat{K}$. Since M is closed in $\mathbb{C}^n$, it follows that $\widehat{K}$ is also closed in $\mathbb{C}^n$.

Next we show $\widehat{K}$ is bounded. Because K is compact, there is some $r > 0$ such that K is contained in the closed ball $\overline{B}_r(0) \subseteq \mathbb{C}^n$. Suppose $w \in M \smallsetminus \overline{B}_r(0)$. Set $R = |w| > r$, and let $f : M \to \mathbb{C}$ be the restriction of the holomorphic linear function $f(z) = z \cdot \overline{w} = z^1\overline{w}^1 + \cdots + z^n\overline{w}^n$. Now $f(w) = |w|^2 = R^2$; but if $z \in K$, then the Cauchy–Schwartz inequality gives

$$|f(z)| = |z \cdot \overline{w}| \leq |z|\,|w| \leq rR < R^2 = |f(w)|.$$

This shows $|f(w)| > \sup_K |f|$, so $w \notin \widehat{K}$. The contrapositive of this statement is $\widehat{K} \subseteq \overline{B}_r(0)$. Since $\widehat{K}$ is closed and bounded in $\mathbb{C}^n$, it is compact. □

The most important fact about Stein manifolds is the following converse to the preceding proposition, proved in 1961 by Errett Bishop and Raghavan Narasimhan. We will neither prove nor use it, but you can find a proof in [**Hör90**] or [**GR09**].

Theorem 2.46 (Stein Embedding Theorem). *Every Stein manifold admits a proper holomorphic embedding into $\mathbb{C}^n$ for some n.*

It is easy to come up with examples of noncompact complex manifolds that are not Stein.

Example 2.47 (Non-Stein Manifolds).

(a) Suppose M is a complex manifold that contains a compact complex submanifold $S \subseteq M$ of positive dimension. Then every $f \in \mathcal{O}(M)$ restricts to a constant function on each connected component of S, so $\mathcal{O}(M)$ does not separate points. Thus, for example, no compact complex manifold of positive dimension is Stein, and no product of such a manifold with any other complex manifold is Stein.

(b) Let U be an arbitrary open subset of $\mathbb{C}^n$ with $n \geq 2$, $p \in U$, and $M = U \smallsetminus \{p\}$. We will show that M is not holomorphically convex, and therefore not Stein. Choose $\varepsilon > 0$ such that $\overline{B}_\varepsilon(p) \subseteq U$, and let $K = \partial B_\varepsilon(p)$. Suppose $f \in \mathcal{O}(M)$ is arbitrary. Then the Hartogs extension theorem shows that f has a holomorphic extension (still denoted by f) to all of U. The restriction of $|f|$ to the compact set $\overline{B}_\varepsilon(p)$ achieves a maximum, and the maximum principle guarantees that this maximum occurs on K.

It follows that for every $z \in \overline{B}_\varepsilon(p) \cap M$, we have $|f(z)| \le \sup_K |f|$, so $\overline{B}_\varepsilon(p) \cap M \subseteq \widehat{K}$. But any sequence in $\overline{B}_\varepsilon(p) \cap M$ that converges to p has no convergent subsequence in $\widehat{K}$, so $\widehat{K}$ is not compact. //

On the other hand, some open submanifolds of $\mathbb{C}^n$ are Stein (in addition to $\mathbb{C}^n$ itself). If $U \subseteq \mathbb{C}^n$ is open, using the restrictions of the coordinate functions $z^1, \dots, z^n$ to U, we can show that $\mathcal{O}(U)$ separates points and directions as in the proof of Proposition 2.45, so an open submanifold of $\mathbb{C}^n$ is Stein if and only if it is holomorphically convex. Here is a class of examples for which we can show that is the case.

Example 2.48 (Convex Open Submanifolds Are Stein). Suppose $U \subseteq \mathbb{C}^n$ is a convex open subset. To see that U is holomorphically convex, let $K \subseteq U$ be an arbitrary compact subset. We begin by showing that the holomorphic hull $\widehat{K}$ is contained in the convex hull of K (considered as a subset of $\mathbb{R}^{2n}$), which we denote by $\operatorname{ch}(K)$. Suppose $p \in U \smallsetminus \operatorname{ch}(K)$. It follows from Problem 2-12 that there exists a real-linear function $f : \mathbb{C}^n \to \mathbb{R}$ such that $f(p) > \sup_K f$. Define $\alpha : \mathbb{C}^n \to \mathbb{C}$ by $\alpha(z) = f(z) - i f(iz)$; a simple computation shows that $\alpha(iz) = i\alpha(z)$, so α is a complex-linear functional whose real part is f. Then $z \mapsto e^{\alpha(z)}$ restricts to a holomorphic function on U, which satisfies

$$\left|e^{\alpha(p)}\right| = e^{f(p)} > \sup_K e^f = \sup_K |e^\alpha|,$$

so $p \notin \widehat{K}$.

Since K is compact, it is easy to verify that $\operatorname{ch}(K)$ is also compact, and since U is convex, $\operatorname{ch}(K)$ is contained in U. Now $\widehat{K}$ is closed in U by continuity, so it is a closed subset of the compact set $\operatorname{ch}(K)$ and thus compact. //

Convexity is an easy way to recognize some Stein manifolds among the open submanifolds of $\mathbb{C}^n$. But convexity is not a necessary condition, for the simple reason that it is not biholomorphically invariant, while the condition of being a Stein manifold is. For example, consider $\mathbb{C}^2$ with holomorphic coordinates (z, w). The open subset $U = \{(z, w) : \operatorname{Re} w > 0\}$ is convex and thus is a Stein manifold. But now consider the holomorphic map $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\varphi(z, w) = (z, w + z^2)$. It is a biholomorphism with inverse $\varphi^{-1}(z, w) = (z, w - z^2)$. Thus $\varphi(U)$ is also a Stein manifold. But $\varphi(U)$ is the subset consisting of all (z, w) such that $\operatorname{Re} w > (\operatorname{Re} z)^2 - (\operatorname{Im} z)^2$, which is not convex.

There is a biholomorphically invariant property called ***pseudoconvexity*** (which is implied by convexity but more general), and it can be shown that a connected open subset of $\mathbb{C}^n$ is a Stein manifold if and only if it is pseudoconvex. Since the deep properties of Stein manifolds and pseudoconvex domains belong more properly to the theory of complex analysis in several variables, we will have no more to say about them in this book; but you can find a discussion of these matters in [**Kra01**], starting in Chapter 3.

Problems

2-1. For each $n \geq 1$, let $\mathrm{U}(n) \subseteq \mathrm{GL}(n,\mathbb{C})$ be the n-dimensional ***unitary group***, that is, the subgroup of matrices $A \in \mathrm{GL}(n,\mathbb{C})$ that satisfy $A^*A = \mathrm{Id}$. (Here A^* denotes the ***Hermitian adjoint of A***, that is, the transposed conjugate of A.) Show that $\mathrm{U}(n)$ is not a complex submanifold of $\mathrm{GL}(n,\mathbb{C})$.

2-2. Prove Lemma 2.27 (a subset of $\mathbb{CP}^n$ is a complex submanifold if and only its preimage in $\mathbb{C}^{n+1} \smallsetminus \{0\}$ is a complex submanifold.)

2-3. For positive integers m, n, let $S\colon \mathbb{CP}^m \times \mathbb{CP}^n \to \mathbb{CP}^{mn+m+n}$ be the map

$$\begin{aligned} S([w^0,\dots,w^m],[z^0,\dots,z^n]) \\ = [w^0z^0, w^0z^1,\dots,w^0z^n, w^1z^0,\dots,w^1z^n,\dots,w^mz^0,\dots,w^mz^n], \end{aligned}$$

where the products on the right-hand side are listed in lexicographic order. Show that S is a holomorphic embedding, called the ***Segre embedding***. Use this to conclude that every product of projective complex manifolds is projective.

2-4. Prove that the following formulas determine a well-defined diffeomorphism $F\colon \mathbb{S}^2 \to \mathbb{CP}^1$:

$$F(x,y,z) = \begin{cases} [x+iy,\ 1-z], & (x,y,z) \neq (0,0,1), \\ [1+z,\ x-iy], & (x,y,z) \neq (0,0,-1). \end{cases}$$

2-5. Show that every nonsingular quadric in $\mathbb{CP}^2$ is biholomorphic to $\mathbb{CP}^1$. [Hint: Start by thinking about how to parametrize the projective completion of the affine curve $w = z^2$.]

2-6. Show that every nonsingular quadric in $\mathbb{CP}^3$ is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. [Hint: Use Problem 2-3.]

2-7. For $n \geq 3$, let $\mathrm{G}_2^+(\mathbb{R}^n)$ be the ***Grassmannian of oriented 2-planes in $\mathbb{R}^n$***, that is, the set of 2-dimensional linear subspaces of $\mathbb{R}^n$ together with a choice of orientation for each. The natural action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ induces a transitive action on $\mathrm{G}_2^+(\mathbb{R}^n)$ with a closed isotropy group, giving it a smooth manifold structure as a homogeneous space by [**LeeSM**, Thm. 21.20]. (You do not need to prove this.) Prove that every nonsingular quadric in $\mathbb{CP}^{n-1}$ is diffeomorphic to $\mathrm{G}_2^+(\mathbb{R}^n)$. [Hint: Consider the quadric Q defined by $(w^1)^2 + \cdots + (w^n)^2 = 0$, and show that the map $Q \to \mathrm{G}_2^+(\mathbb{R}^n)$ by $[w] \mapsto \operatorname{span}(\operatorname{Re} w, \operatorname{Im} w)$ is an $\mathrm{SO}(n)$-equivariant diffeomorphism.]

2-8. Prove that every Fermat hypersurface (Example 2.39) is nonsingular.

2-9. (a) Show that every automorphism of $\mathbb{CP}^1$ is a projective transformation. [Hint: First show that it suffices to assume $F([1,0]) = [1,0]$ and $F([0,1] = [0,1]$.]

(b) Show that every automorphism of $\mathbb{C}$ is an affine function of the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$. [Hint: Begin by extending f to a bijection $F : \mathbb{CP}^1 \to \mathbb{CP}^1$, and show that F has a removable singularity at infinity.]

2-10. Show that every holomorphic map from $\mathbb{CP}^1$ to itself can be written in the form $F([z,w]) = [p(z,w), q(z,w)]$, where p and q are homogeneous polynomials of the same degree with no common zeros except the origin.

2-11. Suppose M is a disconnected compact complex manifold. Show that M is projective if and only if each of its connected components is projective.

2-12. Suppose K is an arbitrary subset of $\mathbb{R}^n$. The ***convex hull of K*** is the intersection of all convex subsets of $\mathbb{R}^n$ containing K. Show that the convex hull of K is equal to the set

$$\left\{ x \in \mathbb{R}^n : f(x) \leq \sup_K f \text{ for all linear functions } f : \mathbb{R}^n \to \mathbb{R} \right\}.$$

2-13. Let V be a finite-dimensional complex vector space. Show that each Grassmannian $\mathrm{G}_k(V)$ is projective. [Hint: Let $\Lambda^k V$ be the set of alternating contravariant k-tensors on V, and define a map from $\mathrm{G}_k(V)$ to $\mathbb{P}(\Lambda^k V)$ by sending the span of $(v_1, \ldots, v_k)$ to $[v_1 \wedge \cdots \wedge v_k]$. Show that this is a holomorphic embedding, called a ***Plücker embedding***.]

2-14. Let $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ be the upper half-plane. Define an action of $\mathbb{Z}^2$ on $\mathbb{C} \times \mathbb{H}$ by $(m,n) \cdot (z,w) = (z + m + nw, w)$.

(a) Show that this action is free, proper, and holomorphic, so $M = (\mathbb{C} \times \mathbb{H})/\mathbb{Z}^2$ is a complex 2-manifold.

(b) Show that the projection $\pi_2 : \mathbb{C} \times \mathbb{H} \to \mathbb{H}$ descends to a surjective holomorphic submersion $\Pi : M \to \mathbb{H}$.

(c) Show that every 1-dimensional complex torus is biholomorphic to at least one of the fibers of Π. [Hint: Use Problem 1-4.]

10.1090/gsm/244/03

Chapter 3

Holomorphic Vector Bundles

Recall that in Chapter 1 we defined a holomorphic vector bundle as a complex vector bundle $\pi : E \to M$ in which E and M are complex manifolds, π is a holomorphic map, and the local trivializations can be chosen to be biholomorphisms. In this chapter we will delve much more deeply into the theory of holomorphic vector bundles.

Holomorphic Bundle Tools

Most of the basic tools for working with smooth vector bundles have direct analogues in the holomorphic category. Here we summarize the main ones.

Lemma 3.1 (Transition Functions of Holomorphic Bundles). *Suppose $\pi : E \to M$ is a holomorphic vector bundle of rank k, and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ and $\Phi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{C}^k$ are two holomorphic local trivializations with $U_\alpha \cap U_\beta \neq \varnothing$. Then for all $(p, v) \in (U_\alpha \cap U_\beta) \times \mathbb{C}^k$, we have*

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v) \tag{3.1}$$

for some holomorphic map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$ called the ***transition function from Φ_β to Φ_α****.*

Proof. Exactly the same as the corresponding proof for smooth bundles, with “holomorphic” substituted for “smooth”; see [**LeeSM**, Lemma 10.5]. □

For a holomorphic vector bundle $E \to M$, a local or global section is called a ***holomorphic section*** if it is holomorphic as a map between complex manifolds. In terms of any holomorphic local frame $(s_1, \dots, s_k)$ over an open set $U \subseteq M$, we can write a rough section σ locally as $\sigma(p) = \sigma^j(p)s_j(p) = \sigma^1(p)s_1(p) + \cdots + \sigma^k(p)s_k(p)$ for some complex-valued functions $\sigma^1, \dots, \sigma^k : U \to \mathbb{C}$, called the ***component functions of*** $\boldsymbol{\sigma}$ with respect to the given frame. We let $\mathcal{O}(U; E)$ denote the complex vector space of local holomorphic sections of E over U, so that $\mathcal{O}(M; E)$ is the space of global holomorphic sections.

Lemma 3.2 (Local Frame Criterion for Holomorphicity). *Let $E \to M$ be a holomorphic vector bundle. Given a rough local section $\sigma : U \to E$ and a holomorphic local frame for E over U, the section σ is holomorphic on U if and only if its component functions are holomorphic.*

Proof. Just the same as the corresponding proof for smooth bundles [**LeeSM**, Prop. 10.22]. □

Lemma 3.3 (Holomorphic Local Frames and Local Trivializations). *For a holomorphic vector bundle $\pi : E \to M$, there is a one-to-one correspondence between holomorphic local frames and holomorphic local trivializations. The local frame $(s_1, \dots, s_k)$ corresponding to a local trivialization (U, Φ) is defined by $s_j(p) = \Phi^{-1}(p, e_j)$, where e_j is the jth standard basis vector for $\mathbb{C}^k$.*

Proof. Again, just like its smooth counterpart [**LeeSM**, Prop. 10.19]. □

Lemma 3.4 (Holomorphic Vector Bundle Chart Lemma). *Suppose M is a complex manifold, and for each $p \in M$ we are given a k-dimensional complex vector space E_p. Let $E = \coprod_{p \in M} E_p$ (the disjoint union of the spaces E_p), and let $\pi : E \to M$ be the obvious projection. Suppose further that we are given*

(i) *an indexed open cover $\{U_\alpha\}_{\alpha \in A}$ of M;*

(ii) *for each $\alpha \in A$, a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ whose restriction to each E_p is a complex-linear isomorphism from E_p to $\{p\} \times \mathbb{C}^k$;*

(iii) *for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, a holomorphic map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$ such that $\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$ for all $(p, v) \in (U_\alpha \cap U_\beta) \times \mathbb{C}^k$.*

Then $\pi : E \to M$ has a unique structure as a holomorphic vector bundle with $\{(U_\alpha, \Phi_\alpha)\}$ as holomorphic local trivializations, and with $\tau_{\alpha\beta}$ as the transition function from Φ_β to Φ_α for each α and β.

Proof. Just like its smooth analogue [**LeeSM**, Lemma 10.6]. □

In the vector bundle chart lemma, we construct a vector bundle from vector spaces E_p that are given in advance. The next important proposition gives a way to construct a holomorphic vector bundle out of thin air, given only its transition functions. To motivate the hypothesis in the following proposition, observe that it follows from equation (3.1) that whenever the domains of three local trivializations $\Phi_\alpha, \Phi_\beta, \Phi_\gamma$ overlap, the transition functions satisfy the equation

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p)$$

for all p in the common domain, where the juxtaposition on the left-hand side represents matrix multiplication. This equation is called the ***cocycle condition***, for reasons that will become clear in Chapter 6 (see Example 6.3).

Proposition 3.5 (Holomorphic Vector Bundle Construction Theorem). *Let M be a complex manifold and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an indexed open cover of M. Suppose for each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$ we are given a holomorphic map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$ such that the following cocycle condition is satisfied for all $\alpha, \beta, \gamma \in A$:*

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \qquad p \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{3.2}$$

Then there is a rank-k holomorphic vector bundle $E \to M$ that admits a holomorphic trivialization over each set U_α, with transition functions given by $\tau_{\alpha\beta}$.

Proof. Let $\widetilde{E}$ be the disjoint union $\coprod_{\alpha\in A}(U_\alpha \times \mathbb{C}^k)$, with a point $(p, v) \in U_\alpha \times \mathbb{C}^k$ denoted in the disjoint union by (p, v, α). Define a relation on $\widetilde{E}$ by $(p, v, \alpha) \sim (p', v', \alpha')$ if $p = p'$ and $v' = \tau_{\alpha\alpha'}(p)v$. By taking two or three of the indices equal to each other, we see that the relation (3.2) implies also that $\tau_{\beta\alpha}(p) = \tau_{\alpha\beta}(p)^{-1}$ and $\tau_{\alpha\alpha}(p) = \mathrm{Id}$, which shows that $\sim$ is an equivalence relation. Let E denote the set of equivalence classes, and define $\pi : E \to M$ by $\pi[(p, v, \alpha)] = p$; the definition of the equivalence relation shows that this is well defined. For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is the set of all equivalence classes of the form $[(p, v, \alpha)]$ for v arbitrary and α such that $p \in U_\alpha$. We can define a complex vector space structure on E_p by choosing a fixed U_α containing p and setting $c_1[(p, v_1, \alpha)] + c_2[(p, v_2, \alpha)] = [(p, c_1v_1 + c_2v_2, \alpha)]$ for $c_1, c_2 \in \mathbb{C}$; the fact that the maps $v \mapsto \tau_{\alpha\beta}(p)v$ are all linear isomorphisms guarantees that this is independent of the choice of α.

Now for each α, define a map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ by $\Phi_\alpha[(p, v, \alpha)] = (p, v)$. It is then straightforward to check that these maps satisfy all of the hypotheses of the chart lemma and thus define a holomorphic vector bundle structure on E. □

► **Exercise 3.6.** Complete the proof of this proposition by verifying that the chart lemma hypotheses are satisfied.

Just as the preceding proposition allows us to construct a bundle given only the transition functions, the next one allows us to detect when two bundles are isomorphic from the same data.

Proposition 3.7 (Holomorphic Vector Bundle Isomorphism Criterion). *Suppose $E \to M$ and $E' \to M$ are holomorphic rank-k vector bundles that both have holomorphic local trivializations $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in A}$ and $\{(U_\alpha, \Phi'_\alpha)\}_{\alpha\in A}$ over the same trivializing cover, with transition functions $\tau_{\alpha\beta}$ and $\tau'_{\alpha\beta}$, respectively. Then E and E' are isomorphic over M if and only if for each $\alpha \in A$ there exists a holomorphic map $\psi_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{C})$ such that for $\alpha, \beta \in A$ we have*

$$\tau_{\alpha\beta}(p) = \psi_\alpha(p)^{-1}\tau'_{\alpha\beta}(p)\psi_\beta(p), \qquad p \in U_\alpha \cap U_\beta. \tag{3.3}$$

Proof. First suppose there exist such maps ψ_α. For each α, define a holomorphic bundle homomorphism $\Psi_\alpha : U_\alpha \times \mathbb{C}^k \to U_\alpha \times \mathbb{C}^k$ by

$$\Psi_\alpha(p, v) = (p, \psi_\alpha(p)v);$$

it has a holomorphic inverse given by replacing $\psi_\alpha(p)$ with $\psi_\alpha(p)^{-1}$. Then let $F_\alpha : E|_{U_\alpha} \to E'|_{U_\alpha}$ be the holomorphic bundle isomorphism $F_\alpha = \Phi'^{-1}_\alpha \circ \Psi_\alpha \circ \Phi_\alpha$.

To see what happens when two of these definitions overlap, recall that the transition functions satisfy $\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$ for $p \in U_\alpha \cap U_\beta$, with a similar formula for $\tau'_{\alpha\beta}$. Thus (3.3) implies the following formula on $(U_\alpha \cap U_\beta) \times \mathbb{C}^k$:

$$\Phi_\alpha \circ \Phi_\beta^{-1} = \Psi_\alpha^{-1} \circ \Phi'_\alpha \circ \Phi'^{-1}_\beta \circ \Psi_\beta.$$

It follows that F_α and F_β agree where both are defined, so they piece together to create a holomorphic bundle homomorphism $F : E \to E'$. It is an isomorphism because its inverse can be constructed in the same way using $\psi_\alpha(p)^{-1}$ in place of $\psi_\alpha(p)$.

Conversely, if $F : E \to E'$ is a bundle isomorphism, for each $\alpha \in A$ the composite map $\Psi_\alpha = \Phi'_\alpha \circ F \circ \Phi_\alpha^{-1}$ is a bundle isomorphism from $U_\alpha \times \mathbb{C}^k$ to itself, so it has the form $\Psi_\alpha(p, v) = (p, \sigma_\alpha(p, v))$ for some holomorphic map $\sigma_\alpha : U_\alpha \times \mathbb{C}^k \to \mathbb{C}^k$. Since $v \mapsto \sigma_\alpha(p, v)$ is a complex-linear isomorphism for each p, there is some function $\psi_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{C})$ such that $\sigma_\alpha(p, v) = \psi_\alpha(p)v$. If we let (e_j) denote the standard basis for $\mathbb{C}^k$ and (e^j) the associated dual basis, the matrix entries of ψ_α satisfy

$$(\psi_\alpha)^l_j(p) = e^l(\sigma_\alpha(p, e_j)),$$

so they are holomorphic by composition. To prove (3.3), we compute

$$\begin{aligned}
(p, \tau_{\alpha\beta}(p)v) &= \Phi_\alpha \circ \Phi_\beta{}^{-1}(p, v) \\
&= \left(\Phi'_\alpha \circ F \circ \Phi_\alpha^{-1}\right)^{-1} \circ \left(\Phi'_\alpha \circ \Phi'^{-1}_\beta\right) \circ \left(\Phi'_\beta \circ F \circ \Phi_\beta^{-1}\right)(p, v) \\
&= \left(p, \psi_\alpha(p)^{-1}\tau'_{\alpha\beta}(p)\psi_\beta(p)v\right).
\end{aligned}$$

□

The next corollary shows that the bundle produced by the vector bundle construction theorem is unique up to isomorphism.

Corollary 3.8. *Suppose $E \to M$ and $E' \to M$ are holomorphic vector bundles of rank k over the same base, and both admit local trivializations over the same open cover with the same transition functions. Then $E \cong E'$.*

Proof. Just apply Proposition 3.7 with $\psi_\alpha \equiv \mathrm{Id}$ for every α. □

Corollary 3.9. *Suppose $E \to M$ is a holomorphic rank-k vector bundle, and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ is a trivializing cover for E with transition functions $\tau_{\alpha\beta}$. Then E is trivial if and only if for each $\alpha \in A$ there is a holomorphic map $\psi_\alpha : U_\alpha \to \mathrm{GL}(k,\mathbb{C})$ such that $\tau_{\alpha\beta} = \psi_\alpha^{-1}\psi_\beta|_{U_\alpha\cap U_\beta}$ for each α and β.*

Proof. This is the special case of Proposition 3.7 when each transition function $\tau'_{\alpha\beta}$ is the identity matrix. □

We need to introduce one more important tool for constructing holomorphic vector bundles. Suppose $\pi : E \to M$ is a rank-k holomorphic vector bundle, N is a complex manifold, and $f : N \to M$ is a holomorphic map. We define a vector bundle $f^*E \to N$, called the ***pullback of E by f***, as follows. The total space of f^*E is the following subset of $N \times E$, called the ***fiber product of N with E over M***:

$$f^*E = \{(p, v) \in N \times E : \pi(v) = f(p)\}.$$

The projection $f^*E \to N$ is just the restriction of the projection on the first factor: $(p, v) \mapsto p$. The fiber of f^*E over a point $p \in N$ is then $\{p\} \times E_{f(p)}$, which inherits a complex vector space structure from that of $E_{f(p)}$. To make it into a holomorphic vector bundle, we construct local frames and use the chart lemma.

Given $p \in N$, choose a holomorphic local frame $(b_1, \dots, b_k)$ for E on a neighborhood of $f(p)$, and define a rough local frame $(\tilde{b}_1, \dots, \tilde{b}_k)$ for f^*E on a neighborhood of p by

$$\tilde{b}_j(p) = (p, b_j(f(p))).$$

Where two such frames (b_j) and (b'_j) overlap, there is a holomorphic $\mathrm{GL}(k,\mathbb{C})$-valued transition function τ such that $b'_j(q) = \tau_j^k(q)b_k(q)$, and therefore

$$\tilde{b}'_j(p) = \big(p, \tau_j^k(f(p))\tilde{b}_k(p)\big).$$

In other words, the frames $(\tilde{b}_j)$ and $(\tilde{b}'_j)$ overlap with the transition function $\tau \circ f$, which is holomorphic by composition. It follows from the chart lemma that f^*E is a holomorphic vector bundle.

In the situation described above, there is a pullback operator on local or global sections, defined as follows. For $\sigma \in \mathcal{O}(U; E)$ over an open subset $U \subseteq M$, we define $f^*\sigma \in \mathcal{O}(f^{-1}(U); f^*E)$ by

$$f^*\sigma(x) = (x, \sigma(f(x))).$$

Proposition 3.10 (Properties of Pullbacks of Sections). *Let $E \to M$ be a holomorphic vector bundle and $f : N \to M$ a holomorphic map.*

(a) *For each open set $U \subseteq N$, pullback defines a complex-linear map $f^* : \mathcal{O}(U; E) \to \mathcal{O}(f^{-1}(U); f^*E)$.*

(b) *If $(s_1, \dots, s_k)$ is a holomorphic local frame for E over $U \subseteq M$, then $(f^*s_1, \dots, f^*s_k)$ is a holomorphic local frame for f^*E over $f^{-1}(U)$.*

(c) *If f is surjective, then $f^* : \mathcal{O}(U; E) \to \mathcal{O}(f^{-1}(U); f^*E)$ is injective for each U.*

► **Exercise 3.11.** Prove the preceding proposition.

Given a holomorphic vector bundle $\pi : E \to M$, a subset $D \subseteq E$ is called a ***holomorphic subbundle of E*** if it is an embedded complex submanifold with the property that each fiber $D_p = D \cap E_p$ is a complex-linear subspace of some fixed dimension m, and the restricted projection $\pi|_D : D \to M$ turns D into a holomorphic vector bundle over M.

Lemma 3.12 (Local Frame Criterion for Subbundles). *Let $\pi : E \to M$ be a holomorphic vector bundle, and let $D \subseteq E$ be a subset whose intersection with each fiber of E is an m-dimensional complex subspace. Then D is a holomorphic subbundle if and only if each $p \in M$ has a neighborhood over which there are holomorphic sections $\sigma_1, \dots, \sigma_m$ of E such that $(\sigma_1(q), \dots, \sigma_m(q))$ forms a basis for D_q for each $q \in M$.*

Proof. Just like the proof of its smooth counterpart [**LeeSM**, Lemma 10.32]. □

Recall that a ***Hermitian inner product*** on a complex vector space V is a map from $V \times V$ to $\mathbb{C}$, often denoted by $(v, w) \mapsto \langle v, w \rangle$, that has the following properties for all $v, w \in V$ and $a \in \mathbb{C}$:

- Conjugate Symmetry: $\langle v, w \rangle = \overline{\langle w, v \rangle}$,
- Sesquilinearity: $\langle av, w \rangle = a\langle v, w \rangle$ and $\langle v, aw \rangle = \bar{a}\langle v, w \rangle$,
- Positive Definiteness: $\langle v, v \rangle > 0$ unless $v = 0$.

Suppose M is a smooth manifold and $E \to M$ is a smooth complex vector bundle of rank m. A ***Hermitian fiber metric*** h on E is a choice h_p of Hermitian inner product on each fiber E_p that is smoothly varying in the sense that for any pair of smooth sections σ, τ of E over an open subset $U \subseteq M$, the value $h_p(\sigma(p), \tau(p))$ depends smoothly on $p \in U$. When it will not cause confusion, we will often denote a Hermitian fiber metric by $\langle v, w \rangle \in \mathbb{C}$ for $v, w \in E_p$, or $\langle \sigma, \tau \rangle \in C^\infty(M; \mathbb{C})$ for $\sigma, \tau \in \Gamma(E)$. Given a Hermitian fiber metric on E, we define the ***norm*** of a vector $v \in E_p$ as $|v| = \langle v, v \rangle^{1/2}$. A smooth complex vector bundle endowed with a Hermitian fiber metric is called a ***Hermitian vector bundle***. An easy partition-of-unity argument shows that every smooth complex vector bundle admits a Hermitian fiber metric. (Note that if E is a holomorphic bundle, it does not make sense to ask that a fiber metric be holomorphic, because of the conjugate linearity in the second argument. Hermitian fiber metrics on holomorphic bundles are merely required to be smooth.)

The Space of Holomorphic Sections

If $E \to M$ is a holomorphic vector bundle, recall that $\mathcal{O}(M; E)$ denotes the complex vector space of global holomorphic sections of E. The next theorem illustrates one of the most dramatic differences between smooth bundles and holomorphic ones.

Theorem 3.13. *Suppose M is a compact complex manifold and $E \to M$ is a holomorphic vector bundle over M. Then $\mathcal{O}(M; E)$ is finite-dimensional.*

We present here a proof using complex analysis. A different proof based on the theory of partial differential equations will be presented in Chapter 9 (just following Theorem 9.35).

This proof will be based on the following standard lemma, which is proved in most functional analysis texts. For convenience, we include a proof here.

Lemma 3.14. *In a real or complex normed linear space $\mathcal{X}$, the closed unit ball is compact if and only if $\mathcal{X}$ is finite-dimensional.*

Proof. Because all norms on a finite-dimensional vector space are equivalent, if $\mathcal{X}$ is finite-dimensional, compactness of the closed unit ball follows from the analogous result for $\mathbb{R}^n$ or $\mathbb{C}^n$ with its standard norm.

Conversely, suppose $\mathcal{X}$ is infinite-dimensional. We will construct by induction an infinite sequence $\{x_j\}_{j=1}^{\infty}$ in $\mathcal{X}$ with $\|x_j\| = 1$ and $\|x_j - x_k\| > \frac{1}{2}$ for all $j \neq k$; because such a subsequence can have no convergent subsequence, it shows that the closed unit ball in $\mathcal{X}$ is not compact.

To begin the induction, just choose $x_1 \in \mathcal{X}$ arbitrarily with $\|x_1\| = 1$. Assuming $x_1, \dots, x_n$ have been chosen, let $\mathcal{X}_n = \operatorname{span}(x_1, \dots, x_n)$. Because $\mathcal{X}$ is infinite-dimensional, there is some $y \in \mathcal{X} \smallsetminus \mathcal{X}_n$. Since every finite-dimensional subspace of a normed linear space is a closed subset, there is some $\varepsilon > 0$ such that the open ball $B_\varepsilon(y)$ is contained in $\mathcal{X} \smallsetminus \mathcal{X}_n$. Let ε_1 be the supremum of such ε's. This means $\|y - x\| \geq \varepsilon_1$ for all $x \in \mathcal{X}_n$, and there is some $x^* \in \mathcal{X}_n$ such that $\|y - x^*\| < 2\varepsilon_1$. Set $x_{n+1} = (y - x^*)/\|y - x^*\|$, so that $\|x_{n+1}\| = 1$ (see Fig. 3.1).

To show that x_{n+1} satisfies the required conditions, suppose $j \in \{1, \dots, n\}$ and compute

$$\|x_{n+1} - x_j\| = \left\| \frac{y - x^*}{\|y - x^*\|} - \frac{\|y - x^*\|\, x_j}{\|y - x^*\|} \right\| = \frac{\left\|y - (x^* + \|y - x^*\|\, x_j)\right\|}{\|y - x^*\|}.$$

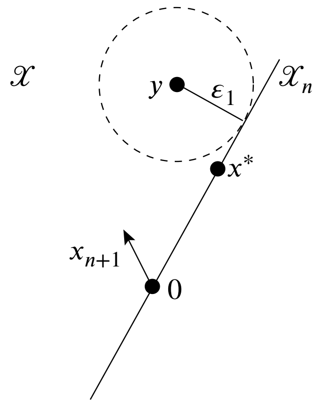

Figure 3.1. Proof of Lemma 3.14

In the last expression on the right, the numerator is greater than or equal to ε_1 because $(x^* + \|y - x^*\|\, x_j) \in \mathscr{X}_n$; and the denominator is less than $2\varepsilon_1$ by our choice of x^*. Therefore, $\|x_{n+1} - x_j\| > \frac{1}{2}$, thus completing the induction. □

Proof of Theorem 3.13. Choose a Hermitian fiber metric $\langle\cdot,\cdot\rangle_h$ on E, and use it to define a global norm on $\mathscr{O}(M;E)$ by

$$\|\sigma\| = \sup_{p\in M} |\sigma(p)|_h.$$

We will prove the theorem by showing that the closed unit ball with respect to this norm is compact. Let $\{\sigma_k\}_{k=1}^{\infty}$ be any sequence in $\mathscr{O}(M;E)$ with $\|\sigma_k\| \le 1$.

For each $p \in M$, there is a neighborhood W_p of p on which E admits a holomorphic local frame. For each p, choose open sets U_p, V_p such that $p \in U_p \subseteq \overline{U}_p \subseteq V_p \subseteq \overline{V}_p \subseteq W_p$. The collection of all such U_p is an open cover of M, so we can choose finitely many sets $W_1, \dots, W_m$ such that the corresponding sets $U_1, \dots, U_m$ cover M.

Consider the first of these sets W_1 with its holomorphic local frame $(s_1, \dots, s_k)$. On the restriction of E to $\overline{V}_1$ we have two fiber norms: the given one $|\cdot|_h$, and the "Euclidean" one determined by the local frame, namely

$$|\sigma(p)|_e = \left|\sigma^j(p)s_j(p)\right|_e = \left(|\sigma^1(p)|^2 + \cdots + |\sigma^k(p)|^2\right)^{1/2}.$$

Let S be the compact set $\overline{V}_1 \times \mathbb{S}^{2k-1} \subseteq \overline{V}_1 \times \mathbb{C}^k$. The function

$$(p,(z^1,\dots,z^k)) \mapsto \left|z^j s_j(p)\right|_h$$

determined by the given fiber metric is continuous and positive on this compact set, and thus attains maximum and minimum positive values Λ and λ there.

For any $p \in V_1$, if $\sigma(p) \neq 0$, then the components of $\sigma(p)/|\sigma(p)|_e$ lie in $\mathbb{S}^{2n-1}$, so

$$|\sigma(p)|_h = |\sigma(p)|_e \left| \frac{\sigma(p)}{|\sigma(p)|_e} \right|_h \geq |\sigma(p)|_e \lambda,$$

and a similar computation shows that $|\sigma(p)|_h \leq \Lambda |\sigma(p)|_e$. Of course, the same inequalities hold trivially when $\sigma(p) = 0$.

Returning to our sequence $\{\sigma_k\}$ with $\|\sigma_k\| \leq 1$, we conclude from the estimate above that $|\sigma_k(p)|_e \leq (1/\lambda)$ for $p \in V_1$, and thus each component function of σ_k with respect to the given frame is uniformly bounded on V_1. Montel's theorem (Prop. 1.35) then implies that a subsequence converges uniformly on the compact set $\overline{U}_1 \subseteq V_1$ to a section that is holomorphic there.

Proceeding similarly, we can choose a sub-subsequence that also converges on $\overline{U}_2$, and so on for $\overline{U}_3, \dots, \overline{U}_m$. The final subsequence converges in the global $\|\cdot\|_h$ norm to a global holomorphic section, showing that the unit ball in $\mathscr{O}(M; E)$ is compact. □

Examples of Holomorphic Vector Bundles

Every complex manifold plays host to one important holomorphic vector bundle, from which many others can be constructed.

Example 3.15 (The Holomorphic Tangent Bundle). Let M be a complex n-manifold and $T'M$ be its holomorphic tangent bundle. Each local holomorphic coordinate chart (z^j) provides a local frame $(\partial/\partial z^1, \dots, \partial/\partial z^n)$ for $T'M$. To see how two such frames overlap, let (w^j) be another local holomorphic coordinate chart. The chain rule (Prop. 1.47) gives

$$\frac{\partial}{\partial w^k} = \frac{\partial z^j}{\partial w^k}\frac{\partial}{\partial z^j} + \frac{\partial \bar{z}^j}{\partial w^k}\frac{\partial}{\partial \bar{z}^j}.$$

Because each z^j is holomorphic, the second term on the right is identically zero, so the transition function from the frame $(\partial/\partial z^j)$ to the frame $(\partial/\partial w^k)$ is the matrix-valued function $(\partial z^j/\partial w^k)$, which is holomorphic. Thus the chart lemma gives $T'M$ the structure of a holomorphic vector bundle. //

Example 3.16 (Dual Bundles). Now suppose $E \to M$ is an arbitrary holomorphic vector bundle. The ***dual bundle of E*** is the bundle E^* whose fiber at each point $p \in M$ is the space E_p^* of complex-linear functionals $E_p \to \mathbb{C}$. Given a local holomorphic frame $(s_1, \dots, s_m)$ for E, we can form the dual frame $(\varepsilon^1, \dots, \varepsilon^m)$ for E^*, satisfying $\varepsilon^j(p)(s_k(p)) = \delta_{jk}$ for all p. If $(\tilde{s}_1, \dots, \tilde{s}_m)$ is another holomorphic local frame for E with transition function τ satisfying

$$\tilde{s}_j = \tau_j^k s_k,$$

then a little linear algebra shows that

$$\tilde{\varepsilon}^k = (\tau^{-1})_l^k \varepsilon^l,$$

so the chart lemma shows that E^* is a holomorphic vector bundle whose transition functions are the transposed inverses (called the ***contragredients***) of those of E. //

Example 3.17 (Bundles Over 0-Manifolds). If M is a 0-manifold and $E \to M$ is any complex vector bundle, for each $p \in M$ we can choose a basis $(s_1(p), \dots, s_k(p))$ for the fiber E_p, and the maps $p \mapsto s_j(p)$ are vacuously holomorphic. Thus $(s_1, \dots, s_k)$ is a holomorphic global frame, so every bundle over M is holomorphically trivial. //

Example 3.18 (Whitney Sums). If $E \to M$ and $E' \to M$ are holomorphic vector bundles of ranks k and k', respectively, their ***Whitney sum*** is the bundle $E \oplus E'$ whose fiber at each $p \in M$ is the direct sum $E_p \oplus E'_p$. Given holomorphic local frames $(s_1, \dots, s_k)$ for E and $(s'_1, \dots, s'_{k'})$ for E', we get a local frame $(s_1, \dots, s_k, s'_1, \dots, s'_{k'})$ for $E \oplus E'$. If τ and τ' are transition functions for overlapping local frames for E and E', respectively, then the transition function for $E \oplus E'$ is the $\mathrm{GL}(k + k', \mathbb{C})$-valued matrix function $\left(\begin{smallmatrix} \tau & 0 \\ 0 & \tau' \end{smallmatrix}\right)$, which is holomorphic. Thus by the chart lemma, $E \oplus E'$ is a holomorphic vector bundle of rank $k + k'$. //

Example 3.19 (Tensor Product Bundles). With E, E' as in the previous example, we define the ***tensor product bundle*** $E \otimes E'$ similarly, as the bundle whose fiber at p is $E_p \otimes E'_p$. Any section of $E \otimes E'$ can be written locally as a finite sum $\sum_j \sigma_j \otimes \sigma'_j$, where each σ_j is a local section of E and each σ'_j is a local section of E'. Local frames (s_i) for E and (s'_j) for E' yield a local frame $(s_i \otimes s'_j)$ for $E \otimes E'$, with holomorphic transition functions, so $E \otimes E'$ is a holomorphic vector bundle of rank kk'. //

Example 3.20 (Determinant Bundles). Suppose $E \to M$ is a holomorphic vector bundle of rank k. The ***determinant bundle of E*** is the complex line bundle $\det E \to M$ whose fiber at a point $p \in M$ is the 1-dimensional vector space $\Lambda^k(E_p)$ of alternating k-vectors (i.e., contravariant alternating k-tensors) in E_p. Any holomorphic local frame $(s_1, \dots, s_k)$ for E yields a local frame $s_1 \wedge \dots \wedge s_k$ for $\det E$. If τ is a transition function between two local trivializations of E, then $\det(\tau)$ is the corresponding transition function for $\det E$, so the determinant bundle is holomorphic. //

Example 3.21 (Restriction of a Bundle). If $\pi : E \to M$ is a holomorphic vector bundle and $S \subseteq M$ is a complex submanifold, the set $E|_S = \pi^{-1}(S)$ together with the projection inherited from E is called the ***restriction of E to S***. Each holomoorphic local trivialization for E restricts to a local trivialization for $E|_S$, with transition functions that are restrictions of those of E, so $E|_S$ is a holomorphic vector bundle over S. //

Example 3.22 (Quotient Bundles). Suppose $E \to M$ is a holomorphic vector bundle of rank n and $F \subseteq E$ is a holomorphic rank-k subbundle. The ***quotient bundle*** $E/F \to M$ is the bundle whose fiber at $p \in M$ is the quotient space E_p/F_p.

Given a holomorphic local frame $(s_1, \dots, s_k)$ for F, we can complete it to a holomorphic local frame $(s_1, \dots, s_n)$ for E (perhaps after shrinking the domain), and then consider $(s_{k+1}, \dots, s_n) \pmod{F}$ as a local frame for E/F. Where two such frames (s_i) and $(\tilde{s}_j)$ overlap, the holomorphic transition matrix has the form

$$\tau^i_j = \begin{pmatrix} \alpha^i_j & 0 \\ \beta^i_j & \gamma^i_j \end{pmatrix},$$

and we have $\tilde{s}_j = \sum_{i=k+1}^n \gamma^i_j s_i \pmod{F}$ for $j = k+1, \dots, n$, so E/F is a holomorphic bundle of rank $n-k$ by the chart lemma. //

Example 3.23 (Normal Bundles). Suppose M is a complex n-manifold and $S \subseteq M$ is k-dimensional complex submanifold. The ***holomorphic normal bundle of S in M*** is the bundle $NS \to S$ defined by $NS = (T'M|_S)/T'S$. The result of Example 3.22 shows that it is a holomorphic bundle of rank $n-k$ on S. (It is important to observe that the ***geometric normal bundle*** that can be defined as the set of tangent vectors that are orthogonal to S with respect to some Riemannian metric on M will not in general have holomorphic transition functions.) //

Example 3.24 (Homomorphism and Endomorphism Bundles). Again let E, E' be holomorphic vector bundles over M. We can form the bundle $\operatorname{Hom}(E, E')$ whose fiber at a point $p \in M$ is the space of complex-linear maps from E_p to E'_p. There is a canonical bijection from $E' \otimes E^*$ to $\operatorname{Hom}(E, E')$ which sends an element $\sigma'_j \otimes \varphi^j \in E'_x \otimes E^*_x$ to the linear map from E_x to E'_x given by $e \mapsto \varphi^j(e)\sigma'_j$. Therefore, $\operatorname{Hom}(E, E')$ has local trivializations with the same transition functions as $E' \otimes E^*$, and is thus a holomorphic vector bundle canonically isomorphic to $E' \otimes E^*$. In the special case $E = E'$, the bundle $\operatorname{Hom}(E, E)$ is denoted by $\operatorname{End}(E)$ and called the ***endomorphism bundle of E***. //

Holomorphic Line Bundles

For the remainder of the chapter, we focus on holomorphic line bundles (i.e., holomorphic vector bundles of rank 1). Local trivializations of line bundles are most easily expressed in terms of local frames, which in this case are just nonvanishing local sections; and transition functions are $\mathrm{GL}(1, \mathbb{C})$-valued holomorphic functions, which we can consider as nonvanishing holomorphic scalar-valued functions. The next lemma shows how transition functions are related to local frames.

Lemma 3.25. *Let $L \to M$ be a holomorphic line bundle. Suppose (U_α, Φ_α) and (U_β, Φ_β) are holomorphic local trivializations of L, with transition function $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(1, \mathbb{C})$ from Φ_β to Φ_α. Then the holomorphic local frames s_α and s_β associated with these local trivializations are related on $U_\alpha \cap U_\beta$ by*

$$s_\beta = \tau_{\alpha\beta} s_\alpha. \tag{3.4}$$

Proof. By Lemma 3.3, for all $p \in U_\alpha \cap U_\beta$ we have

$$s_\alpha(p) = \Phi_\alpha^{-1}(p, e_1), \qquad s_\beta(p) = \Phi_\beta^{-1}(p, e_1), \tag{3.5}$$

and by Lemma 3.1,

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, e_1) = \big(p, \tau_{\alpha\beta}(p) e_1\big).$$

Therefore,

$$\Phi_\alpha\big(s_\beta(p)\big) = \Phi_\alpha \circ \Phi_\beta^{-1}(p, e_1) = \big(p, \tau_{\alpha\beta}(p) e_1\big).$$

On the other hand, since Φ_α is linear on fibers, (3.5) implies

$$\Phi_\alpha\big(\tau_{\alpha\beta}(p) s_\alpha(p)\big) = \tau_{\alpha\beta}(p)\Phi_\alpha\big(s_\alpha(p)\big) = \big(p, \tau_{\alpha\beta}(p) e_1\big).$$

Comparing the last two equations and using the fact that Φ_α is injective proves the result. □

To help remember (3.4), observe that while the transition function $\tau_{\alpha\beta}$ expresses the transition from the Φ_β trivialization to the Φ_α trivialization (see (3.1)), when expressed in terms of frames, the same function goes the other way: it expresses the transition from the α frame to the β frame.

As a consequence, the transition functions of some related bundles can be expressed easily in terms of those of the original bundles.

Lemma 3.26. *Suppose L and L' are holomorphic line bundles over M, and both have local trivializations over the same trivializing cover with transition functions $\tau_{\alpha\beta}$ and $\tau'_{\alpha\beta}$, respectively.*

(a) *L^* has transition functions $1/\tau_{\alpha\beta}$.*

(b) *$L \otimes L'$ has transition functions $\tau_{\alpha\beta}\tau'_{\alpha\beta}$.*

(c) *$L^k = L \otimes \cdots \otimes L$ has transition functions $(\tau_{\alpha\beta})^k$.*

► **Exercise 3.27.** Prove this lemma.

Next we note some important properties of the tensor product operation on holomorphic line bundles.

Lemma 3.28. *Let M be a complex manifold, and let L, L', L'' be holomorphic line bundles over M.*

(a) *$(L \otimes L') \otimes L'' \cong L \otimes (L' \otimes L'')$.*

(b) *$L \otimes L' \cong L' \otimes L$.*

(c) *If L_1 and L_1' are holomorphic line bundles isomorphic to L and L', respectively, then $L_1 \otimes L_1' \cong L \otimes L'$.*

(d) *$L \otimes L^*$ is a trivial bundle.*

(e) *If L' is trivial, then $L \otimes L' \cong L$.*

Proof. These all follow from the formulas given in Lemma 3.26 together with the result of Corollary 3.8. □

Thanks to these results, we can make the following definition. Let M be a complex manifold, and let $\operatorname{Pic}(M)$ denote the set of isomorphism classes of holomorphic line bundles over M. (Although the class of all holomorphic line bundles over M is too big to be a set, the fact that isomorphism classes are determined by their transition functions implies that the isomorphism classes do constitute a set.) The next theorem shows that it is an abelian group in a natural way, called the ***Picard group of M*** after the French mathematician Charles Émile Picard, who first defined it in the early twentieth century.

Theorem 3.29 (The Picard Group). *If M is a complex manifold, the product operation defined by $[L_1] \cdot [L_2] = [L_1 \otimes L_2]$ turns the set $\operatorname{Pic}(M)$ of isomorphism classes of holomorphic line bundles over M into an abelian group. The identity element is the isomorphism class of any trivial bundle, and the inverse of $[L]$ is $[L^*]$.*

Proof. This follows easily from Lemma 3.28. □

The next lemma gives another way of understanding tensor power bundles.

Lemma 3.30. *Let $L \to M$ be a holomorphic line bundle. For each positive integer d, the tensor power bundle $(L^*)^d$ is naturally isomorphic to the bundle whose fiber at a point $p \in M$ is the space of functions $\varphi\colon L_p \to \mathbb{C}$ that are **homogeneous of degree d**, meaning that $\varphi(\lambda v) = \lambda^d \varphi(v)$ for all $\lambda \in \mathbb{C}$ and $v \in L_p$. If $s\colon U \to L$ is a holomorphic local frame for L, then a local section φ of $(L^*)^d$ is holomorphic on U if and only if the function $p \mapsto \varphi(s(p))$ is holomorphic.*

Proof. Just as in smooth manifold theory, the fiber of $(L^*)^d$ over p is naturally isomorphic to the space of d-linear maps from $L_p \times \cdots \times L_p$ to $\mathbb{C}$ (see [**LeeSM**, Prop. 12.10]). Since L_p is 1-dimensional, each such map φ gives rise to a homogeneous map $\widetilde{\varphi}\colon L_p \to \mathbb{C}$ of degree d by $\widetilde{\varphi}(v) = \varphi(v, \dots, v)$; conversely, given a homogeneous map $\widetilde{\varphi}$, we can recover φ by choosing a basis vector b for L_p and defining $\varphi(b, \dots, b) = \widetilde{\varphi}(b)$, and noting that a multilinear map is determined by its action on basis vectors.

If s is a holomorphic local frame for L and ε is the dual frame for L^*, then $\varepsilon^d = \varepsilon \otimes \cdots \otimes \varepsilon$ is a holomorphic local frame for $(L^*)^d$ whose action on s as a homogeneous function is $\varepsilon^d(s) = 1$. Therefore an arbitrary section φ of $(L^*)^d$ is a holomorphic multiple of this frame if and only if its action on $s(p)$ depends holomorphically on p. □

Line Bundles over Projective Space

There is one particularly important line bundle over $\mathbb{CP}^n$. We define $T \to \mathbb{CP}^n$ as the following subbundle of the product bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$:

$$T = \left\{(\xi, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \,:\, z \in \xi\right\}.$$

Proposition 3.31. *$T \to \mathbb{CP}^n$ is a holomorphic line bundle.*

Proof. We will use the local frame criterion (Lemma 3.12). Because $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ is a product bundle, local sections of it are determined by functions from open subsets of $\mathbb{CP}^n$ to $\mathbb{C}^{n+1}$. On the open subset $U_\alpha \subseteq \mathbb{CP}^n$ consisting of points $[w^0, \dots, w^n]$ with $w^\alpha \neq 0$, we have a local section $s_\alpha : U_\alpha \to T$ defined by

$$s_\alpha([w^0, \dots, w^n]) = \left([w],\, \frac{1}{w^\alpha}(w^0, \dots, w^n)\right). \tag{3.6}$$

This local section is nowhere vanishing and spans the fiber of T at each point of U_α. In terms of affine coordinates $(z^1, \dots, z^n) \leftrightarrow [z^1, \dots, 1, \dots, z^n]$, it has the local representation $(z^1, \dots, z^n) \mapsto (z^1, \dots, 1, \dots, z^n)$, so it is holomorphic. The local frame criterion shows that T is a holomorphic line bundle. □

The bundle $T \to \mathbb{CP}^n$ is called the ***tautological bundle*** because the fiber of T over each $\xi \in \mathbb{CP}^n$ is exactly the line ξ itself.

It will be useful to have explicit expressions for the transition functions for this bundle and related ones.

Proposition 3.32 (Transition Functions on $\mathbb{CP}^n$). *On each open subset $U_\alpha \subseteq \mathbb{CP}^n$ where $w^\alpha \neq 0$, let $s_\alpha : U_\alpha \to T$ be the holomorphic section defined by* (3.6). *On $U_\alpha \cap U_\beta$, these sections satisfy $s_\beta = \tau_{\alpha\beta} s_\alpha$, where*

$$\tau_{\alpha\beta}([w]) = w^\alpha / w^\beta. \tag{3.7}$$

Thus T^*, T^k and $(T^*)^k$ have trivializations over the same open sets, with transition functions

$$\tau_{\alpha\beta}([w])^k = (w^\alpha / w^\beta)^k \text{ for } T^k; \tag{3.8}$$

$$\tau_{\alpha\beta}([w])^{-1} = w^\beta / w^\alpha \text{ for } T^*; \tag{3.9}$$

$$\tau_{\alpha\beta}([w])^{-k} = (w^\beta / w^\alpha)^k \text{ for } (T^*)^k. \tag{3.10}$$

Proof. The formulas (3.8)–(3.10) follow from (3.7) and Lemma 3.26, so we need only calculate the transition functions for T. Let $U_\beta \subseteq \mathbb{CP}^n$ be the set where $w^\beta \neq 0$ and let $s_\beta : U_\beta \to T$ be the corresponding holomorphic section. Then (assuming $\alpha < \beta$ for simplicity), on $U_\alpha \cap U_\beta$ we have

$$s_\beta([w]) = \left([w], \left(\frac{w^0}{w^\beta}, \dots, \frac{w^\alpha}{w^\beta}, \dots, \frac{w^\beta}{w^\beta}, \dots, \frac{w^n}{w^\beta}\right)\right) = \frac{w^\alpha}{w^\beta} s_\alpha([w]).$$

This proves (3.7). □

Let us look more carefully at the structure of the tautological bundle. The first thing to note is that because it is a holomorphic subbundle of the product bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, it is an embedded complex submanifold of dimension $n+1$ (the dimension of the base plus the dimension of the fiber). To further understand its structure, we use the fact that in addition to the bundle projection $T \to \mathbb{CP}^n$, which is the restriction of the projection $\pi_1 : \mathbb{CP}^n \times \mathbb{C}^{n+1} \to \mathbb{CP}^n$, we have another projection from T to $\mathbb{C}^{n+1}$, namely the restriction of $\pi_2 : \mathbb{CP}^n \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$.

Proposition 3.33. *Let $T_0 \subseteq T$ be the image of the zero section of T, and let $\Pi : T \to \mathbb{C}^{n+1}$ be the restriction of $\pi_2 : \mathbb{CP}^n \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. Then $\Pi^{-1}(0) = T_0$, which is a complex submanifold of T biholomorphic to $\mathbb{CP}^n$, and Π restricts to a biholomorphism from $T \smallsetminus T_0$ to $\mathbb{C}^{n+1} \smallsetminus \{0\}$.*

Proof. Directly from the definition of Π, it follows that $\Pi^{-1}(0) = \{([w], 0) : [w] \in \mathbb{CP}^n\}$, which is exactly the image of the zero section $\zeta : \mathbb{CP}^n \to T$. Because the bundle projection $\pi : T \to \mathbb{CP}^n$ is a holomorphic left inverse for ζ, it follows that ζ is a holomorphic embedding whose image is $\Pi^{-1}(0)$.

On the other hand, the restriction $\Pi|_{T \smallsetminus T_0} : T \smallsetminus T_0 \to \mathbb{C}^{n+1} \smallsetminus \{0\}$ has a holomorphic inverse given by $v \mapsto ([v], v)$, so it is a biholomorphism. □

Blowing Up

Proposition 3.33 shows that if T is the tautological bundle over $\mathbb{CP}^{n-1}$, we can think of the total space of T as a copy of $\mathbb{C}^n$ in which the origin has been replaced by a copy of $\mathbb{CP}^{n-1}$. This leads to a new way of constructing complex manifolds out of other complex manifolds, by replacing a point in an n-manifold with a copy of $\mathbb{CP}^{n-1}$, modeled locally on the total space of T.

Let M be a complex n-manifold and $p \in M$. The ***blowup of M at p*** is a complex manifold $\widetilde{M}$ together with a surjective holomorphic map $\pi : \widetilde{M} \to M$, defined as follows. As a set, $\widetilde{M}$ is the disjoint union of $M \smallsetminus \{p\}$ with $\mathbb{P}(T'_p M)$ (the projectivization of the holomorphic tangent space $T'_p M$). Let $E = \mathbb{P}(T'_p M) \subseteq \widetilde{M}$ (called the ***exceptional hypersurface*** of the blowup), and define $\pi : \widetilde{M} \to M$ (called the ***blowdown map***) by

$$\pi(z) = \begin{cases} z, & z \in \widetilde{M} \smallsetminus E, \\ p, & z \in E. \end{cases}$$

Let T be the tautological bundle over $\mathbb{CP}^{n-1}$, and let $\Pi : T \to \mathbb{C}^n$ be the restriction to T of $\pi_2 : \mathbb{CP}^{n-1} \times \mathbb{C}^n \to \mathbb{C}^n$, called the ***model blowdown map***.

Given a holomorphic coordinate chart (U, φ) for M centered at p, with coordinate functions $(z^1, \dots, z^n)$, choose $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq \varphi(U)$. Let

$$U(\varepsilon) = \varphi^{-1}(B_\varepsilon(0)) \subseteq M, \tag{3.11}$$

$$\widetilde{U}(\varepsilon) = \pi^{-1}(U(\varepsilon)) \subseteq \widetilde{M}, \tag{3.12}$$

$$T(\varepsilon) = \Pi^{-1}(B_\varepsilon(0)) \subseteq T. \tag{3.13}$$

Define a map $\Phi\colon T(\varepsilon) \to \widetilde{U}(\varepsilon)$ by

$$\Phi([w], z) = \begin{cases} \varphi^{-1}(z), & z \neq 0, \\ \left[w^1 \left.\dfrac{\partial}{\partial z^1}\right|_p + \cdots + w^n \left.\dfrac{\partial}{\partial z^n}\right|_p \right], & z = 0. \end{cases} \tag{3.14}$$

In the case when M has complex dimension 1, $\mathbb{P}(T'_p M)$ is a 0-dimensional projective space, which is just a single point; in that case we just define $\widetilde{M} = M$ and π to be the identity map.

Proposition 3.34 (Blowups Are Complex Manifolds). *Let M be a complex manifold, $\pi\colon \widetilde{M} \to M$ the blowup of M at a point $p \in M$, and $E = \pi^{-1}(p)$. There is a unique complex manifold structure on $\widetilde{M}$ with the property that for each holomorphic coordinate chart for M centered at p, the map Φ defined above is a biholomorphism onto a neighborhood of E. The blowdown map π is holomorphic, and E is an embedded compact complex hypersurface.*

Proof. We will construct coordinate charts for $\widetilde{M}$ and use the complex manifold chart lemma to give it the structure of a complex manifold.

Since the blowdown map π restricts to a bijection from $\widetilde{M} \smallsetminus E$ to $M \smallsetminus \{p\}$, we can begin with all of the given holomorphic coordinate charts on $M \smallsetminus \{p\}$, pulled back to $\widetilde{M} \smallsetminus E$ via π. To define holomorphic coordinates on a neighborhood of E, choose a holomorphic coordinate chart (U, φ) for M centered at p and define $\Phi\colon T(\varepsilon) \to \widetilde{U}(\varepsilon)$ as above. This is a bijection, so we can use holomorphic coordinates on $T(\varepsilon)$ composed with Φ^{-1} to define holomorphic coordinates on $\widetilde{U}(\varepsilon)$. These are certainly all holomorphically compatible with each other, and because the restriction of Φ to $T(\varepsilon) \smallsetminus T_0$ is a biholomorphism onto $\widetilde{U}(\varepsilon) \smallsetminus E \approx U(\varepsilon) \smallsetminus \{p\}$, these coordinates are also holomorphically compatible with the given coordinates on $\widetilde{M} \smallsetminus E$. The only remaining thing to check is that the coordinates defined by any other holomorphic chart $(\widetilde{U}, \widetilde{\varphi})$ centered at p are compatible with these.

To that end, suppose $\left(\widetilde{U}, \widetilde{\varphi}\right)$ is another chart centered at p, and let $\widetilde{\Phi}$ be the corresponding map defined in the same way as Φ. Write the component functions of $\widetilde{\varphi} \circ \varphi^{-1}(z)$ as $\left(\tilde{z}^1(z), \dots, \tilde{z}^n(z)\right)$. The composite map $\widetilde{\Phi}^{-1} \circ \Phi$ has the following form:

$$\widetilde{\Phi}^{-1} \circ \Phi([w], z) = \begin{cases} \Big(\big[\tilde{z}^1(z), \dots, \tilde{z}^n(z)\big],\ \big(\tilde{z}^1(z), \dots, \tilde{z}^n(z)\big) \Big), & z \neq 0, \\ \Big(\Big[\sum_j w^j \dfrac{\partial \tilde{z}^1}{\partial z^j}(0), \dots, \sum_j w^j \dfrac{\partial \tilde{z}^n}{\partial z^j}(0) \Big],\ 0 \Big), & z = 0. \end{cases}$$

To see that this is holomorphic, we will express it in another way. Using the Taylor series for $\tilde{z}^k$ together with the fact that $\tilde{z}^k(0) = 0$, we can write

$$\tilde{z}^k(z) = \sum_{j=1}^n z^j f_j^k(z)$$

for some holomorphic functions f_j^k, with

$$f_j^k(0) = \frac{\partial \tilde{z}^k}{\partial z^j}(0).$$

Thus the composite map $\widetilde{\Phi}^{-1} \circ \Phi$ is the restriction to $T(\varepsilon)$ of the holomorphic map from $\mathbb{CP}^{n-1} \times \mathbb{C}^n$ to itself given by

$$([w], z) \mapsto \Big(\Big[\sum_j w^j f_j^1(z), \dots, \sum_j w^j f_j^n(z) \Big],\ \big(\tilde{z}^1(z), \dots, \tilde{z}^n(z)\big) \Big).$$

This shows that $\widetilde{\Phi}^{-1} \circ \Phi$ is holomorphic, and the same argument applies to its inverse. Thus we have made $\widetilde{M}$ into a complex manifold, and each of the maps Φ defined above is a biholomorphism onto a neighborhood of E as claimed. Since any other such holomorphic structure must contain all of these charts, it must be equal to this one.

To see that π is holomorphic, just note that its restriction to $\widetilde{M} \smallsetminus E$ is a biholomorphism onto its image, and its restriction to $\Phi(T(\varepsilon))$ is equal to the holomorphic composition $\varphi^{-1} \circ \Pi \circ \Phi^{-1}$. Because E is the image of the compact complex hypersurface T_0 under the diffeomorphism $\Phi\colon T(\varepsilon) \to \widetilde{U}(\varepsilon)$, it is itself an embedded compact complex hypersurface. □

Similarly, we can define the blowup of M at finitely many points $p_1, \dots, p_m \in M$ by letting $\widetilde{M}$ be the set

$$\widetilde{M} = M \smallsetminus \{p_1, \dots, p_m\} \amalg \mathbb{P}(T'_{p_1} M) \amalg \cdots \amalg \mathbb{P}(T'_{p_m} M),$$

with the obvious blowdown map $\pi\colon \widetilde{M} \to M$, and with the holomorphic structure on $\widetilde{M}$ defined by applying the above construction in a neighborhood of each exceptional hypersurface $\pi^{-1}(p_i)$.

This blowup construction will play a major role in our proof of the Kodaira embedding theorem in Chapter 10.

Holomorphic Sections over Projective Space

We can describe explicitly the spaces of holomorphic sections of various line bundles over $\mathbb{CP}^n$. We begin with the tautological bundle.

Proposition 3.35. *The tautological bundle $T \to \mathbb{CP}^n$ has no nontrivial global holomorphic sections.*

Proof. Suppose $\sigma : \mathbb{CP}^n \to T$ is a holomorphic section. Composing with the model blowdown map $\Pi : T \to \mathbb{C}^{n+1}$, we get a holomorphic map $\Pi \circ \sigma : \mathbb{CP}^n \to \mathbb{C}^{n+1}$. Because $\mathbb{CP}^n$ is compact, each of the component functions of this map must be constant. Thus there is some $c \in \mathbb{C}^{n+1}$ such that $\Pi \circ \sigma(\mathbb{CP}^n) = \{c\}$.

For each $\xi \in \mathbb{CP}^n$, the value $\sigma(\xi)$ lies in the fiber of T over ξ, and thus the point $c = \Pi(\sigma(\xi))$ lies in the 1-dimensional subspace $\xi \subseteq \mathbb{C}^{n+1}$. The only point in $\mathbb{C}^{n+1}$ that lies in every 1-dimensional subspace is the origin, and the preimage of 0 in T contains only the zero point of each fiber, so σ is equal to the zero section. □

Next we look at the dual bundle of T. Because it will turn out to be even more important than T itself, we use the symbol H to denote the holomorphic line bundle T^*; the significance of this choice of notation will be explained in the next section. Similarly, we use the notations

H^0 for the trivial bundle $\mathbb{CP}^n \times \mathbb{C} \to \mathbb{CP}^n$,

H^d for the d-fold tensor power bundle $H \otimes \cdots \otimes H$ $(d \geq 1)$,

H^{-1} for $T \cong H^*$,

H^{-d} for the d-fold tensor power bundle $T \otimes \cdots \otimes T$ $(d \geq 1)$.

The notation for negative powers of H is motivated by the fact that $H^{-d} = T^d$ represents the inverse of H^d in the Picard group. (In the algebraic geometry literature, the notation $\mathcal{O}(d)$ is often used to denote the line bundle H^d; see the discussion following Exercise 5.15 for an explanation of the reason for this notation.)

We can construct some nontrivial sections of H and its positive tensor powers in the following way. By Lemma 3.30, for $d \geq 1$, the fiber of H^d at $\xi \in \mathbb{CP}^n$ can be identified with the set of functions from the line ξ to $\mathbb{C}$ that are homogeneous of degree d. One way of constructing a section of H^d is to start with a homogeneous holomorphic degree-d polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and restrict it to each 1-dimensional subspace of $\mathbb{C}^{n+1}$. This yields a rough section $\varphi_f : \mathbb{CP}^n \to H^d$, given explicitly by

$$\varphi_f(\xi) = f|_\xi. \tag{3.15}$$

To see that this section is holomorphic, choose affine coordinates $(z^1, \dots, z^n) \leftrightarrow [z^1, \dots, 1, \dots, z^n]$ on the open set $U_\alpha \subseteq \mathbb{CP}^n$ where the αth homogeneous coordinate is nonzero. Over U_α, we have a local holomorphic frame $s_\alpha : U_\alpha \to T$ defined by (3.6). The section φ_f applied to s_α gives

$$\varphi_f(s_\alpha([z^1, \ldots, 1, \ldots, z^n])) = f(z^1, \ldots, 1, \ldots, z^n),$$

which is holomorphic, so it follows from Lemma 3.30 that φ_f is a holomorphic section.

The next theorem shows that we have produced all of the nontrivial holomorphic sections of powers of H.

Theorem 3.36 (Holomorphic Sections of Projective Line Bundles). *Let $H \to \mathbb{CP}^n$ be the dual of the tautological bundle. The global holomorphic sections of powers of H are described as follows. Let d be a positive integer.*

$\mathscr{O}(\mathbb{CP}^n; H^{-d}) = \{0\}$.

$\mathscr{O}(\mathbb{CP}^n; H^0) = \{\text{constants}\}$.

$\mathscr{O}(\mathbb{CP}^n; H^d) = \{\varphi_f : f \text{ a homogeneous holomorphic degree-}d\text{ polynomial}\}$.

Proof. Proposition 3.35 showed that the only global holomorphic section of $T = H^{-1}$ is the zero section. Assume for the sake of contradiction that σ is a nontrivial holomorphic section of H^{-d} for some $d \geq 2$. We showed above that H^{d-1} has a nontrivial holomorphic section φ_f. Then $\sigma \otimes \varphi_f$ is a nontrivial holomorphic section of $H^{-d} \otimes H^{d-1} \cong H^{-1}$, which is a contradiction.

Because a holomorphic section of the trivial bundle H^0 is just a scalar-valued holomorphic function, it follows from Corollary 1.33 that the only global holomorphic sections of H^0 are constants.

Now consider H^d for $d > 0$. We observed above that every homogeneous holomorphic degree-d polynomial f gives rise to a holomorphic section φ_f, so we just need to show that every holomorphic section is of this form. Let $\sigma \in \mathscr{O}(\mathbb{CP}^n; H^d)$ be arbitrary. We can define a holomorphic function $f : \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{C}$ by

$$f(z) = \sigma([z])(z),$$

where we are viewing $\sigma([z]) \in (H_{[z]})^d$ as a homogeneous function from the line $[z]$ to $\mathbb{C}$. By Hartogs's extension theorem, f extends to a holomorphic function on all of $\mathbb{C}^{n+1}$. It satisfies $f(\lambda z) = \lambda^d f(z)$ for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^{n+1} \smallsetminus \{0\}$, and thus also for $z = 0$ by continuity; in other words, it is homogeneous of degree d.

We will show that f is actually a polynomial. For nonzero $z \in \mathbb{C}^{n+1}$, homogeneity implies

$$|f(z)| = |z|^d \left| f\left(\frac{z}{|z|}\right) \right| \leq C|z|^d,$$

where C is the supremum of $|f|$ on the unit sphere. This implies that the Taylor series of f at the origin has no terms of order less than d. Let p be the polynomial function obtained by considering only the terms of degree d in the Taylor series for f at the origin:

$$p(z) = \sum_{k_1+\cdots+k_n=d} a_{k_1\ldots k_n}(z^1)^{k_1}\cdots(z^n)^{k_n},$$

and let $r(z) = f(z) - p(z)$. Then r is homogeneous of degree d because f and p are. Since the Taylor series of r starts with terms of order $d+1$, there is some constant C' such that $|r(z)| \le C'|z|^{d+1}$ for all z in the closed unit ball. For any $z \in \mathbb{C}^{n+1}$ and $\varepsilon > 0$ small enough that $|\varepsilon z| \le 1$, we have

$$|r(z)| = \varepsilon^{-d}|r(\varepsilon z)| \le \varepsilon^{-d}C'|\varepsilon z|^{d+1} = \varepsilon C'|z|.$$

Taking the limit as $\varepsilon \to 0$, we find that $r(z) \equiv 0$. Thus f is equal to the polynomial p, and $\sigma = \varphi_p$. □

We will see later that every holomorphic line bundle on $\mathbb{CP}^n$ is isomorphic to H^d for some integer d (see Proposition 9.51).

Applications of Holomorphic Line Bundles

We end the chapter by describing two significant applications of holomorphic line bundles.

Line Bundles and Hypersurfaces

In $\mathbb{C}^n$, many complex hypersurfaces can be written as regular level sets of globally defined holomorphic functions. But in a compact complex manifold, this is never possible, because all global holomorphic functions are constants. Instead, we can use sections of line bundles.

Suppose M is a complex manifold, $L \to M$ is a holomorphic line bundle, and $\sigma \in \mathscr{O}(M; L)$. The ***variety determined by σ*** is the set $V_\sigma = \{p \in M : \sigma(p) = 0\}$. We say the section σ ***vanishes simply*** if whenever s is a local holomorphic frame for L, we can write $\sigma = fs$ where $df_p \ne 0$ whenever $f(p) = 0$. It is easy to check that this condition is independent of the choice of local frame, and if σ is a holomorphic section that vanishes simply, then V_σ is a closed complex hypersurface (i.e., a codimension-1 complex submanifold that is a closed subset of M).

Example 3.37. Suppose $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a homogeneous holomorphic polynomial of degree d, and $V \subseteq \mathbb{CP}^n$ is the algebraic variety determined by f. The polynomial f also defines a holomorphic section $\varphi_f : \mathbb{CP}^n \to H^d$ as in (3.15), and V is exactly the variety determined by this section.

In particular, in the case $d = 1$, holomorphic sections of H are of the form φ_f where f is a complex-linear functional on $\mathbb{C}^{n+1}$, and the varieties determined by such sections are exactly the projective hyperplanes in $\mathbb{CP}^n$. For this reason, the bundle H is called the ***hyperplane bundle*** (which explains our choice of the notation H). //

It turns out that every closed complex hypersurface is cut out by a holomorphic section of a line bundle in this way. If M is a complex manifold and $S \subseteq M$ is a closed complex hypersurface, let us say an open cover $\{U_\alpha\}_{\alpha \in A}$ of M together with holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$ is a ***system of local defining functions for S*** if each function f_α vanishes simply on $U_\alpha \cap S$ and nowhere else. Every closed complex hypersurface has such a system: Corollary 2.13 shows that there is a local defining function in a neighborhood of each point of S, and because S is closed we can obtain an open cover of M by adding in the open set $U_0 = M \smallsetminus S$ with $f_0 \equiv 1$.

The next lemma provides a weak sort of uniqueness for local defining functions.

Lemma 3.38. *Suppose M is a complex manifold and $S \subseteq M$ is a complex hypersurface. If $U \subseteq M$ is open and $f_1, f_2 : U \to \mathbb{C}$ are holomorphic functions that vanish simply on $U \cap S$ and nowhere else, then there is a nonvanishing holomorphic function $h : U \to \mathbb{C}$ such that $f_2 = hf_1$.*

Proof. Given f_1, f_2 as in the statement of the lemma, for each point $p \in U \cap S$ (if there are any), we can choose local holomorphic slice coordinates $(z^1, \dots, z^n)$ on a neighborhood W of p such that $S \cap W$ is the set where $z^n = 0$. Since $f_1(z^1, \dots, z^{n-1}, 0) \equiv 0$, the Taylor series of f_1 centered at any point of $W \cap S$ has no terms that are of order zero in the variable z^n, and we can factor out z^n to write $f_1(z) = z^n g_1(z)$ for some holomorphic function g_1. The fact that f_1 vanishes simply on $U \cap S$ implies that df_1 is nonvanishing at points where $z^n = 0$, so we must have $g_1 \neq 0$ on on $W \cap S$. Similarly, $f_2(z) = z^n g_2(z)$ with g_2 nonzero on $W \cap S$. It follows that everywhere in $W \smallsetminus S$,

$$\frac{f_2(z)}{f_1(z)} = \frac{g_2(z)}{g_1(z)},$$

which extends to a nonvanishing holomorphic function on all of W. Since the complement of S is dense in U, the extension is unique by continuity. Thus f_2/f_1 extends uniquely to a nonvanishing holomorphic function in a neighborhood of each point of U, and by uniqueness the extensions all fit together to determine a nowhere-vanishing holomorphic function $h : U \to \mathbb{C}$ satisfying $f_2 = hf_1$. □

Theorem 3.39 (Line Bundle Associated with a Hypersurface). *Suppose M is a complex manifold and $S \subseteq M$ is a closed complex hypersurface. Then there exist a holomorphic line bundle $L_S \to M$, called the **associated line bundle for S**, and a holomorphic section $\sigma \in \mathcal{O}(M; L_S)$ that vanishes simply on S and nowhere else. Any two line bundles that admit holomorphic sections vanishing simply only on S are isomorphic.*

Proof. Let $\{(U_\alpha, f_\alpha)\}_{\alpha \in A}$ be a system of local defining functions for S. Whenever $U_\alpha \cap U_\beta \neq \varnothing$, Lemma 3.38 shows that that there is a nowhere-vanishing holomorphic function $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(1, \mathbb{C})$ such that $\tau_{\alpha\beta} = f_\alpha / f_\beta$ on the complement of S.

When $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$, on the complement of S we have

$$\tau_{\alpha\beta}\tau_{\beta\gamma} = \frac{f_\alpha}{f_\beta}\frac{f_\beta}{f_\gamma} = \tau_{\alpha\gamma},$$

and the same holds by continuity on all of $U_\alpha \cap U_\beta \cap U_\gamma$. Thus we have the data to apply the bundle construction theorem and obtain a holomorphic line bundle $L_S \to M$, with a trivialization over each set U_α and transition functions $\tau_{\alpha\beta}$. The associated local frames $s_\alpha : U_\alpha \to L_S$ satisfy (3.4).

We define a holomorphic section $\sigma : M \to L_S$ by setting

$$\sigma(p) = f_\alpha(p)s_\alpha(p) \text{ for } p \in U_\alpha.$$

Equation (3.4) guarantees that these definitions agree where they overlap, and it follows immediately that S is the variety determined by σ, and σ vanishes simply on S.

To show that any two such bundles are isomorphic, suppose $L' \to M$ is another holomorphic line bundle and $\sigma' : M \to L'$ is a holomorphic section that vanishes simply on S and nowhere else. We can choose an open cover $\{U_\alpha\}_{\alpha\in A}$ that is a trivializing cover for both bundles. Let $s_\alpha : U_\alpha \to L_S$ and $s'_\alpha : U_\alpha \to L'$ be holomorphic local frames, and let $\tau_{\alpha\beta}$ and $\tau'_{\alpha\beta}$ be the associated transition functions so that

$$s_\beta = \tau_{\alpha\beta}s_\alpha, \qquad s'_\beta = \tau'_{\alpha\beta}s'_\alpha. \tag{3.16}$$

In each open set U_α, we can write $\sigma = f_\alpha s_\alpha$ and $\sigma' = f'_\alpha s'_\alpha$ for some holomorphic functions f_α, f'_α that vanish simply on S. Lemma 3.38 shows that $\psi_\alpha = f'_\alpha / f_\alpha$ extends to a nonvanishing holomorphic function on all of U_α.

On $U_\alpha \cap U_\beta$, we have

$$f_\alpha s_\alpha = \sigma = f_\beta s_\beta = f_\beta \tau_{\alpha\beta} s_\alpha,$$

which implies $f_\alpha = f_\beta \tau_{\alpha\beta}$, and similarly $f'_\alpha = f'_\beta \tau'_{\alpha\beta}$. Therefore, on $(U_\alpha \cap U_\beta) \setminus S$, we have

$$\tau_{\alpha\beta} = \frac{(f'_\alpha)(f_\beta\tau_{\alpha\beta})}{f'_\alpha f_\beta} = \frac{(f'_\beta\tau'_{\alpha\beta})(f_\alpha)}{f'_\alpha f_\beta} = \psi_\alpha^{-1}\tau'_{\alpha\beta}\psi_\beta,$$

and the same formula holds on all of $U_\alpha \cap U_\beta$ by continuity. Thus $L' \cong L_S$ by the isomorphism criterion (Prop. 3.7). □

Example 3.40 (Line Bundles Associated with Algebraic Hypersurfaces). Suppose $S \subseteq \mathbb{CP}^n$ is a nonsingular projective algebraic hypersurface defined by a single homogeneous polynomial of degree d. Example 3.37 showed there is a global holomorphic section of H^d that vanishes simply on S and nowhere else; thus by uniqueness, $L_S \cong H^d$. //

Line Bundles and Divisors on Riemann Surfaces

The association between line bundles and hypersurfaces is particularly important for Riemann surfaces (complex 1-manifolds). If M is a compact Riemann surface, a closed hypersurface in M is just a finite collection of points. A bundle $L_{\{p\}}$ associated with a single point is called a ***point bundle***.

As we have seen, some holomorphic line bundles on Riemann surfaces (such as the tautological bundle on $\mathbb{CP}^1$) have no nontrivial holomorphic sections at all. But we can get more information about the relationship between line bundles and hypersurfaces in this case by by expanding the type of sections we consider.

Let M be a Riemann surface. A ***meromorphic function*** on M is a holomorphic function $f : M \smallsetminus P \to \mathbb{C}$, where P is a closed discrete subset of M, such that for each $p \in P$, the function $1/f$ extends to a holomorphic function on a neighborhood of p that vanishes at p. Each $p \in P$ is called a ***pole of f***, and the order of the zero of $1/f$ at p is called the ***order of the pole***. To put it another way, for each $p \in M$, if z is a holomorphic coordinate centered at p, we can write f in a punctured neighborhood of p in the form $f(z) = z^k h(z)$ for some integer k and some nonvanishing holomorphic function h defined on a neighborhood of p; if $k > 0$, then p is a zero of order k, and if $k < 0$, then p is a pole of order $|k|$.

Similarly, if $L \to M$ is a holomorphic line bundle, a ***meromorphic section of L*** is a holomorphic section defined on the complement of a closed discrete subset $P \subseteq M$ such that for each $p \in P$, if s is a holomorphic local frame for L on a neighborhood of p, then $\sigma(z) = f(z)s(z)$ for some function f that has a pole at p. It is an easy exercise to check that in both cases, the order of a pole is independent of the choice of coordinates or local frame.

We will be primarily concerned with the case in which M is compact. In that case, the set of poles of a meromorphic function or section is finite, as is the set of zeros.

There is a simple algebraic construction that can be used to keep track of the locations and orders of zeros and poles of meromorphic functions or sections. For a compact Riemann surface M, a ***divisor*** on M is a finite formal linear combination of points of M with integer coefficients (that is, an element of the free abelian group on the set of points of M; see [**LeeTM**, p. 244]). A divisor $D = \sum_j n_j p_j$ is said to be ***effective*** if each of the integers n_j is nonnegative. The set of divisors on M forms an abelian group under addition, the free abelian group on the points of M. It is denoted by $\operatorname{Div}(M)$.

If f is a meromorphic function on M, we define the ***divisor of f***, denoted by (f), to be the sum

$$(f) = \sum_{i=1}^{k} n_i p_i - \sum_{j=1}^{l} m_j q_j,$$

where $p_1, \dots, p_k$ are the zeros of f, $q_1, \dots, q_l$ are its poles, and the integers n_i and m_j are their respective orders. A divisor of the form $D = (f)$ for some meromorphic function f is called a ***principal divisor***. The divisor of a product of meromorphic functions is the sum of the divisors of the factors: $(fg) = (f) + (g)$, with the understanding that a product like $(z-a)^k(z-a)^l$ results in a removable singularity at $z = a$ if $k + l \geq 0$. Therefore, the set of principal divisors forms a subgroup of $\operatorname{Div}(M)$. Two divisors are said to be ***linearly equivalent*** if their difference is a principal divisor. The quotient of the group of divisors modulo principal divisors is called the ***divisor class group of M***, and denoted by $\operatorname{Cl}(M)$.

The divisor of a meromorphic section σ of a holomorphic line bundle is defined similarly and denoted by (σ). Such a divisor is effective if and only if σ is a holomorphic section.

Here is a generalization of Theorem 3.39.

Theorem 3.41 (Line Bundle Associated with a Divisor). *Suppose M is a compact Riemann surface. For each divisor $D \in \operatorname{Div}(M)$, there exist a holomorphic line bundle $L_D \to M$, called the **associated line bundle for D**, and a meromorphic section of L_D, unique up to a constant multiple, whose divisor is equal to D. The bundle is unique in the sense that any other holomorphic line bundle that admits a meromorphic section with divisor equal to D is isomorphic to L_D. The map sending D to the isomorphism class of L_D is a homomorphism from $\operatorname{Div}(M)$ to $\operatorname{Pic}(M)$, whose kernel is the group of principal divisors; thus it descends to an injective homomorphism from $\operatorname{Cl}(M)$ to $\operatorname{Pic}(M)$.*

Proof. Given $D \in \operatorname{Div}(M)$, write $D = \sum_{\alpha=1}^{k} n_\alpha p_\alpha$. For each α such that $n_\alpha \neq 0$, choose a neighborhood U_α of p_α and a meromorphic function f_α on U_α that is holomorphic and nonvanishing on $U_\alpha \smallsetminus \{p_\alpha\}$ and has a zero of order n_α if $n_\alpha > 0$ and a pole of order $|n_\alpha|$ if $n_\alpha < 0$. Let $U_0 = M \smallsetminus \{p_1, \dots, p_k\}$, and $f_0 \equiv 1$. Whenever $U_\alpha \cap U_\beta \neq \varnothing$, there is a nowhere-vanishing holomorphic function $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(1, \mathbb{C})$ such that $\tau_{\alpha\beta} = f_\alpha / f_\beta$ on the complement of $\{p_\alpha, p_\beta\}$. As in the proof of Theorem 3.39, when $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$, we have $\tau_{\alpha\beta}\tau_{\beta\gamma} = \tau_{\alpha\gamma}$. The bundle construction theorem yields a holomorphic line bundle $L_D \to M$, with a holomorphic local frame s_α on each set U_α and transition functions $\tau_{\alpha\beta}$. The meromorphic section σ is defined as before by setting $\sigma = f_\alpha s_\alpha$ on U_α, and noting that these sections agree on overlaps. It follows immediately from the definition that $(\sigma) = D$. If τ is another meromorphic section with the same divisor, then the ratio σ/τ has only removable singularities; thus it can be considered as a globally defined holomorphic function, and is therefore constant.

If $L' \to M$ is another holomorphic line bundle that admits a meromorphic section with the same divisor, the same argument as in the proof of Theorem 3.41 shows that $L' \cong L_D$.

To see that the map $D \mapsto [L_D]$ is a homomorphism, suppose D and D' are divisors. Then there are meromorphic sections σ of L_D and σ' of $L_{D'}$ whose divisors are D and D', respectively. Then $\sigma \otimes \sigma'$ is a section of $L_D \otimes L_{D'}$ whose divisor is $D + D'$, so the uniqueness argument above shows that $L_D \otimes L_{D'} \cong L_{D+D'}$.

To verify the statement about the kernel of this homorphism, first suppose that D is in the kernel. That means L_D is a trivial bundle, and it has a meromorphic section σ whose divisor is D. Since a meromorphic section of the trivial bundle is just a meromorphic function, this shows that D is principal. Conversely, if D is principal, there is a global meromorphic function f such that $D = (f)$. Since a meromorphic function is the same as a meromorphic section of the trivial bundle, the uniqueness statement in the first part of the proof shows that L_D is isomorphic to the trivial bundle. □

We will show in Chapter 9 (see Thm. 9.61) that in fact the homomorphism from $\mathrm{Div}(M)$ to $\mathrm{Pic}(M)$ is surjective, so it induces an isomorphism between the divisor class group $\mathrm{Cl}(M)$ and $\mathrm{Pic}(M)$.

We should note that there is also a theory of divisors on higher-dimensional complex manifolds; but because zeros and poles of meromorphic functions are no longer isolated in that case, the theory is somewhat more complicated technically, and we will not pursue it.

Line Bundles and Projective Embeddings

Our next major application of holomorphic line bundles will be as a tool for constructing embeddings into projective spaces.

Suppose M is a compact complex manifold and $L \to M$ is a holomorphic line bundle. As Theorem 3.13 showed, the space $\mathcal{O}(M; L)$ of global holomorphic sections of L is finite-dimensional. A point $p \in M$ is called a ***base point for L*** if every global holomorphic section of L vanishes at p. The set of base points is denoted by $B(L)$ and called the ***base locus of L***.

If L has at least one nontrivial holomorphic section, by choosing a basis $(s_0, \dots, s_m)$ for $\mathcal{O}(M; L)$ we can define a map from $M \smallsetminus B(L)$ to $\mathbb{CP}^m$ as follows. Given $p_0 \in M \smallsetminus B(L)$, choose a local frame s for L in a neighborhood U of p_0, express each section s_j locally as $s_j = f_j s$ for some holomorphic function $f_j \in \mathcal{O}(U)$, and map each $p \in U$ to the point $[f_0(p), \dots, f_m(p)] \in \mathbb{CP}^n$. To see that this does not depend on the choice of local frame, let $\tilde{s}$ be any other local frame for L, so there is a nonvanishing transition function τ satisfying $\tilde{s} = \tau s$ in a neighborhood of p_0. If we write $s_j = \widetilde{f}_j \tilde{s}$, then $\widetilde{f}_j \tau = f_j$, and therefore $[f_0(p), \dots, f_m(p)] = [\tau(p)\widetilde{f}_0(p), \dots, \tau(p)\widetilde{f}_m(p)] = [\widetilde{f}_0(p), \dots, \widetilde{f}_m(p)]$. Thus it makes sense to introduce the notation $[s_0(p), \dots, s_m(p)]$ to denote the point $[f_0(p), \dots, f_m(p)]$ with respect to any local frame, and define a holomorphic map $F : M \smallsetminus B(L) \to \mathbb{CP}^m$ by $F(p) = [s_0(p), \dots, s_m(p)]$. Any two bases for $\mathcal{O}(M; L)$ differ by a complex-linear isomorphism, and therefore the corresponding maps differ by a projective transformation. Any one of these maps is called an ***associated map*** for L.

There are simple criteria for deciding when a map associated to a line bundle $L \to M$ is a global embedding, based on the following lemma.

Lemma 3.42. *Suppose M is a compact complex manifold and $L \to M$ is a holomorphic line bundle that admits at least one nontrivial global holomorphic section. Let $F : M \smallsetminus B(L) \to \mathbb{CP}^m$ be an associated map.*

(a) *Given distinct points $p, q \in M \smallsetminus B(L)$, $F(p) \neq F(q)$ if and only if there exists a holomorphic section σ of L such that $\sigma(p) = 0$ and $\sigma(q) \neq 0$.*

(b) *Given $p \in M \smallsetminus B(L)$ and $v \in T'_pM$, $D'F(p)v \neq 0$ if and only if there is a holomorphic section σ of L such that $\sigma(p) = 0$ and $v\widehat{\sigma}(p) \neq 0$, where $\widehat{\sigma}$ is the component function of σ with respect to some holomorphic local frame.*

Proof. If M is a one-point space, then both claims are vacuously true, so assume henceforth that M contains at least two points. Choose a basis $(s_0, \dots, s_m)$ for $\mathcal{O}(M; L)$ and write the associated map as $F(p) = [s_0(p), \dots, s_m(p)]$.

To prove (a), let $p, q \in M \smallsetminus B(L)$ be arbitrary distinct points, and suppose first that there exists a section $\sigma = \sum_{j=0}^m a_j s_j \in \mathcal{O}(M; L)$ with $\sigma(p) = 0$ and $\sigma(q) \neq 0$. It follows that $F(p)$ lies in the projective hyperplane defined by the linear function $f(w) = \sum_{j=0}^m a_j w^j$ but $F(q)$ does not, so $F(p) \neq F(q)$.

Conversely, suppose there is no section σ satisfying $\sigma(p) = 0$ and $\sigma(q) \neq 0$. Since p is not a base point, there is some j such that $s_j(p) \neq 0$; after rearranging the basis (which just changes F by a projective transformation), we may assume $s_0(p) \neq 0$. For each k, let $\sigma_k = s_k - (s_k(p)/s_0(p))s_0$, where the notation $s_k(p)/s_0(p)$ means the complex number b_k such that $s_k(p) = b_k s_0(p)$. Then $\sigma_k(p) = 0$ for each k, and our hypothesis implies that $\sigma_k(q) = 0$ as well. This means

$$s_k(q) = \left(\frac{s_k(p)}{s_0(p)}\right) s_0(q) = \lambda s_k(p) \text{ for each } k, \text{ where } \lambda = \frac{s_0(q)}{s_0(p)}.$$

Thus $[s_0(q), \dots, s_m(q)] = [\lambda s_0(p), \dots, \lambda s_m(p)] = [s_0(p), \dots, s_m(p)]$, which means $F(q) = F(p)$.

To prove (b), let $p \in M \smallsetminus B(L)$ and $v \in T'_pM$ be arbitrary. As above, we can arrange that $s_0(p) \neq 0$. We use s_0 as a local frame for L in a neighborhood of p, and in that neighborhood write $s_j = f_j s_0$ for some holomorphic functions $f_0, \dots, f_m$ with $f_0 \equiv 1$. Choose any holomorphic coordinates (u^k) for M on a neighborhood of p, and use affine coordinates $(z^1, \dots, z^m) \leftrightarrow [1, z^1, \dots, z^m]$ on a neighborhood of $F(p)$. Then F has the coordinate representation $\widehat{F}(u) = (f_1(u), \dots, f_n(u))$, and for any $v = \sum_k v^k \partial/\partial u^k \in T'_p M$ we have

$$D' F(p)v = \sum_{j,k} \frac{\partial f_j}{\partial u^k}(p) v^k \left.\frac{\partial}{\partial z^j}\right|_{F(p)} = \sum_j v(f_j) \left.\frac{\partial}{\partial z^j}\right|_{F(p)}. \tag{3.17}$$

Suppose there is a section $\sigma = \sum_{j=0}^m a_j s_j$ such that $\sigma(p) = 0$ and $v(\widehat{\sigma}) \neq 0$. Note that our choice of s_0 as a local frame implies

$$\sigma = \sum_{j=0}^m a_j s_j = \Big(\sum_{j=0}^m a_j f_j\Big) s_0,$$

so the component function of σ is $\widehat{\sigma} = \sum_{j=0}^m a_j f_j$. The fact that $v(\widehat{\sigma}) \neq 0$ implies that $v(f_j) \neq 0$ for some $j \geq 1$, and therefore (3.17) shows that $D' F(p)v \neq 0$.

Conversely, assume $D' F(p)v \neq 0$. Then (3.17) implies that

$$v(f_j) = v^k \frac{\partial f^j}{\partial u^k}(p) \neq 0 \text{ for some } j,$$

which proves that the condition of (b) is satisfied with $\sigma = s_j$. □

Given a holomorphic line bundle L over a complex manifold M, we say $\mathcal{O}(M; L)$ ***separates points*** if for every pair of distinct points $p, q \in M$, there exists a global holomorphic section σ of L such that $\sigma(p) = 0$ and $\sigma(q) \neq 0$; and it ***separates directions*** if for every $p \in M$ and every nonzero $v \in T'_p M$, there is a global holomorphic section σ such that $\sigma(p) = 0$ and $v\widehat{\sigma}(p) \neq 0$, where $\widehat{\sigma}$ is the component function of σ with respect to some holomorphic local frame. (The reason for insisting that $\sigma(p) = 0$ is because that guarantees the condition $v\widehat{\sigma}(p) = 0$ will be independent of the choice of local frame, as you can check.)

A holomorphic line bundle L over a compact complex manifold M is said to be ***very ample*** if

(i) $\mathcal{O}(M; L)$ separates points, and

(ii) $\mathcal{O}(M; L)$ separates directions.

(You will notice the strong resemblance to the definition of a Stein manifold. There is no requirement analogous to holomorphic convexity in this case, though, because it would be vacuous on a compact manifold. Compactness takes its place.)

Theorem 3.43. *Suppose M is a compact complex manifold and $L \to M$ is a holomorphic line bundle. Each associated map for L is a global embedding of M into a projective space if and only L is very ample.*

Proof. First assume M has dimension 0. Then L is trivial, and we can write $M = \{p_0, \dots, p_k\}$. For each k there is a section s_k that is nonzero at p_k and zero at the other points, and is vacuously holomorphic. Thus L separates points, and it vacuously separates directions because there are no nonzero tangent vectors. Therefore, every line bundle is very ample, and every associated map is an embedding.

Assume henceforth that M has positive dimension. Suppose $L \to M$ is a very ample holomorphic line bundle. First we note that L has no base points. Let $q \in M$ be arbitrary. Then for any other point $p \neq q$, there is a holomorphic section $\sigma : M \to L$ such that $\sigma(p) = 0$ and $\sigma(q) \neq 0$, so not all sections vanish at q. Thus the associated map F is defined on all of M. It follows from the definition of *very ample* and Lemma 3.42 that F is an injective holomorphic immersion into $\mathbb{CP}^n$, and the closed map lemma [**LeeTM**, Lemma 4.50] shows that F is an embedding.

Conversely, suppose the associated map is a global holomorphic embedding $F : M \to \mathbb{CP}^n$. The fact that F is globally defined means that L has no base points. Because F is an injective holomorphic immersion, Lemma 3.42 shows that $\mathscr{O}(M; L)$ separates points and directions, so L is very ample. □

Somewhat more generally, a holomorphic line bundle L is said to be ***ample*** if for some positive integer k, the tensor power L^k is very ample. The reason for this seemingly trivial distinction is that, as we will see in Chapter 10, there is a relatively simple criterion for determining whether a particular bundle is ample, whereas determining if it is very ample is not so straightforward.

Corollary 3.44. *A compact complex manifold is projective if and only if it admits an ample holomorphic line bundle.*

Proof. A compact 0-dimensional manifold certainly admits an embedding into projective space, and the only holomorphic line bundle on it is the trivial one, which is very ample by the argument in the proof of Theorem 3.43. So assume henceforth that M is a compact complex manifold of positive dimension.

If M admits an ample line bundle L, then Theorem 3.43 shows that the associated map for some positive power of L is a holomorphic embedding.

Conversely, suppose M is projective, so it is biholomorphic to a complex submanifold of $\mathbb{CP}^n$ for some n; we may as well assume M is itself a compact complex submanifold of $\mathbb{CP}^n$. Let $[w^0, \dots, w^n]$ denote homogeneous coordinates on $\mathbb{CP}^n$, and let $L \to M$ be the restriction to M of the hyperplane bundle $H \to \mathbb{CP}^n$. We will show that L is very ample.

To show that $\mathscr{O}(M; L)$ separates points, suppose p, q are distinct points in M, represented in homogeneous coordinates by $[p^0, \dots, p^n]$ and $[q^0, \dots, q^n]$, respectively. Because the vectors $P = (p^0, \dots, p^n)$ and $Q = (q^0, \dots, q^n)$ are linearly independent in $\mathbb{C}^{n+1}$, there is a linear function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ such that $f(P) = 0$ and $f(Q) \neq 0$. Let $\sigma = \varphi_f|_M$ (using the notation of (3.15)); this is a holomorphic section of L that satisfies $\sigma(p) = 0$ and $\sigma(q) \neq 0$.

To show that $\mathscr{O}(M; L)$ separates directions, let $p \in M$ and $v \in T'_p M$ with $v \neq 0$. There is some affine coordinate chart containing p; after renumbering the coordinates if necessary, we may assume it is the chart U_0 defined by $w^0 \neq 0$, and write $p = [1, p^1, \dots, p^n]$. In affine coordinates $(z^1, \dots, z^n) \leftrightarrow [1, z^1, \dots, z^n]$, we can write $v = v^j \partial/\partial z^j|_p$, and since $v \neq 0$, there is some component v^k that is nonzero. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be the linear function $f(w^0, \dots, w^n) = w^k - p^k w^0$, let $\varphi_f : \mathbb{CP}^n \to H$ be the corresponding section, and let $\sigma = \varphi_f|_M$. Then $\sigma(p) = 0$. The section $\varphi_{w^0} : \mathbb{CP}^n \to H$ is nonvanishing on U_0, so it gives a local frame for H there, and it restricts to a local frame for L. The component function of σ with respect to this frame is

$$\widehat{\sigma}([w]) = \frac{w^k - p^k w^0}{w^0},$$

which has the coordinate representation

$$\widehat{\sigma}(z^1, \dots, z^n) = z^k - p^k,$$

so $v\widehat{\sigma} = v^k \neq 0$. □

The next proposition describes an important property of very ample line bundles.

Proposition 3.45. *Suppose M is a compact complex manifold, $L \to M$ is a very ample holomorphic line bundle, and $F : M \to \mathbb{CP}^n$ is its associated map. Then $L \cong F^*H$, where $H \to \mathbb{CP}^n$ is the hyperplane bundle.*

Proof. Problem 3-13. □

Problems

3-1. Let $H \to \mathbb{CP}^n$ be the hyperplane bundle. For any distinct integers k, l, show that the tensor powers H^k and H^l are not isomorphic to each other.

3-2. Let $m > n$ and let $F : \mathbb{CP}^n \to \mathbb{CP}^m$ be the holomorphic embedding

$$F([w^0, \dots, w^n]) = [w^0, \dots, w^n, 0, \dots, 0].$$

Let $H \to \mathbb{CP}^m$ denote the hyperplane bundle of $\mathbb{CP}^m$. Show that F^*H is isomorphic to the hyperplane bundle of $\mathbb{CP}^n$.

3-3. Let $H \to \mathbb{CP}^1$ be the hyperplane bundle. Show that $H^2 \cong T'\mathbb{CP}^1$.

3-4. Let M be a complex manifold. A ***holomorphic vector field*** on M is a holomorphic section of $T'M$. Let Z be a smooth section of $T'M$ and let θ_t denote the flow of $\operatorname{Re} Z$. Show that Z is holomorphic if and only if θ_t is a holomorphic map (where it is defined) for each t.

3-5. For $n \geq 1$, let $U_0 \subseteq \mathbb{CP}^n$ be the image of the standard affine embedding $(z^1, \dots, z^n) \mapsto [1, z^1, \dots, z^n]$. Show that there are global holomorphic vector fields $Z_1, \dots, Z_n$ on $\mathbb{CP}^n$ whose coordinate representations in U_0 are $Z_j = \partial/\partial z^j$.

3-6. Suppose M is a complex manifold. Let $\pi : \widetilde{M} \to M$ be the blowup of M at a point $p \in M$, and let $E = \pi^{-1}(p) \subseteq \widetilde{M}$ be the exceptional hypersurface.

(a) Show that if N is another complex manifold and $F : \widetilde{M} \to N$ is a holomorphic map that is constant on E, then there is a unique holomorphic map $f : M \to N$ such that $f \circ \pi = F$.

(b) Show that the pullback $\pi^* : \mathcal{O}(M) \to \mathcal{O}(\widetilde{M})$ is an isomorphism.

3-7. Suppose M and N are complex manifolds and $F : M \to N$ is a holomorphic map. Show that the global holomorphic Jacobian map $D' : T'M \to T'N$ is holomorphic.

3-8. Suppose G is a connected, compact, complex Lie group. Prove that G is abelian, as follows. Let $\mathfrak{g}$ be the Lie algebra of left-invariant vector fields on G, and let $e \in G$ be the identity.

(a) Let $C : G \times G \to G$ be the holomorphic map $C(g,h) = ghg^{-1}$. For each $g \in G$, the map $C_g : G \to G$ defined by $C_g(h) = C(g,h)$ is called ***conjugation by g***. Show that $D(C_g)(e)v = DC(g,e)(0,v)$ for each $v \in T'G$.

(b) Show that the map from G to $\mathrm{GL}(T'_eG)$ given by $g \mapsto D'(C_g)(e)$ is holomorphic, and conclude that $D'(C_g)(e) = \mathrm{Id}$ for all g.

(c) The ***adjoint representation of G*** is the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ given by $\mathrm{Ad}(g) = (C_g)_*$ (the Lie algebra isomorphism induced by the group isomorphism C_g). Show that $\mathrm{Ad}(g) = \mathrm{Id}$ for all $g \in G$.

(d) The ***adjoint representation of $\mathfrak{g}$*** is the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by $\mathrm{ad}(X)(Y) = [X,Y]$. Using the fact that $\mathrm{Ad}_* = \mathrm{ad}$ [**LeeSM**, Thm. 20.27], show that $\mathfrak{g}$ is abelian and therefore G is abelian.

3-9. Let $T \to \mathbb{CP}^n$ be the tautological bundle. Show that the total space T is diffeomorphic to $\mathbb{CP}^{n+1}$ minus a point, by considering the smooth map $F : T \to \mathbb{CP}^{n+1}$ given by

$$F([w],v) = \left[w^0, \dots, w^n, w \cdot \bar{v}\right],$$

where $w \cdot \bar{v} = \sum_{j=0}^{n} w^j \bar{v}^j$.

3-10. If M and N are connected smooth oriented n-manifolds, their ***oriented connected sum*** is a manifold $M \# N$ obtained as follows. Choose a positively oriented smooth coordinate chart (U,φ) for M and a negatively oriented one (V,ψ) for N, and choose $\varepsilon > 0$ such that both $\varphi(U)$ and $\psi(V)$ contain $B_{2\varepsilon}(0)$. Let $M' = M \smallsetminus \varphi^{-1}(\overline{B}_{\varepsilon/2}(0))$, $N' = N \smallsetminus \psi^{-1}(\overline{B}_{\varepsilon/2}(0))$, and let $M \# N$ be the quotient space obtained from the disjoint union $M' \amalg N'$ by identifying $\varphi^{-1}(x)$ with $\psi^{-1}(\varepsilon^2 x/|x|^2)$ for all $x \in B_{2\varepsilon}(0) \smallsetminus \overline{B}_{\varepsilon/2}(0)$. Then $M \# N$ is a connected n-manifold with a unique smooth structure determined by the requirement that the quotient map restricts to smooth embeddings of M' and N' into $M \# N$, and an orientation consistent with those of M' and N' (you do not need to prove this).

Suppose M is a connected complex n-manifold and $\widetilde{M}$ is the blowup of M at a point. Use the result of Problem 3-9 to show that $\widetilde{M}$ is diffeomorphic to the oriented connected sum $M \# \overline{\mathbb{CP}^n}$, where $\overline{\mathbb{CP}^n}$ denotes the smooth manifold $\mathbb{CP}^n$ with the opposite orientation.

3-11. Let M be a complex manifold. Suppose $S_1, \dots, S_m \subseteq M$ are disjoint closed complex hypersurfaces and $S = S_1 \cup \cdots \cup S_m$. Show that the line bundle L_S associated with S is isomorphic to $L_{S_1} \otimes \cdots \otimes L_{S_m}$.

3-12. Suppose $\widetilde{M}$ and M are complex manifolds, $S \subseteq M$ is a closed complex hypersurface, and $f : \widetilde{M} \to M$ is a holomorphic map that is transverse to S, so that $\widetilde{S} = f^{-1}(S)$ is a closed complex hypersurface in $\widetilde{M}$ (see Example 2.23). Show that $f^*L_S \cong L_{\widetilde{S}}$. [Hint: Consider a holomorphic section of L_S that vanishes simply on S.]

3-13. Prove Proposition 3.45 (a very ample line bundle is isomorphic to the pullback of the hyperplane bundle).

3-14. Show that the action of $\mathbb{Z}/2$ on $\mathbb{C}^n$ generated by $z \mapsto -z$ lifts to a free, proper, and holomorphic action on the blowup of $\mathbb{C}^n$ at the origin.

10.1090/gsm/244/04

Chapter 4

The Dolbeault Complex

For a complex manifold M, the decomposition of $T_{\mathbb{C}}M$ into $T'M \oplus T''M$ leads to a related decomposition of all complex-valued differential forms, from which we can build new biholomorphic invariants called the Dolbeault cohomology groups.

Decomposing Differential Forms by Type

Suppose M is a complex n-manifold. For $0 \leq k \leq n$, we define the ***bundle of complex k-forms***, denoted by $\Lambda^k_{\mathbb{C}}M$, to be the complexification of $\Lambda^k M$. We can view the fiber of $\Lambda^k_{\mathbb{C}}M$ at a point $a \in M$ as the space of alternating complex-multilinear maps from $(T_aM)_{\mathbb{C}}$ to $\mathbb{C}$. Just as for any complexified bundle, every smooth section of $\Lambda^k_{\mathbb{C}}M$ can be written uniquely as a sum $\omega + i\eta$, where ω and η are ordinary smooth real k-forms, and $\Lambda^k_{\mathbb{C}}M$ has a natural conjugation operator given by $\overline{\omega + i\eta} = \omega - i\eta$. A complex differential form ω is said to be ***real*** if $\bar{\omega} = \omega$. The exterior derivative operator d extends immediately by complex linearity to complex differential forms, and it still satisfies the antiderivation rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ when α is a complex k-form and β is a complex l-form. The integral of a complex k-form over a k-dimensional (real) manifold is defined by integrating the real and imaginary parts separately, provided either the manifold or the support of the k-form is compact, and Stokes's theorem holds for such forms.

The complex structure on TM yields another way to decompose complex forms, which is more useful than real and imaginary parts. In the domain of any holomorphic local coordinates $(z^1, \dots, z^n)$, the 1-forms $(dz^1, \dots, dz^n, d\bar{z}^1, \dots, d\bar{z}^n)$ provide a local frame for the complexified cotangent bundle. Thus the following collection of forms constitutes a smooth local frame for $\Lambda^k_{\mathbb{C}}M$:

$$\left\{ dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q} : \right.$$
$$\left. p + q = k,\ j_1 < \cdots < j_p, \text{ and } l_1 < \cdots < l_q \right\}.$$

We wish to separate out the complex differential forms with the property that each term has exactly p of the dz^j factors and q of the $d\bar{z}^l$ factors when it is expressed in terms of holomorphic local coordinates. But to make this definition meaningful, we need to verify that it is independent of coordinates. That follows from the next lemma.

Lemma 4.1. *Suppose α is a complex k-form on a complex manifold, and $p+q=k$. Then the following are equivalent:*

(a) *In every local holomorphic coordinate chart $(z^1,\dots,z^n)$, α can be expressed as a sum of terms, each of which has exactly p of the dz^j factors and q of the $d\bar{z}^l$ factors.*

(b) *If $V_1,\dots,V_k$ are complex vector fields on an open subset of M, then $\alpha(V_1,\dots,V_k)=0$ if more than p of the vector fields are sections of $T'M$, or more than q of them are sections of $T''M$.*

► **Exercise 4.2.** Prove this lemma.

Thus for $p+q=k$, we say a complex k-form is of ***type*** (p,q) or ***bidegree*** (p,q), or is a (p,q)***-form*** for short, if it satisfies either of the equivalent conditions in the preceding lemma. Let $\Lambda^{p,q}M \subseteq \Lambda^k_{\mathbb{C}}M$ be the subset consisting of forms of type (p,q). Lemma 4.1 shows that $\Lambda^{p,q}M$ is locally spanned by smooth sections of $\Lambda^k_{\mathbb{C}}M$, so it is a smooth subbundle. Moreover, because every complex k-form is a sum of forms of type (p,q) for $p+q=k$, and $\Lambda^{p,q}M \cap \Lambda^{p',q'}M$ contains only the zero form unless $p=p'$ and $q=q'$, we have a Whitney sum decomposition

$$\Lambda^k_{\mathbb{C}}M = \bigoplus_{p+q=k} \Lambda^{p,q}M.$$

Thus for each (p,q) there is a coordinate-independent projection operator

$$\pi^{p,q} : \Lambda^k_{\mathbb{C}}M \to \Lambda^{p,q}M.$$

We use the notation $\mathscr{E}^k(M)$ to denote the space of smooth sections of $\Lambda^k_{\mathbb{C}}M$, and $\mathscr{E}^{p,q}(M)$ for the space of smooth sections of $\Lambda^{p,q}M$. By convention, $\mathscr{E}^{p,q}(M)=0$ on a complex n-manifold except for $0 \le p,q \le n$, and similarly $\mathscr{E}^k(M)=0$ except for $0 \le k \le 2n$.

Proposition 4.3. *Let M be a complex manifold, $\alpha \in \mathscr{E}^{p,q}(M)$, and $\beta \in \mathscr{E}^{p',q'}(M)$.*

(a) $\bar{\alpha} \in \mathscr{E}^{q,p}(M)$.

(b) $\alpha \wedge \beta \in \mathscr{E}^{p+p',q+q'}(M)$.

► **Exercise 4.4.** Prove this proposition.

The real usefulness of the decomposition of forms by types is based on the following proposition.

Proposition 4.5. *If M is a complex manifold, then*

$$d(\mathscr{E}^{p,q}(M)) \subseteq \mathscr{E}^{p+1,q}(M) \oplus \mathscr{E}^{p,q+1}(M).$$

Proof. Suppose $\alpha \in \mathscr{E}^{p,q}(M)$. This is a local question, so we may choose holomorphic local coordinates $(z^1, \dots, z^n)$ and write

$$\alpha = \sum_{J,L}{}' \alpha_{JL}\, dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q},$$

where $J = (j_1, \dots, j_p)$ and $L = (l_1, \dots, l_q)$ are multi-indices, and the primed summation sign denotes a sum over only increasing multi-indices, that is, those that satisfy $j_1 < \cdots < j_p$ and $l_1 < \cdots < l_q$. It follows that

$$d\alpha = \sum_{J,L}{}' \sum_r \left(\frac{\partial \alpha_{JL}}{\partial z^r} dz^r + \frac{\partial \alpha_{JL}}{\partial \bar{z}^r} d\bar{z}^r \right) \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q},$$

which expands to a sum of $(p+1, q)$-forms and $(p, q+1)$-forms. □

Thanks to this proposition, we can make the following definitions. We continue to let M be a complex n-manifold. For each $p, q \in \{0, \dots, n\}$, define the ***Dolbeault operator*** $\bar{\partial}\colon \mathscr{E}^{p,q}(M) \to \mathscr{E}^{p,q+1}(M)$ and its conjugate $\partial\colon \mathscr{E}^{p,q}(M) \to \mathscr{E}^{p+1,q}(M)$ by

$$\bar{\partial} = \pi^{p,q+1} \circ d, \qquad \partial = \pi^{p+1,q} \circ d.$$

More generally, if α is any complex differential form, we define $\partial\alpha$ and $\bar{\partial}\alpha$ by decomposing α into terms of type (p, q) and applying the formulas above to each term separately. These operators, first introduced by the English mathematician William V. D. Hodge [**Hod41**], were later named after Pierre Dolbeault in recognition of his use of them in proving the theorem now known as the *Dolbeault theorem* (Thm. 6.19 below).

Example 4.6 (Dolbeault Operators in Coordinates). If u is a smooth complex-valued function (a $(0,0)$-form), we have the following formulas in holomorphic coordinates (using the summation convention):

$$\partial u = \frac{\partial u}{\partial z^j} dz^j, \qquad \bar{\partial} u = \frac{\partial u}{\partial \bar{z}^j} d\bar{z}^j. \tag{4.1}$$

And for a $(1,0)$-form $\alpha = \alpha_j dz^j$,

$$\partial\alpha = \frac{\partial \alpha_j}{\partial z^l} dz^l \wedge dz^j, \qquad \bar{\partial}\alpha = \frac{\partial \alpha_j}{\partial \bar{z}^l} d\bar{z}^l \wedge dz^j.$$

More generally, if

$$\alpha = \sum_{J,L}{}' \alpha_{JL}\, dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q},$$

then

$$\partial\alpha = {\sum_{J,L}}' \sum_r \frac{\partial\alpha_{JL}}{\partial z^r} dz^r \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q}, \text{ and} \tag{4.2}$$

$$\bar\partial\alpha = {\sum_{J,L}}' \sum_r \frac{\partial\alpha_{JL}}{\partial \bar{z}^r} d\bar{z}^r \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q}, \tag{4.3}$$

as you can check. //

The operator $\bar\partial$ is also sometimes called the ***Cauchy–Riemann operator***, in part because of the following result.

Proposition 4.7. *On a complex manifold, a smooth function f is holomorphic if and only if $\bar\partial f \equiv 0$.*

Proof. This follows immediately from (4.1) and Proposition 1.42(a). □

Proposition 4.8. *On a complex manifold M, if $\alpha \in \mathscr{E}^k(M)$ and $\beta \in \mathscr{E}^l(M)$, then*

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta, \qquad \bar\partial(\alpha \wedge \beta) = \bar\partial\alpha \wedge \beta + (-1)^k \alpha \wedge \bar\partial\beta.$$

Proof. This follows from Propositions 4.5 and 4.3. □

Proposition 4.9. *Let M be a complex manifold. For every complex differential form α on M, the following identities hold:*

$$d\alpha = \partial\alpha + \bar\partial\alpha, \tag{4.4}$$

$$\overline{\partial\alpha} = \bar\partial(\bar\alpha), \tag{4.5}$$

$$\partial\partial\alpha = \bar\partial\bar\partial\alpha = 0, \tag{4.6}$$

$$\partial\bar\partial\alpha = -\bar\partial\partial\alpha. \tag{4.7}$$

Proof. By decomposing α into types and working with each type separately, we see that it suffices to prove these identities under the assumption that α is a (p,q)-form. Equation (4.4) follows directly from Proposition 4.5 and the definition of ∂ and $\bar\partial$, and (4.5) follows from the coordinate formulas (4.2) and (4.3) because dz^j and $d\bar{z}^j$ are conjugates of each other.

For (4.6) and (4.7), note that

$$0 = d(d\alpha) = (\partial + \bar\partial)(\partial + \bar\partial)\alpha = \partial\partial\alpha + (\partial\bar\partial\alpha + \bar\partial\partial\alpha) + \bar\partial\bar\partial\alpha.$$

On the right-hand side, the first term is in $\mathscr{E}^{p+2,q}(M)$, the term in parentheses is in $\mathscr{E}^{p+1,q+1}(M)$, and the last term is in $\mathscr{E}^{p,q+2}(M)$. Since these spaces intersect only in the zero form, each of those three terms must be zero. □

The importance of the Dolbeault operators stems from the fact that they are preserved by holomorphic maps.

Proposition 4.10. *Suppose M and N are complex manifolds and $F : M \to N$ is a holomorphic map. Then for all p and q and all $\alpha \in \mathscr{E}^{p,q}(N)$,*

$$F^*(\mathscr{E}^{p,q}(N)) \subseteq \mathscr{E}^{p,q}(M), \tag{4.8}$$

$$F^*(\partial\alpha) = \partial(F^*\alpha), \tag{4.9}$$

$$F^*(\bar\partial\alpha) = \bar\partial(F^*\alpha). \tag{4.10}$$

Proof. Again, these are local assertions, so for each $p \in M$ we may choose holomorphic coordinates $(z^1, \dots, z^m)$ on a neighborhood of p and $(w^1, \dots, w^n)$ on a neighborhood of $F(p)$, and compute

$$F^* dw^j = \frac{\partial F^j}{\partial z^l} dz^l, \qquad F^* d\overline{w}^j = \frac{\partial \overline{F}^j}{\partial \bar z^l} d\bar z^l.$$

Inserting these into the coordinate formula for $F^*\alpha$ when α is a (p,q)-form shows that the number of dz^j and $d\bar z^j$ factors in each term of $F^*\alpha$ is exactly the same as the number of dw^j and $d\overline{w}^j$ factors, respectively, in the expression for α, so $F^*\alpha$ is also a (p,q)-form. This proves (4.8). It follows from this that $F^* \circ \pi^{p,q} = \pi^{p,q} \circ F^*$, and then (4.9) and (4.10) follow because F^* commutes with d and with the projections $\pi^{p+1,q}$ and $\pi^{p,q+1}$. □

This allows us to define a new set of biholomorphic invariants. First, here are a few definitions that may be familiar from algebraic topology. A ***cochain complex*** is a sequence of abelian groups (or real or complex vector spaces) indexed by the integers, $\{A^q : q \in \mathbb{Z}\}$, together with homomorphisms $d^q : A^q \to A^{q+1}$ for each q, such that the composition of any two successive homomorphisms is zero. (Frequently in practice, the groups are defined only for values of q in a certain range, in which case we just take $A^q = 0$ for other values of q.) We often denote such a complex by A^*, with the homomorphisms understood from the context. The ***qth cohomology group*** of A^*, denoted by $H^q(A^*)$, is the quotient of the kernel of the qth homomorphism by the image of the previous one:

$$H^q(A^*) = \frac{\operatorname{Ker}\left(d^q : A^q \to A^{q+1}\right)}{\operatorname{Im}\left(d^{q-1} : A^{q-1} \to A^q\right)}.$$

If A^* is a cochain complex of vector spaces, then the homology groups are objects in the same category. If A^* and B^* are cochain complexes, a ***cochain map*** $\varphi : A^* \to B^*$ is a collection of homomorphisms $\varphi^q : A^q \to B^q$ that satisfy $\varphi^{q+1} \circ d^q = d^q \circ \varphi^q$ for all q; any such map descends to a homomorphism $\varphi^* : H^q(A^*) \to H^q(B^*)$, called the ***induced cohomology homomorphism***. (For completeness, we remark that a ***chain complex*** is a sequence of abelian groups (or real or complex vector spaces) $A_* = \{A_q : q \in \mathbb{Z}\}$, with homomorphisms $\partial_q : A_q \to A_{q-1}$ going in the direction of decreasing indices and satisfying $\partial_{q-1} \circ \partial_q = 0$. The corresponding quotient groups in that case are called the ***homology groups*** of the chain complex, denoted by $H_q(A_*)$. A ***chain map*** between chain complexes A_* and

B_* is a collection of homomorphisms $\varphi_q : A_q \to B_q$ satisfying $\varphi_{q-1} \circ \partial_q = \partial_q \circ \varphi_q$, which descend to homomorphisms $\varphi_* : H_q(A_*) \to H_q(B_*)$.)

Now let M be a complex n-manifold. Because $\bar\partial \circ \bar\partial \equiv 0$, for each p we obtain a cochain complex called the ***pth Dolbeault complex***:

$$0 \to \mathscr{E}^{p,0}(M) \xrightarrow{\bar\partial} \mathscr{E}^{p,1}(M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{E}^{p,n}(M) \to 0.$$

Then we define the ***Dolbeault cohomology groups*** $H^{p,q}(M)$ as the cohomology groups of this complex, which are the following complex vector spaces:

$$H^{p,q}(M) = \frac{\operatorname{Ker}\left(\bar\partial : \mathscr{E}^{p,q}(M) \to \mathscr{E}^{p,q+1}(M)\right)}{\operatorname{Im}\left(\bar\partial : \mathscr{E}^{p,q-1}(M) \to \mathscr{E}^{p,q}(M)\right)}.$$

They are zero outside of the range $0 \le p, q \le n$. (In principle, these spaces could have been defined using ∂ instead of $\bar\partial$, but $\bar\partial$ is preferred because it characterizes holomorphic functions.)

Theorem 4.11 (Functoriality of Dolbeault Cohomology). *If $F : M \to N$ is a holomorphic map, then for each p and q, the pullback $F^* : \mathscr{E}^{p,q}(N) \to \mathscr{E}^{p,q}(M)$ descends to a linear map, also denoted by F^*, from $H^{p,q}(N)$ to $H^{p,q}(M)$. It satisfies*

(4.11) $\quad (\mathrm{Id}_M)^* = \mathrm{Id} : H^{p,q}(M) \to H^{p,q}(M),$

(4.12) $\quad (G \circ F)^* = F^* \circ G^*$ *if $F : M \to N$ and $G : N \to P$ are holomorphic.*

Proof. The pullback satisfies $F^* \circ \bar\partial = \bar\partial \circ F^*$, so it defines a cochain map and therefore descends to cohomology. Then (4.11) and (4.12) hold on cohomology because they already hold when applied to (p, q)-forms. □

Corollary 4.12 (Biholomorphism Invariance of Dolbeault Cohomology). *The Dolbeault cohomology groups are biholomorphism invariants: If $F : M \to N$ is a biholomorphism, then for all p and q, F^* descends to an isomorphism from $H^{p,q}(N)$ to $H^{p,q}(M)$.*

Proof. This follows from the fact that $(F^{-1})^*$ is an inverse for F^*. □

If M is a complex manifold whose Dolbeault cohomology groups are finite-dimensional (as we will see they always are when M is compact), we define the ***Hodge numbers of M*** to be $h^{p,q}(M) = \dim H^{p,q}(M)$. These are similar to the ***Betti numbers*** of a smooth manifold, $b^k(M) = \dim H^k_{\mathrm{dR}}(M)$, where $H^k_{\mathrm{dR}}(M)$ denotes the kth de Rham cohomology group. But unlike the Betti numbers, which are topological invariants, the Hodge numbers in general depend on the holomorphic structure of the manifold.

A Poincaré Lemma for the Dolbeault Operator

The Dolbeault cohomology groups provide answers to the question "which $\bar\partial$-closed forms are $\bar\partial$-exact," just as the de Rham cohomology groups answer "which d-closed forms are d-exact." An important feature of the de Rham groups is that d-closed forms on smooth manifolds are always *locally* exact. (This is a direct consequence of the *Poincaré lemma* [**LeeSM**, Thm. 17.14], which says that every d-closed form on a star-shaped open subset of $\mathbb{R}^N$ is exact.) Thus the de Rham groups really are reflecting global properties of the manifold. The next theorem shows that an analogous fact is also true for $\bar\partial$-closed forms.

Theorem 4.13 (The $\bar\partial$-Poincaré Lemma). *Suppose M is a complex manifold and ω is a smooth (p,q)-form on M that satisfies $\bar\partial\omega = 0$, with $q \geq 1$. Then in a neighborhood of each point there is a smooth $(p,q-1)$-form η with $\bar\partial\eta = \omega$.*

Before beginning the proof, we need to establish the following analytic lemma.

Lemma 4.14 (The Inhomogeneous Cauchy–Riemann Equations). *Suppose U is an open subset of $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ is smooth. For each point $a \in U$ and each $k \in \{1,\dots,n\}$, there is a neighborhood of a on which there exists a smooth solution g to the equation*

$$\frac{\partial g}{\partial \bar z^k} = f. \tag{4.13}$$

If f is holomorphic with respect to one of the variables z^j for $j \neq k$, then so is g.

Proof. To simplify the notation a bit, we will carry out the proof in the case $k = 1$. Given $a = (a^1,\dots,a^n) \in U$, choose $r > 0$ such that the closed polydisk $\overline{D}{}^n_r(a)$ is contained in U. We wish to define $g : D^n_r(a) \to \mathbb{C}$ by

$$g(z^1,\dots,z^n) = \frac{1}{2\pi i}\int_{\overline{D}_r(a^1)} \frac{f(w,z^2,\dots,z^n)}{w-z^1}\, dw \wedge d\bar w. \tag{4.14}$$

Because the integrand is not continuous on the domain of integration, we must first make sure that the integral makes sense. Note that $dw\wedge d\bar w$ is a constant multiple of $du\wedge dv$ (where we write $w = u+iv$), and $|f(w,z^2,\dots,z^n)|/|w-z^1| \leq C/|w-z^1|$ for some constant C. The function $w \mapsto C/|w-z^1|$ is an integrable function of w on any compact subset of the plane (whether interpreted as a Lebesgue integral or as an improper Riemann integral), as can be verified by expressing the integral in polar coordinates centered at z^1. Thus the integral makes sense for each z, and g is a well-defined function.

Now we show that g is smooth in all variables, is holomorphic in z^j if f is, and satisfies (4.13). Given a complex number $b^1 \in D_r(a^1)$, choose $\varepsilon > 0$ such that $D_\varepsilon(b^1) \subseteq D_r(a^1)$, and let $\varphi : \mathbb{C} \to \mathbb{R}$ be a smooth bump function such that $\varphi \equiv 1$ on $D_{\varepsilon/2}(b^1)$ and $\operatorname{supp}\varphi \subseteq D_\varepsilon(b^1)$. Let $f_1(z) = (1-\varphi(z^1))f(z)$ and $f_2(z) = \varphi(z^1)f(z)$, so $f = f_1 + f_2$ with f_1 identically zero whenever $z^1 \in D_{\varepsilon/2}(b^1)$, and the function

$w \mapsto f_2(w, z^2, \ldots, z^n)$ is supported in $D_\varepsilon(b^1)$ for each $(z^2, \ldots, z^n)$. Note also that if f is holomorphic with respect to one of the variables z^j for $j \geq 2$, then so are f_1 and f_2 because φ depends only on z^1. Correspondingly, let $g = g_1 + g_2$, where g_i is defined as in (4.14) but with f_i in place of f.

For g_1, the integrand is smooth in all variables and the domain of integration is compact, so g_1 is smooth in all variables. In particular, as long as $|z^1 - b^1| < \varepsilon/2$, we can differentiate under the integral sign to conclude

$$\frac{\partial g_1}{\partial \bar{z}^1}(z) = 0 = f_1(z).$$

Moreover, if f_1 is holomorphic with respect to z^j for some $j \geq 2$, then differentating under the integral sign shows that g_1 satisfies the Cauchy–Riemann equations with respect to that variable and thus is also holomorphic.

Now consider g_2. Choose R large enough that $\overline{D}_R(z_1) \supseteq D_\varepsilon(b_1)$ for every $z^1 \in D_\varepsilon(b^1)$. Because the function $w \mapsto f_2(w, z^2, \ldots, z^n)$ is smooth and compactly supported in $D_\varepsilon(b^1)$ for each $(z^2, \ldots, z^n)$, for any given $z^1 \in D_\varepsilon(b^1)$ we might as well compute g_2 by integrating over the entire disk $\overline{D}_R(z^1)$. Given such a z^1, we make the change of variables $w = z^1 + re^{i\theta}$, $dw = e^{i\theta}(dr + ir\,d\theta)$, $d\overline{w} = e^{-i\theta}(dr - ir\,d\theta)$ to conclude

$$\begin{aligned} g_2(z^1, \ldots, z^n) &= \frac{1}{2\pi i} \int_{\overline{D}_R(z^1)} \frac{f_2(w, z^2, \ldots, z^n)}{w - z^1}\, dw \wedge d\overline{w} \\ &= \frac{1}{2\pi i} \int_{\overline{D}_R(0)} \frac{f_2(z^1 + re^{i\theta}, z^2, \ldots, z^n)}{re^{i\theta}} (-2ir)\, dr \wedge d\theta \\ &= \frac{-1}{\pi} \int_0^{2\pi} \int_0^R f_2(z^1 + re^{i\theta}, z^2, \ldots, z^n) e^{-i\theta}\, dr\, d\theta. \end{aligned}$$

This holds for all $z \in D^n_R(a)$ such that $|z^1 - b^1| < \varepsilon$. This integrand is smooth in all variables and the domain of integration is compact, so once again we conclude that g_2 is smooth, and if f_2 is holomorphic with respect to z^j, $j \geq 2$, then so is g_2.

To compute $\partial g_2/\partial \bar{z}^1$, we differentiate under the integral sign:

$$\frac{\partial g_2}{\partial \bar{z}^1}(z^1, \ldots, z^n) = \frac{-1}{\pi} \int_0^{2\pi} \int_0^R \frac{\partial f_2}{\partial \bar{z}^1}(z^1 + re^{i\theta}, z^2, \ldots, z^n) e^{-i\theta}\, dr\, d\theta.$$

Now change variables back to $w = z^1 + re^{i\theta}$:

$$\frac{\partial g_2}{\partial \bar{z}^1}(z^1, \ldots, z^n) = \frac{1}{2\pi i} \int_{\overline{D}_R(z^1)} \frac{\partial f_2}{\partial \bar{z}^1}(w, z^2, \ldots, z^n) \frac{dw \wedge d\overline{w}}{w - z^1}.$$

Choose $\delta > 0$ smaller than R. For w in the compact annulus $A_{R,\delta} = \overline{D}_R(z^1) \smallsetminus D_\delta(z^1)$ and $(z^2, \ldots, z^n)$ arbitrary, the integrand is smooth in w and equal to $d\eta$, where

$$\eta = -\frac{f_2(w, z^2, \dots, z^n)\, dw}{w - z^1}.$$

By Stokes's theorem, therefore,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{A_{R,\delta}} \frac{\partial f_2}{\partial \bar z^1}(w, z^2, \dots, z^n)\frac{dw \wedge d\overline{w}}{w - z^1} &= \frac{1}{2\pi i}\int_{A_{R,\delta}} d\eta \\
&= \frac{1}{2\pi i}\int_{\partial D_R(z^1)} \eta - \frac{1}{2\pi i}\int_{\partial D_\delta(z^1)} \eta \\
&= -\frac{1}{2\pi i}\int_{\partial D_\delta(z^1)} \eta,
\end{aligned}$$

where the negative sign in the next-to-last equality results from the Stokes orientation on the inner circle, and the last equality follows because we chose R large enough that η is identically zero on the outer circle. To compute this last integral, parametrize $\partial D_\delta(z^1)$ by $w = z^1 + \delta e^{i\theta}$ for $\theta \in [0, 2\pi]$, so the pullback of η to the parameter domain $[0, 2\pi]$ is $-f_2(z^1 + \delta e^{i\theta}, z^2, \dots, z^n) i\, d\theta$. Therefore,

$$-\frac{1}{2\pi i}\int_{\partial D_\delta(z^1)} \eta = \frac{1}{2\pi}\int_0^{2\pi} f_2(z^1 + \delta e^{i\theta}, z^2, \dots, z^n)\, d\theta.$$

As $\delta \to 0$, the integrand converges uniformly to $f_2(z^1, \dots, z^n)$, so the limit of the integral is equal to $\frac{1}{2\pi}\int_0^{2\pi} f_2(z)\, d\theta = f_2(z)$. Putting this all together, we conclude that

$$\begin{aligned}
\frac{\partial g_2}{\partial \bar z^1}(z) &= \frac{1}{2\pi i}\int_{\overline{D}_r(a^1)} d\eta \\
&= \lim_{\delta\to 0}\frac{1}{2\pi i}\int_{A_{R,\delta}} d\eta = -\lim_{\delta\to 0}\frac{1}{2\pi i}\int_{\partial D_\delta(z)} \eta = f_2(z).
\end{aligned}$$

Combining this with the computation for f_1, we find that for $z \in D_r^n(a)$ such that $|z^1 - b^1| < \varepsilon$,

$$\frac{\partial f}{\partial \bar z}(z) = \frac{\partial f_1}{\partial \bar z}(z) + \frac{\partial f_2}{\partial \bar z}(z) = g_1(z) + g_2(z) = g(z).$$

Since b_1 was arbitrary, the same formula holds on the entire polydisk $D_r^n(a)$, thus completing the proof. □

Proof of the $\bar\partial$-Poincaré lemma. Let $n = \dim M$. This is purely a local question, so for each point $a \in M$, we can choose holomorphic coordinates $(z^1, \dots, z^n)$ centered at a and use the coordinate map to consider ω as a $\bar\partial$-closed (p, q)-form on a neighborhood U of 0 in $\mathbb{C}^n$. We make this assumption henceforth.

We begin with the special case $p = 0$. When $n = 1$, only the $(0, 1)$ case is nontrivial. Thus suppose $\omega = f\, d\bar z$ is a smooth $(0, 1)$-form on a neighborhood of 0 in $\mathbb{C}$. It is automatically $\bar\partial$-closed because there are no nonzero $(0, 2)$-forms.

Lemma 4.14 shows that there is a smooth function g on a neighborhood of 0 such that $\partial g/\partial \bar z = f$, which is equivalent to $\bar\partial g = \omega$.

Now let $n > 1$. Suppose ω is a smooth $\bar\partial$-closed $(0,q)$-form on a neighborhood U of 0 in $\mathbb{C}^n$. For each $k \in \{0,\dots,q\}$, let $\Lambda_k^{0,q}(U)$ denote the smooth subbundle of $\Lambda^{0,q}(U)$ spanned by q-fold wedge products involving only $d\bar z^1,\dots,d\bar z^k$; and let $\mathscr{E}_k^{0,q}(U)$ denote the space of smooth sections of $\Lambda_k^{0,q}(U)$. We will prove by induction on k that whenever $\omega \in \mathscr{E}_k^{0,q}(U)$ and $\bar\partial\omega = 0$, there exists $\eta \in \mathscr{E}^{0,q-1}(U)$ such that $\bar\partial\eta = \omega$. When $k = n$, this is the desired conclusion.

For $k = 0$, $\mathscr{E}_k^{0,q}(M)$ contains only $\omega = 0$, so there is nothing to prove. Suppose $k \ge 1$ and the claim is true for $\mathscr{E}_{k-1}^{0,q}(U)$, and let ω be an element of $\mathscr{E}_k^{0,q}(U)$ satisfying $\bar\partial\omega = 0$. By separating out the terms in ω that contain $d\bar z^k$, we can write $\omega = \alpha + d\bar z^k \wedge \beta$, where $\alpha \in \mathscr{E}_{k-1}^{0,q}(U)$ and $\beta \in \mathscr{E}_{k-1}^{0,q-1}(U)$. Write

$$\alpha = \sum_I{}' \alpha_I d\bar z^{i_1} \wedge \cdots \wedge d\bar z^{i_q}, \qquad \beta = \sum_J{}' \beta_J d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_{q-1}},$$

where the only multi-indices I and J that occur have all indices less than k. Our assumption is

$$\begin{aligned} 0 = \bar\partial\omega &= \bar\partial\alpha - d\bar z^k \wedge \bar\partial\beta \\ &= \sum_I{}' \sum_{r=1}^n \frac{\partial\alpha_I}{\partial\bar z^r} d\bar z^r \wedge d\bar z^{i_1} \wedge \cdots \wedge d\bar z^{i_q} \\ &\quad - d\bar z^k \wedge \sum_J{}' \sum_{s=1}^n \frac{\partial\beta_J}{\partial\bar z^s} d\bar z^s \wedge d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_{q-1}}. \end{aligned}$$

Let $J = (j_1,\dots,j_{q-1})$ be any increasing multi-index of length $q-1$ with all indices less than k, and let $s > k$. When we evaluate the right-hand side of the above expression on the basis vectors $\left(\partial/\partial\bar z^k, \partial/\partial\bar z^s, \partial/\partial\bar z^{j_1},\dots,\partial/\partial\bar z^{j_{q-1}}\right)$, only one term produces a nonzero result, and that result is $-\partial\beta_J/\partial\bar z^s$. Thus each coefficient β_J is holomorphic in the variables $z^{k+1},\dots,z^n$. Lemma 4.14 shows that for each such J, in some neighborhood U_0 of 0 there is a smooth function γ_J satisfying $\partial\gamma_J/\partial\bar z^k = \beta_J$, and $\partial\gamma_J/\partial\bar z^s = 0$ for $s \ge k+1$. Let γ be the $(0,q-1)$-form

$$\gamma = \sum_J{}' \gamma_J d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_{q-1}},$$

so that

$$\begin{aligned} \bar\partial\gamma &= \sum_J{}' \sum_{s=1}^k \frac{\partial\gamma_J}{\partial\bar z^s} d\bar z^s \wedge d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_{q-1}} \\ &= d\bar z^k \wedge \beta \mod \mathscr{E}_{k-1}^{0,q}(U_0). \end{aligned}$$

Therefore, $\omega - \bar\partial\gamma \in \mathscr{E}^{0,q}_{k-1}(U_0)$. Since $\omega - \bar\partial\gamma$ is $\bar\partial$-closed, the inductive hypothesis guarantees that there is a smooth $(0, q-1)$-form σ such that $\bar\partial\sigma = \omega - \bar\partial\gamma$, so we have $\bar\partial(\sigma + \gamma) = \omega$. This completes the inductive step and thus proves the lemma in the $p = 0$ case.

Finally, let p be arbitrary, and suppose ω is a $\bar\partial$-closed (p, q)-form. We can write

$$\omega = \sum_{I,J}{}' \omega_{IJ} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_q} = \sum_I{}' \alpha_I \wedge \beta_I,$$

where

$$\alpha_I = dz^{i_1} \wedge \cdots \wedge dz^{i_p},$$
$$\beta_I = \sum_J{}' \omega_{IJ} d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_q}.$$

Note that $\bar\partial\alpha_I = 0$ because $d\alpha_I = 0$, so

$$0 = \bar\partial\omega = (-1)^p \sum_I{}' \alpha_I \wedge \bar\partial\beta_I.$$

Choose a particular multi-index $I = (i_1, \dots, i_p)$, and take the interior product of both sides of this equation with the vector fields $\partial/\partial z^{i_1}, \dots, \partial/\partial z^{i_p}$ in turn, using the fact that interior multiplication is an ***antiderivation*** (meaning that $v \lrcorner (\omega \wedge \eta) = (v \lrcorner \omega) \wedge \eta + (-1)^k \omega \wedge (v \lrcorner \eta)$ when ω and η are forms of degrees k and l, respectively [**LeeSM**, Lemma 14.13]). This yields

$$0 = \frac{\partial}{\partial z^{i_p}} \lrcorner \cdots \lrcorner \frac{\partial}{\partial z^{i_1}} \lrcorner \bar\partial\omega = (-1)^p \bar\partial\beta_I,$$

so $\bar\partial\beta_I = 0$ for each I. Therefore, by the $p = 0$ case proved above, for each I there is a $(0, q-1)$-form σ_I on a neighborhood of 0 such that $\bar\partial\sigma_I = \beta_I$, and it follows that $\bar\partial\big(\sum_I' \alpha_I \wedge \sigma_I\big) = \omega$. □

Corollary 4.15 (Local $\partial\bar\partial$-Lemma). *Suppose θ is a smooth, closed (p, q)-form on a complex manifold M, with p and q both positive. In a neighborhood of each point of M, there exists a smooth $(p-1, q-1)$-form α such that $\theta = i\partial\bar\partial\alpha$. If θ is a real (p, p)-form, then α can be chosen to be real.*

Proof. Because θ is a closed $(p+q)$-form, in a neighborhood of each point there is a complex $(p+q-1)$-form η such that $d\eta = \theta$ by the ordinary Poincaré lemma. (The Poincaré lemma applies to complex-valued forms by applying it separately to the real and imaginary parts.) Since the only parts of η that can contribute to the (p, q)-part of $d\eta$ are the $(p, q-1)$ and $(p-1, q)$ parts, we may as well assume that η decomposes as $\eta = \eta^{(p,q-1)} + \eta^{(p-1,q)}$.

Using the fact that $d = \partial + \bar\partial$, we can decompose the equation $d\eta = \theta$ as follows:

$$\begin{aligned} \bar\partial\eta^{(p-1,q)} &= 0 && ((p-1,q+1) \text{ part}),\\ \bar\partial\eta^{(p,q-1)} + \partial\eta^{(p-1,q)} &= \theta && ((p,q) \text{ part}),\\ \partial\eta^{(p,q-1)} &= 0 && ((p+1,q-1) \text{ part}). \end{aligned}$$

Now apply the $\bar\partial$-Poincaré lemma to conclude that (after shrinking the neighborhood if necessary) there exists a $(p-1,q-1)$-form β such that $\bar\partial\beta = \eta^{(p-1,q)}$. Similarly, since $\overline{\eta^{(p,q-1)}}$ is $\bar\partial$-closed, there exists a $(q-1,p-1)$-form γ such that $\bar\partial\gamma = \overline{\eta^{(p,q-1)}}$.

Set $\alpha = i\bar\gamma - i\beta$, which is a $(p-1,q-1)$-form. Using the fact that $\partial\bar\partial = -\bar\partial\partial$, we compute

$$\begin{aligned} i\partial\bar\partial\alpha &= i\partial\bar\partial(i\bar\gamma) - i\partial\bar\partial(i\beta)\\ &= \bar\partial(\partial\bar\gamma) + \partial(\bar\partial\beta)\\ &= \bar\partial\eta^{(p,q-1)} + \partial\eta^{(p-1,q)}\\ &= \theta. \end{aligned}$$

In case θ is a real (p,p)-form, we can choose η to be real, which means $\overline{\eta^{(p,p-1)}} = \eta^{(p-1,p)}$; and then we can choose $\gamma = \beta$, so that $\alpha = i\bar\beta - i\beta$ is real. □

Bundle-Valued Forms

For holomorphic vector bundles, there is a generalized Dolbeault complex built out of "bundle-valued differential forms." Here is how that works.

We begin with a smooth manifold M and a smooth complex vector bundle $E \to M$. Let $\operatorname{End}(E) \to M$ be its endomorphism bundle, which is canonically isomorphic to $E \otimes E^*$. (See Example 3.24 for the holomorphic case; the argument is exactly the same for smooth bundles.) For each nonnegative integer q, we define the ***bundle of E-valued q-forms*** as the tensor product bundle $\Lambda^q_{\mathbb{C}} M \otimes E$. We will use the notation $\mathscr{E}^q(M;E)$ to denote the space of smooth sections of $\Lambda^q_{\mathbb{C}} M \otimes E$, so $\mathscr{E}^0(M;E)$ is just the space of smooth sections of E itself. Similarly, $\Lambda^q_{\mathbb{C}} M \otimes \operatorname{End}(E)$ is the bundle of ***endomorphism-valued q-forms***, and $\mathscr{E}^q(M;\operatorname{End}(E))$ is its space of smooth sections.

There are several natural wedge product operations on bundle-valued and endomorphism-valued forms. Suppose $E \to M$ is a smooth complex vector bundle. The simplest case is the wedge product between ordinary (scalar-valued) differential forms and E-valued differential forms. For $\alpha \in \mathscr{E}^q(M)$ and $\beta \otimes \sigma \in \mathscr{E}^{q'}(M;E)$, we define

$$\alpha \wedge (\beta \otimes \sigma) = (\alpha \wedge \beta) \otimes \sigma \in \mathscr{E}^{q+q'}(M;E),$$

and then extend linearly to sums of such tensor products. Because this expression depends bilinearly on β and σ, it follows from the characteristic property of tensor product spaces [**LeeSM**, Prop. 12.7] (which holds equally well for tensor products over $\mathbb{C}$ or $\mathbb{R}$) that this is well defined. In terms of a local frame (s_j) for E, we can write $\beta = \beta^j \otimes s_j$ for some scalar-valued forms β^j (using the summation convention), and then

$$\alpha \wedge \left(\beta^j \otimes s_j\right) = \left(\alpha \wedge \beta^j\right) \otimes s_j. \tag{4.15}$$

In particular, if β is an E-valued 0-form (i.e., a smooth section of E), then $\alpha \wedge \beta$ is just the same as $\alpha \otimes \beta$; and if α is a 0-form (i.e., a smooth function), then $\alpha \wedge \beta = \alpha\beta$.

There is also a wedge product operation between E-valued forms and E^*-valued forms, which produces scalar forms. We define a wedge operation

$$\wedge : \mathscr{E}^q(M; E^*) \times \mathscr{E}^{q'}(M; E) \to \mathscr{E}^{q+q'}(M)$$

by setting

$$(\gamma \otimes \varphi) \wedge (\beta \otimes \sigma) = \varphi(\sigma)(\gamma \wedge \beta) \in \mathscr{E}^{q+q'}(M) \tag{4.16}$$

for $\gamma \otimes \varphi \in \mathscr{E}^{q'}(M; E^*)$ and $\beta \otimes \sigma \in \mathscr{E}^q(M; E)$, and extending bilinearly. This yields the following expression in terms of a local frame (s_j) for E and its dual frame (ε^k) for E^*:

$$\left(\gamma_k \otimes \varepsilon^k\right) \wedge \left(\beta^j \otimes s_j\right) = \gamma_j \wedge \beta^j. \tag{4.17}$$

Especially important are wedge products with endomorphism-valued forms. For $\alpha \otimes A \in \mathscr{E}^q(M; \operatorname{End}(E))$, $\beta \otimes B \in \mathscr{E}^{q'}(M; \operatorname{End}(E))$, and $\gamma \otimes \sigma \in \mathscr{E}^{q''}(M; E)$, we define

$$(\alpha \otimes A) \wedge (\beta \otimes B) = (\alpha \wedge \beta) \otimes (A \circ B) \in \mathscr{E}^{q+q'}(M; \operatorname{End}(E)),$$
$$(\alpha \otimes A) \wedge (\gamma \otimes \sigma) = (\alpha \wedge \gamma) \otimes A\sigma \in \mathscr{E}^{q+q'}(M; E),$$

and extend bilinearly. To see how to compute these locally, let (s_j) be a local frame for E and (ε^k) the dual frame for E^*. Because of the canonical isomorphism $\operatorname{End}(E) \cong E \otimes E^*$, each section $\omega \in \mathscr{E}^q(\operatorname{End}(E))$ can be expressed locally in the form

$$\omega = \omega^j_k \otimes s_j \otimes \varepsilon^k,$$

for a uniquely determined matrix $\left(\omega^j_k\right)$ of ordinary q-forms. The tensor product $s_j \otimes \varepsilon^k$ represents the endomorphism of E whose action on a basis element s_i is $\left(s_j \otimes \varepsilon^k\right)(s_i) = \delta^k_i s_j$, so the wedge product defined above satisfies

$$\begin{aligned} \omega \wedge \eta &= \left(\omega^j_k \otimes s_j \otimes \varepsilon^k\right) \wedge \left(\eta^l_m \otimes s_l \otimes \varepsilon^m\right) \\ &= \left(\omega^j_k \wedge \eta^l_m\right) \otimes \left(\delta^k_l s_j \otimes \varepsilon^m\right) \\ &= \left(\omega^j_k \wedge \eta^k_m\right) \otimes \left(s_j \otimes \varepsilon^m\right). \end{aligned} \tag{4.18}$$

In other words, the matrix of forms representing $\omega \wedge \eta$ is the matrix product of the ones representing ω and η, with individual entries combined via the wedge product.

(In an expression like ω^j_k, we always interpret the upper index as the row number and the lower index as the column number.) Similarly, if $\gamma = \gamma^m \otimes s_m$ is the local expression for an element of $\mathscr{E}^{q''}(M;E)$, then

$$\omega \wedge \gamma = \left(\omega^j_k \wedge \gamma^k\right) \otimes s_j. \tag{4.19}$$

Now suppose M is a complex manifold. For each p, q, we define the ***bundle of E-valued (p, q)-forms*** as the tensor product bundle $\Lambda^{p,q}M \otimes E$, and $\mathscr{E}^{p,q}(M;E)$ denotes the space of smooth sections of $\Lambda^{p,q}M \otimes E$.

When the bundle E is holomorphic, something special happens.

Proposition 4.16 (Cauchy–Riemann Operator on Bundle-Valued Forms). *Suppose M is a complex manifold and $E \to M$ is a holomorphic vector bundle. There are operators $\bar\partial_E : \mathscr{E}^{p,q}(M;E) \to \mathscr{E}^{p,q+1}(M;E)$ satisfying the following properties:*

(i) *For $\sigma \in \mathscr{E}^{0,0}(M;E) = \Gamma(E)$, $\bar\partial_E\sigma = 0$ if and only if σ is a holomorphic section.*

(ii) *For $\alpha \in \mathscr{E}^{p,q}(M)$ and $\beta \in \mathscr{E}^{p',q'}(M;E)$,*

$$\bar\partial_E(\alpha \wedge \beta) = \bar\partial\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \bar\partial_E\beta.$$

(iii) *For $\gamma \in \mathscr{E}^{p,q}(M;E^*)$ and $\beta \in \mathscr{E}^{p',q'}(M;E)$,*

$$\bar\partial(\gamma \wedge \beta) = \bar\partial_{E^*}\gamma \wedge \beta + (-1)^{p+q}\gamma \wedge \bar\partial_E\beta.$$

(iv) $\bar\partial_E \circ \bar\partial_E = 0$.

(v) *If $\alpha \in \mathscr{E}^{p,q}(M;E)$ satisfies $\bar\partial_E\alpha = 0$, then in a neighborhood of each point there exists $\beta \in \mathscr{E}^{p,q-1}(M;E)$ such that $\bar\partial_E\beta = \alpha$.*

Proof. Suppose $\sigma \in \mathscr{E}^{p,q}(M;E)$. In any open set $U \subseteq M$ over which there is a holomorphic local frame (s_j) for E, we can write $\sigma|_U = \sigma^j \otimes s_j$ for scalar-valued forms σ_j. We wish to define $\bar\partial_E\sigma$ by setting

$$\bar\partial_E\sigma\big|_U = \left(\bar\partial\sigma^j\right) \otimes s_j. \tag{4.20}$$

To ensure this makes sense globally, we need to check that it is independent of the choice of holomorphic local frame. If $(\tilde s_k)$ is another holomorphic local frame, then where the domains overlap we can write $\tilde s_k = \tau^j_k s_j$ for some holomorphic functions τ^j_k. Then $\sigma = \widetilde\sigma^k \tilde s_k$ with $\tau^j_k\widetilde\sigma^k = \sigma^j$. Because $\bar\partial\tau^j_k \equiv 0$, we have

$$\left(\bar\partial\sigma^j\right) \otimes s_j = \bar\partial\left(\tau^j_k\widetilde\sigma^k\right) \otimes s_j = \left(\bar\partial\widetilde\sigma^k\right) \otimes \left(\tau^j_k s_j\right) = \left(\bar\partial\widetilde\sigma^k\right) \otimes \tilde s_k.$$

This proves that $\bar\partial_E$ is well defined.

To prove (i), suppose σ is a smooth section of E. In terms of any local holomorphic frame (s_j), we can write $\sigma = f^j s_j$ for some complex-valued functions f^j. If σ is holomorphic, then each f^j is holomorphic, and (4.20) shows that $\bar\partial_E \sigma = 0$. Conversely, if $\bar\partial_E \sigma = 0$, then $0 = \left(\bar\partial f^j\right) \otimes s_j$; since the sections s_j are linearly independent at each point, this shows that $\bar\partial f^j = 0$ for each j, so σ is holomorphic.

Now (ii) follows by writing $\beta \in \mathscr{E}^{p',q'}(M;E)$ locally as $\beta^j \otimes s_j$, so that

$$\begin{aligned}
\bar\partial_E(\alpha \wedge \beta) &= \bar\partial_E\left(\left(\alpha \wedge \beta^j\right) \otimes s_j\right) \\
&= \left(\bar\partial\alpha \wedge \beta^j + (-1)^{p+q}\alpha \wedge \bar\partial\beta^j\right) \otimes s_j \\
&= \bar\partial\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \bar\partial_E\beta.
\end{aligned}$$

To prove (iii), write β locally as above, and write $\gamma = \gamma_k \otimes \varepsilon^k$, where (ε^k) denotes the local holomorphic frame for E^* dual to (s_j), so that $\gamma \wedge \beta = \gamma_j \wedge \beta^j$. We compute

$$\begin{aligned}
&\bar\partial_{E^*}\gamma \wedge \beta + (-1)^{p+q}\gamma \wedge \bar\partial_E\beta \\
&\quad = \left(\bar\partial\gamma_k \otimes \varepsilon^k\right) \wedge \left(\beta^j \otimes s_j\right) + (-1)^{p+q}\left(\gamma_k \otimes \varepsilon^k\right) \wedge \left(\bar\partial\beta^j \otimes s_j\right) \\
&\quad = \bar\partial\gamma_j \wedge \beta^j + (-1)^{p+q}\gamma_j \wedge \bar\partial\beta^j \\
&\quad = \bar\partial\left(\gamma_j \wedge \beta^j\right) = \bar\partial(\gamma \wedge \beta).
\end{aligned}$$

To prove (iv), let $\sigma \in \mathscr{E}^{p,q}(M;E)$, and let (s_j) be a holomorphic local frame for E on $U \subseteq M$. Writing $\sigma = \sigma^j \otimes s_j$ on U, we conclude from (4.20) that

$$\bar\partial_E\left(\bar\partial_E\sigma\right) = \bar\partial_E\left(\left(\bar\partial\sigma^j\right) \otimes s_j\right) = \left(\bar\partial\bar\partial\sigma^j\right) \otimes s_j = 0.$$

Finally, to prove (v), suppose α satisfies $\bar\partial_E\alpha = 0$. In terms of a holomorphic local frame, we can write $\alpha = \alpha^j \otimes s_j$ for some scalar-valued (p,q)-forms α^j satisfying $\left(\bar\partial\alpha^j\right) \otimes s_j = 0$. As above, the pointwise linear independence of the s_j's implies $\bar\partial\alpha^j = 0$ for each j, and then the $\bar\partial$-Poincaré lemma yields $(p, q-1)$-forms β^j in a neighborhood of each point such that $\bar\partial\beta^j = \alpha^j$ and thus $\bar\partial_E\left(\beta^j \otimes s_j\right) = \alpha$. □

Note that in general, there is no natural ∂ operator on holomorphic vector bundles. What makes it possible to define $\bar\partial_E$ is the fact that the transition functions relating different holomorphic frames are holomorphic and thus killed by the scalar $\bar\partial$ operator.

Thanks to part (iv) of the preceding proposition, we can define the ***Dolbeault cohomology groups with coefficients in E*** as the vector spaces

(4.21) $$H^{p,q}(M;E) = \frac{\operatorname{Ker}\left(\bar\partial_E : \mathscr{E}^{p,q}(M;E) \to \mathscr{E}^{p,q+1}(M;E)\right)}{\operatorname{Im}\left(\bar\partial_E : \mathscr{E}^{p,q-1}(M;E) \to \mathscr{E}^{p,q}(M;E)\right)}.$$

A particularly important special case of this construction applies to the bundles $\Lambda^{p,0}M$. If (z^j) and $(\tilde{z}^j)$ are overlapping holomorphic coordinate charts for M, then the local coordinate frames for $\Lambda^{p,0}M$ are related by

$$d\tilde{z}^{j_1} \wedge \cdots \wedge d\tilde{z}^{j_p} = \sum_{i_1,\ldots,i_p=1}^{n} \frac{\partial \tilde{z}^{j_1}}{\partial z^{i_1}} \cdots \frac{\partial \tilde{z}^{j_p}}{\partial z^{i_p}} dz^{i_1} \wedge \cdots \wedge dz^{i_p},$$

so the transition functions between the two frames are holomorphic. Thus $\Lambda^{p,0}M$ has the structure of a holomorphic vector bundle, and Problem 4-2 shows that the bundle-valued $\bar{\partial}$-operator can be identified with the ordinary $\bar{\partial}$ in this case. (Note that the bundles $\Lambda^{p,q}M$ for $q \neq 0$ do not have natural holomorphic structures, however; they are just smooth bundles.) A section α of $\Lambda^{p,0}M$ is holomorphic if and only if $\bar{\partial}\alpha = 0$.

We denote the space of holomorphic (i.e., $\bar{\partial}$-closed) sections of $\Lambda^{p,0}M$ by $\Omega^p(M)$; these sections are called ***holomorphic p-forms***. One particular case that will be of special importance is the case $p = n$, where $n = \dim M$. In that case $\Lambda^{n,0}M$ is a holomorphic line bundle, called the ***canonical bundle of M***. It is typically denoted by $K_M \to M$, or sometimes just $K \to M$ if it will not cause confusion. Its dual, denoted by K_M^* or K^*, is called the ***anticanonical bundle***.

Proposition 4.17 (Canonical Bundle of Projective Space). *Let $K \to \mathbb{CP}^n$ be the canonical bundle of $\mathbb{CP}^n$ and let $H \to \mathbb{CP}^n$ be its hyperplane bundle. Then $K \cong H^{-(n+1)}$.*

Proof. Problem 4-4. □

More generally, if $E \to M$ is a holomorphic vector bundle, each of the bundles $\Lambda^{p,0} \otimes E$ is also holomorphic, and holomorphic sections of such a bundle are those that can be written locally as a finite sum of tensor products of holomorphic forms with holomorphic sections of E. We denote the space of holomorphic sections of $\Lambda^{p,0} \otimes E$ by $\Omega^p(M; E)$.

Problems

4-1. Let M be a complex manifold and let ω be a 2-form on M. Show that ω is of type $(1,1)$ if and only if $\omega(JX, JY) = \omega(X, Y)$ for all vector fields X and Y, where $J : TM \to TM$ is the canonical almost complex structure on M.

4-2. On a complex manifold M, show that the complex vector bundles $\Lambda^{p,q}M$ and $\Lambda^{p,0}M \otimes \Lambda^{0,q}M$ are isomorphic, and under this isomorphism the operator $\bar{\partial}_E$ for $E = \Lambda^{p,0}M$ corresponds to the standard $\bar{\partial}$ operator.

4-3. Let (M, J) be an almost complex manifold of real dimension $2n$, and define $\mathscr{E}^{p,q}(M)$ to be the space of smooth complex-valued $(p+q)$-forms that vanish if more than p of their arguments are sections of $T'M$ or more than q are sections of $T''M$. Show that the following are equivalent:
(a) J is integrable.
(b) For every pair of integers $p, q \in \{0, \dots, n\}$, the exterior derivative operator d maps $\mathscr{E}^{p,q}(M)$ to $\mathscr{E}^{p+1,q}(M) \oplus \mathscr{E}^{p,q+1}(M)$.
(c) $d\big(\mathscr{E}^{0,1}(M)\big) \subseteq \mathscr{E}^{1,1}(M) \oplus \mathscr{E}^{0,2}(M)$.

4-4. Prove Proposition 4.17 (canonical bundle of projective space).

4-5. Prove that there are no nontrivial global holomorphic p-forms on $\mathbb{CP}^n$ for $p > 0$. [Hint: Proceed by induction on p, using interior products with the vector fields Z_j of Problem 3-5 to prove the inductive step.]

4-6. Prove that every holomorphic map from a complex projective space to a complex torus is constant. [Hint: Show that if $T = \mathbb{C}^m/\Lambda$ is a complex torus, the holomorphic 1-forms $dz^1, \dots, dz^m$ on $\mathbb{C}^m$ descend to holomorphic forms $\zeta^1, \dots, \zeta^n$ on T, and apply the result of Problem 4-5 to $F^*\zeta^j$ for $F : \mathbb{CP}^n \to T$.]

4-7. Let M be a complex manifold. A smooth real-valued function u on M is said to be ***pluriharmonic*** if in every holomorphic chart, u is harmonic (in the usual one-complex-variable sense) as a function of each complex coordinate when the others are held fixed. Show that the following are equivalent.
(a) u is pluriharmonic.
(b) $\partial\bar{\partial}u = 0$.
(c) For every holomorphic embedding $j : \mathbb{D} \hookrightarrow M$ of the unit disk into M, $u \circ j$ is harmonic (in the usual sense) on $\mathbb{D}$.
(d) In a neighborhood of each point of M, u is the real part of a holomorphic function.

4-8. If $S \subseteq M$ is a complex submanifold of a complex manifold M, the ***conormal bundle of S in M*** is the holomorphic bundle $N^*S \to S$ dual to the holomorphic normal bundle (see Example 3.23). Show that N^*S is canonically isomorphic to the holomorphic subbundle of $\Lambda^{1,0}M|_S$ consisting of $(1,0)$-forms that vanish when applied to vectors tangent to S.

4-9. Suppose $S \subseteq M$ is a closed complex hypersurface in a complex manifold M, and let $N^*S \to M$ be its conormal bundle (Problem 4-8). Prove that $N^*S \cong L_S^*\big|_S$, where $L_S \to M$ is the line bundle associated with S.

4-10. Suppose S is a closed complex hypersurface in a complex manifold M. Let K_S and K_M denote the canonical bundles of S and M, respectively, and L_S the line bundle on M associated with the hypersurface S. Prove the following ***adjunction formula***:

$$K_S \cong (K_M \otimes L_S)\big|_S.$$

[Hint: Begin by covering a neighborhood of S with holomorphic coordinate charts in which the first coordinate vanishes simply on S.]

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Chapter 5

Sheaves

There are many situations in differential geometry in which interesting geometric objects can be produced locally (nonvanishing sections of bundles, orientations, Riemannian metrics, logarithms of holomorphic functions, forms whose exterior derivative is a given closed form, to name a few), leading to the question of whether they can be pieced together to form a global object with the same properties. In smooth manifold theory, this can sometimes be accomplished with a partition of unity, as we do when proving the existence of Riemannian metrics. But there are many such problems, such as deciding whether a closed form is exact or trying to produce holomorphic objects of any type, in which multiplying by smooth bump functions destroys the property we are attempting to preserve, so partitions of unity are not useful. Sheaves were invented to systematize the process of going from local objects to global ones in such situations. Their applications to complex manifolds are plentiful and deep.

Sheaves were first introduced in the early 1940s by the French mathematician Jean Leray (while he was a prisoner of war in Austria); the concept was then developed and refined by a number of mathematicians over the next two decades. The article [**Mil00**] gives a good overview of the early history of the subject.

Definitions

The idea of a sheaf begins with a simple and familiar construction. Suppose M is a topological space. A ***presheaf on M*** is an assignment to each open set $U \subseteq M$ of a set $\mathscr{S}(U)$, called the set of ***sections of $\mathscr{S}$ over U***, along with a map $r^U_V : \mathscr{S}(U) \to \mathscr{S}(V)$ called ***restriction*** whenever $U \supseteq V$, with the properties

$$r^U_U = \mathrm{Id}_U \quad \text{for every open set } U, \tag{5.1}$$

$$r^V_W \circ r^U_V = r^U_W \quad \text{whenever } U \supseteq V \supseteq W. \tag{5.2}$$

For our applications, we will require our presheaves to have some algebraic structure. A presheaf $\mathscr{S}$ is called a ***presheaf of abelian groups*** if each $\mathscr{S}(U)$ has an abelian group structure and the restriction maps are all homomorphisms; ***presheaves of rings*** and ***presheaves of real or complex vector spaces*** are defined similarly. All of these are, in particular, presheaves of abelian groups (possibly with extra structure), so sometimes we will concentrate on presheaves of abelian groups and leave it to the reader to fill in the easy generalizations to presheaves with additional structure. Some typical examples to keep in mind on a complex manifold M are the presheaf $\mathscr{O}$ for which $\mathscr{O}(U)$ is the ring of scalar-valued holomorphic functions on U, and the presheaf $\mathscr{E}^{p,q}$ for which $\mathscr{E}^{p,q}(U)$ is the complex vector space of smooth (p,q)-forms on U. (In order for these assignments to satisfy the definition of a presheaf, we must assign meanings to $\mathscr{O}(\varnothing)$ and $\mathscr{E}^{p,q}(\varnothing)$. You can check that $\mathscr{S}(\varnothing) = \{0\}$ satisfies the definitions whenever $\mathscr{S}$ is a presheaf of sets, abelian groups, rings, or vector spaces, and we adopt this convention for all of our sheaves unless otherwise specified.)

Although there is no requirement that the elements of $\mathscr{S}(U)$ be maps of any kind or that the "restriction maps" be actual restrictions of maps, it is convenient to use the notation $s|_V$ for $r^U_V(s)$; property (5.2) guarantees that its value does not depend on which larger open set U we started with.

The definition of a presheaf can be expressed concisely in the language of category theory (see, e.g., [**LeeTM**, pp. 209–214] or [**Hat02**, pp. 162–165]). Given a topological space M, let $\mathsf{Top}(M)$ denote the category whose objects are open subsets of M and whose morphisms are set inclusions. If C is any category whatsoever (typically sets, abelian groups, rings, or vector spaces), a ***presheaf on M with values in*** C is a contravariant functor from $\mathsf{Top}(M)$ to C.

Because the purpose of sheaf theory is to systematize the process of piecing together local data to produce global results, our main objects of study will be presheaves with two additional properties, which guarantee that each section is locally determined and that compatible local sections can be glued together to produce global ones. A ***sheaf*** is a presheaf that satisfies the following two conditions whenever $\{U_\alpha\}_{\alpha\in A}$ is a collection of open subsets of M and $U = \bigcup_\alpha U_\alpha$:

- Locality property: If s and t are elements of $\mathscr{S}(U)$ such that $s|_{U_\alpha} = t|_{U_\alpha}$ for each $\alpha \in A$, then $s = t$.
- Gluing property: If we are given elements $s_\alpha \in \mathscr{S}(U_\alpha)$ for each α such that $s_\alpha|_{U_\alpha\cap U_\beta} = s_\beta|_{U_\alpha\cap U_\beta}$ whenever $U_\alpha \cap U_\beta \neq \varnothing$, then there is an element $s \in \mathscr{S}(U)$ such that $s_\alpha = s|_{U_\alpha}$ for every $\alpha \in A$.

► **Exercise 5.1.** Show that the following presheaves (with the restriction maps given by actual restriction) satisfy the locality and gluing properties.

(a) Given a topological, smooth, or holomorphic vector bundle $E \to M$, $\mathscr{S}(U)$ is the vector space of continuous, smooth, or holomorphic sections of E over U, respectively (with $\mathscr{S}(\varnothing) = \{0\}$ as explained above).
(b) Given any continuous map $\pi : X \to M$ between topological spaces, $\mathscr{S}(U)$ is the set of ***local sections of π*** over U, that is, continuous maps $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{Id}_U$. (In this case, $\mathscr{S}(\varnothing) = \{\varnothing\}$, because the empty map $\varnothing \subseteq \varnothing \times X$ is the unique map from $\varnothing$ to any set.)
(c) With X, M, and π as above, $\widehat{\mathscr{S}}(U)$ is the set of ***rough local sections of π*** over U, that is, maps $\sigma : U \to X$ that satisfy $\pi \circ \sigma = \mathrm{Id}_U$ but are not necessarily continuous.

Here are some general constructions we will use. Let M be a topological space.

If $\mathscr{S}$ is a sheaf on M and $V \subseteq M$ is an open subset, we define the ***restriction of $\mathscr{S}$ to V***, denoted by $\mathscr{S}|_V$, to be the presheaf

$$\mathscr{S}|_V(U) = \mathscr{S}(U) \text{ for all open sets } U \subseteq V, \tag{5.3}$$

with the restriction maps inherited from $\mathscr{S}$. It follows easily from the definitions that $\mathscr{S}|_V$ satisfies the gluing and locality properties and thus is a sheaf.

Given a sheaf $\mathscr{S}$ on M, a ***subpresheaf $\mathscr{T}$ of $\mathscr{S}$*** is a choice of subset (or subgroup, subring, or vector subspace as appropriate) $\mathscr{T}(U) \subseteq \mathscr{S}(U)$ for each open set $U \subseteq M$, such that $r^U_V(\mathscr{T}(U)) \subseteq \mathscr{T}(V)$ whenever $U \supseteq V$. If $\mathscr{T}$ satisfies the gluing and locality properties with the induced restriction maps, then it is called a ***subsheaf of $\mathscr{S}$***.

If $\mathscr{S}_1, \dots, \mathscr{S}_k$ are sheaves of abelian groups on M, their ***direct sum*** is the sheaf $\mathscr{S}_1 \oplus \cdots \oplus \mathscr{S}_k$ defined by $(\mathscr{S}_1 \oplus \cdots \oplus \mathscr{S}_k)(U) = \mathscr{S}_1(U) \oplus \cdots \oplus S_k(U)$. It is straightforward to check that this is a sheaf.

Examples of sheaves abound.

Example 5.2 (Sheaves). Let M be a smooth manifold and $E \to M$ a smooth complex vector bundle. The sheaves $\mathscr{C}$, $\mathscr{C}^*$, $\mathscr{E}$, $\mathscr{E}^*$, $\mathscr{E}^k$, $\mathscr{Z}^k$, $\mathscr{E}(E)$, $\mathscr{E}^k(E)$, $\underline{G}$, and G_p on M are defined by the following assignments to each open set $U \subseteq M$; in each case, the restriction maps are the obvious ones unless otherwise specified.

(a) $\mathscr{C}(U)$ is the vector space of continuous complex-valued functions on U.

(b) $\mathscr{C}^*(U)$ is the set of nowhere-vanishing continuous complex-valued functions on U, which is an abelian group under pointwise multiplication.

(c) $\mathscr{E}(U)$ is the vector space of smooth complex-valued functions on U.

(d) $\mathscr{E}^*(U)$ is the abelian group of nowhere-vanishing smooth complex functions on U.

(e) $\mathscr{E}^k(U)$ is the vector space of smooth complex k-forms on U.

(f) $\mathscr{Z}^k(U)$ is the vector space of smooth closed complex k-forms on U.

(g) $\mathscr{E}(E)(U) = \mathscr{E}(U; E)$ is the vector space of smooth sections of E over U.

(h) $\mathscr{E}^k(E)(U) = \mathscr{E}^k(U; E)$ is the vector space of smooth E-valued complex k-forms on U.

(i) $\underline{G}(U)$, for any abelian group G, is the set of locally constant functions from U to G (that is, functions that are constant in a neighborhood of each point); it is a group under pointwise addition. The sheaf $\underline{G}$ is called the ***constant sheaf*** with coefficients in G. If G is a ring or vector space, then $\underline{G}$ is a sheaf of rings or vector spaces, respectively.

(j) $G_p(U)$, for a point $p \in M$ and an abelian group G, is defined to be G if $p \in U$, and $\{0\}$ if not. The restriction map r^U_V is the identity if $p \in V \subseteq U$, and otherwise it is the zero map. Any such sheaf is called a ***skyscraper sheaf***.

We have focused on sheaves of complex-valued functions and forms because they are most useful in complex manifold theory; but some of these sheaves have obvious real counterparts, which we will denote by a subscript $\mathbb{R}$ when we have occasion to use them—for example, $\mathscr{E}_{\mathbb{R}}$ is the sheaf of smooth real-valued functions on M.

In addition, when M is a complex manifold, $E \to M$ is a holomorphic vector bundle, and $S \subseteq M$ is a closed subset, the sheaves $\mathscr{O}$, $\mathscr{O}^*$, $\mathscr{E}^{p,q}$, $\mathscr{Z}^{p,q}$, Ω^p, $\mathscr{I}_S$, $\mathscr{I}^2_S$, $\mathscr{O}(E)$, $\mathscr{E}^{p,q}(E)$, $\mathscr{Z}^{p,q}(E)$, $\Omega^p(E)$, $\mathscr{I}_S(E)$, and $\mathscr{I}^2_S(E)$ are defined as follows:

(k) $\mathscr{O}(U)$ is the vector space of holomorphic functions from U to $\mathbb{C}$.

(l) $\mathscr{O}^*(U)$ is the abelian group of nowhere-vanishing holomorphic functions on U.

(m) $\mathscr{E}^{p,q}(U)$ is the space of smooth (p,q)-forms on U.

(n) $\mathscr{Z}^{p,q}(U)$ is the space of smooth $\bar\partial$-closed (p,q)-forms on U.

(o) $\Omega^p(U)$ is the space of holomorphic p-forms on U (the same as $\mathscr{Z}^{p,0}(U)$).

(p) $\mathscr{I}_S(U)$ is the set of holomorphic functions on U that vanish on $S \cap U$.

(q) $\mathscr{I}^2_S(U)$ is the vector space of holomorphic functions on U that ***vanish to second order on $S \cap U$***, which means that in a neighborhood V of each point they can be written as a finite sum $\sum_j u_j v_j$ where u_j and v_j are holomorphic functions that vanish on $S \cap V$.

(r) $\mathscr{O}(E)(U) = \mathscr{O}(U; E)$ is the space of holomorphic sections of E over U.

(s) $\mathscr{E}^{p,q}(E)(U) = \mathscr{E}^{p,q}(U; E)$ is the space of smooth E-valued (p,q)-forms on U.

(t) $\mathscr{Z}^{p,q}(E)(U) = \mathscr{Z}^{p,q}(U; E)$ is the space of smooth $\bar\partial_E$-closed E-valued (p,q)-forms on U.

(u) $\Omega^p(E)(U) = \Omega^p(U; E)$ is the space of holomorphic E-valued p-forms on U.

(v) $\mathscr{I}_S(E)(U) = \mathscr{I}_S(U; E)$ is the set of holomorphic sections of E over U that vanish on $S \cap U$.

(w) $\mathscr{I}_S^2(E)(U) = \mathscr{I}_S^2(U; E)$ is the sheaf of holomorphic sections of E over U whose coefficient functions in any local holomorphic frame vanish to second order on $S \cap U$. //

It is worth noting that the sheaves $\mathscr{C}$, $\mathscr{E}$, and $\mathscr{O}$ defined above also have the structure of sheaves of commutative rings under pointwise multiplication. The sheaf $\mathscr{I}_S$ is called the ***ideal sheaf of S***, because $\mathscr{I}_S(U)$ is an ideal in the ring $\mathscr{O}(U)$.

On the other hand, it is not hard to come up with examples of presheaves that are not sheaves.

Example 5.3 (Presheaves That Are Not Sheaves). Let M be a manifold.

(a) $\mathscr{R}(U)$ is defined to be $\mathbb{R}$ for every nonempty open set $U \subseteq M$, and all restriction maps to nonempty open sets are the identity map. Unless M consists of only a single point, this presheaf is not a sheaf, because it fails to satisfy the gluing property: whenever U_α and U_β are disjoint nonempty open sets, we can choose distinct elements $x_\alpha, x_\beta \in \mathbb{R}$ and define "sections" $x_\alpha \in \mathscr{R}(U_\alpha)$ and $x_\beta \in \mathscr{R}(U_\beta)$, which vacuously satisfy the compatibility condition because $U_\alpha \cap U_\beta = \varnothing$. But there is no section $x \in \mathscr{R}(U_\alpha \cup U_\beta)$ that restricts to these sections.

(b) $\mathscr{R}_0(U) = \mathbb{R}$ for every nonempty open set U, but now all restriction maps are zero maps. This presheaf fails to satisfy the locality property.

(c) $\mathscr{B}(U)$ is the set of bounded continuous complex-valued functions on U. If M is noncompact, this presheaf fails to satisfy the gluing property. //

For some purposes, it is important to consider sheaves of modules over a sheaf of rings. Thus suppose $\mathscr{R}$ is a sheaf of commutative rings on M. (All our rings are assumed to have a multiplicative identity, denoted by 1.) A sheaf $\mathscr{S}$ on M is called a ***sheaf of $\mathscr{R}$-modules*** if for each open set $U \subseteq M$, $\mathscr{S}(U)$ can be given the structure of a module over the ring $\mathscr{R}(U)$, and the restriction maps are compatible with the module structure in the sense that for any open sets $V \subseteq U \subseteq M$ and sections $r \in \mathscr{R}(U)$, $s \in \mathscr{S}(U)$, we have

$$(rs)|_V = (r|_V)(s|_V).$$

Example 5.4 (Sheaves of Modules).

(a) If M is a smooth manifold and $\mathscr{E}$ is its sheaf of smooth complex-valued functions, the sheaves $\mathscr{E}^k$, $\mathscr{E}(E)$, and $\mathscr{E}^k(E)$ defined in Example 5.2 are sheaves of $\mathscr{E}$-modules.

(b) If M is a complex manifold, then $\mathscr{E}^{p,q}$ and $\mathscr{E}^{p,q}(E)$ are likewise sheaves of $\mathscr{E}$-modules on M; while Ω^p, $\mathscr{I}_S$, $\mathscr{I}_S^2$, $\mathscr{O}(E)$, $\Omega^p(E)$, $\mathscr{I}_S(E)$, and $\mathscr{I}_S^2(E)$ are sheaves of $\mathscr{O}$-modules, where $\mathscr{O}$ is the sheaf of holomorphic functions on M.

(c) Every sheaf $\mathcal{S}$ of abelian groups on a topological space M is a sheaf of $\underline{\mathbb{Z}}$-modules, where $\underline{\mathbb{Z}}$ is the constant sheaf on M with coefficients in $\mathbb{Z}$. To verify this, suppose $s \in \mathcal{S}(U)$ and $k \in \underline{\mathbb{Z}}(U)$. Each point $x \in U$ has a neighborhood $V_x \subseteq U$ on which $k|_{V_x}$ is constant, so $(k|_{V_x})(s|_{V_x})$ is an element of the group $\mathcal{S}(V_x)$. By the gluing property, these local sections patch together to determine a global section $ks \in \mathcal{S}(U)$. //

Sheaf Morphisms

Suppose $\mathcal{S}$ and $\mathcal{S}'$ are sheaves on the same space M, with restriction maps denoted by r^U_V and r'^U_V, respectively. A ***sheaf morphism from $\mathcal{S}$ to $\mathcal{S}'$*** is a collection of maps $F_U : \mathcal{S}(U) \to \mathcal{S}'(U)$, one for each open set $U \subseteq M$, with the property that they commute with restriction maps: whenever $U \supseteq V$, we have $r'^U_V \circ F_U = F_V \circ r^U_V$:

$$\begin{array}{ccc} \mathcal{S}(U) & \xrightarrow{F_U} & \mathcal{S}'(U) \\ {\scriptstyle r^U_V}\downarrow & & \downarrow{\scriptstyle r'^U_V} \\ \mathcal{S}(V) & \xrightarrow[F_V]{} & \mathcal{S}'(V). \end{array} \tag{5.4}$$

If $\mathcal{S}$ and $\mathcal{S}'$ are sheaves of abelian groups, rings, or vector spaces, then we require all the maps F_U to be homomorphisms in the appropriate category. In addition, if $\mathcal{S}$ and $\mathcal{S}'$ are both sheaves of modules over a sheaf $\mathcal{R}$ of commutative rings, a morphism $F : \mathcal{S} \to \mathcal{S}'$ is called an ***$\mathcal{R}$-module morphism*** if each F_U is an $\mathcal{R}(U)$-module homomorphism. More generally, if $\mathcal{S}$ and $\mathcal{S}'$ are presheaves, then ***presheaf morphisms*** from $\mathcal{S}$ to $\mathcal{S}'$ are defined in exactly the same way.

Sometimes if there is no risk of confusion, we will just denote all of the homomorphisms F_U by the same symbol F. Using this notation together with the shorthand notation for restrictions we introduced above, (5.4) can be written

$$F(s|_V) = F(s)|_V. \tag{5.5}$$

If $F : \mathcal{S} \to \mathcal{S}'$ and $G : \mathcal{S}' \to \mathcal{S}''$ are sheaf (or presheaf) morphisms, their ***composition*** is the morphism $G \circ F : \mathcal{S} \to \mathcal{S}''$ defined by $(G \circ F)_U = G_U \circ F_U$ for each open set U.

A sheaf morphism $F : \mathcal{S} \to \mathcal{S}'$ is called a ***sheaf isomorphism*** if there is a sheaf morphism $G : \mathcal{S}' \to \mathcal{S}$ satisfying $G \circ F = \mathrm{Id}_{\mathcal{S}}$ and $F \circ G = \mathrm{Id}_{\mathcal{S}'}$, or equivalently if each F_U is an isomorphism in the appropriate category; if there exists such an isomorphism, we say $\mathcal{S}$ and $\mathcal{S}'$ are ***isomorphic sheaves***, denoted by $\mathcal{S} \cong \mathcal{S}'$. ***Presheaf isomorphisms*** and ***isomorphic presheaves*** are defined similarly.

Example 5.5 (Sheaf Morphisms).

(a) If $\mathscr{S}$ is a sheaf on a topological space M and $\mathscr{T}$ is a subsheaf of $\mathscr{S}$, there is an ***inclusion morphism*** $i : \mathscr{T} \hookrightarrow \mathscr{S}$, defined by $i_U(\sigma) = \sigma$ for $\sigma \in \mathscr{T}(U) \subseteq \mathscr{S}(U)$.

(b) If M is a smooth manifold, then for each nonnegative integer k and each open subset $U \subseteq M$, the exterior derivative operator is a linear map $d : \mathscr{E}^k(U) \to \mathscr{E}^{k+1}(U)$. Because d acts locally, it commutes with restrictions, and thus defines a sheaf morphism $d : \mathscr{E}^k \to \mathscr{E}^{k+1}$. Similarly, if M is a complex manifold, there are sheaf morphisms $\bar{\partial} : \mathscr{E}^{p,q} \to \mathscr{E}^{p,q+1}$.

(c) On a complex manifold M, let $\mathscr{O}$ and $\mathscr{O}^*$ be the sheaves of holomorphic functions and nonvanishing holomorphic functions, respectively. There is a morphism $\varepsilon : \mathscr{O} \to \mathscr{O}^*$ given by $\varepsilon_U(f) = e^{2\pi i f}$ for $f \in \mathscr{O}(U)$ on an open subset $U \subseteq M$. Because $\varepsilon_U(f + g) = \varepsilon_U(f)\varepsilon_U(g)$, this is a sheaf morphism. (Recall that $\mathscr{O}^*$ is a sheaf of abelian groups under multiplication.)

(d) Let G and H be abelian groups, and let $\underline{G}$ and $\underline{H}$ be the corresponding constant sheaves on some topological space M. Every group homomorphism $F : G \to H$ defines a sheaf morphism $\underline{F} : \underline{G} \to \underline{H}$ by setting $\underline{F}_U(f) = F \circ f$ for $f \in \underline{G}(U)$. //

For any fixed topological space M, the sheaves on M and sheaf morphisms form a category. Similarly, there is a category of sheaves of abelian groups on M, or similarly sheaves of rings, real or complex vector spaces, or indeed sheaves on M with values in any category.

The Étalé Space of a Presheaf

In order to analyze the behavior of sheaf morphisms, we need to construct certain sets called *stalks* associated with a presheaf or sheaf. Before doing the general construction, let us illustrate the principle in one concrete special case. Suppose M is a complex manifold and $\mathscr{O}$ is its sheaf of holomorphic functions. For each $p \in M$, we define an equivalence relation on the set of all holomorphic functions $f \in \mathscr{O}(U)$ for open sets U that contain p, by saying $f \in \mathscr{O}(U)$ is equivalent to $g \in \mathscr{O}(V)$ if $f \equiv g$ on some neighborhood of p. The equivalence class of $f \in \mathscr{O}(U)$ is called the ***germ of f at p*** and denoted by $[f]_p$; it is not usually necessary to specify a particular open subset because the same germ is represented by the restriction of f to any neighborhood of p, however small. The ***stalk of $\mathscr{O}$ at p*** is the vector space $\mathscr{O}_p$ consisting of germs of all holomorphic functions at p. Addition and scalar multiplication of germs are defined by adding or multiplying any representatives that are defined on the same open set. The same construction can be done with continuous or smooth functions, or with continuous, smooth, or holomorphic sections of a vector bundle.

To define stalks for arbitrary presheaves we use instead the following general algebraic construction. A ***directed set*** is a nonempty set I with a binary relation "$\le$" that is reflexive and transitive, and such that any two elements have a common upper bound. (In many examples, directed sets are actually partially ordered sets, but they may not be, because the relation need not be antisymmetric; see the discussion following the proof of Lemma 6.5 for an example.)

A ***direct system*** of algebraic objects (which in our applications will be abelian groups, rings, or vector spaces) is a family $\{G_\alpha\}_{\alpha\in I}$ of objects indexed by a directed set I, together with a homomorphism $f_{\alpha\beta}\colon G_\alpha \to G_\beta$ whenever $\alpha \le \beta$, satisfying $f_{\alpha\alpha} = \mathrm{Id}_{G_\alpha}$ for each α and $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$ whenever $\alpha \le \beta \le \gamma$.

Given a direct system, define an equivalence relation on the disjoint union of all the G_α's by saying $g_\alpha \in G_\alpha$ is equivalent to $g_\beta \in G_\beta$ if there is some $\gamma \in I$ with $\alpha \le \gamma$ and $\beta \le \gamma$ such that $f_{\alpha\gamma}(g_\alpha) = f_{\beta\gamma}(g_\beta) \in G_\gamma$. The ***direct limit*** of the direct system, denoted by $\varinjlim G_\alpha$, is the set of equivalence classes with addition, for example, defined by

$$[g_\alpha] + [g_\beta] = [f_{\alpha\gamma}(g_\alpha) + f_{\beta\gamma}(g_\beta)],$$

where γ is some upper bound for α and β. This is well defined because all of the maps $f_{\alpha\beta}$ are homomorphisms. Other operations such as ring multiplication or scalar multiplication are defined similarly. For each object G_α, there is a canonical homomorphism from G_α to the direct limit, obtained by sending $g \in G_\alpha$ to its equivalence class.

Suppose $\mathscr{S}$ is a presheaf of abelian groups on a topological space M. For each $p \in M$, the collection of groups $\mathscr{S}(U)$ where U ranges over all open sets containing p, together with the restriction maps, is a direct system with the relation $U \le V$ if $U \supseteq V$. (The intersection of two open sets containing p is a common upper bound.) We define the ***stalk of $\mathscr{S}$ at p***, denoted by $\mathscr{S}_p$, to be the direct limit of this system. The equivalence class of an element $s \in \mathscr{S}(U)$ under this relation is denoted by $[s]_p$; by analogy with the construction for holomorphic functions described above, we call $[s]_p$ the ***germ of s at p***. You should convince yourself that this construction applied to the sheaf $\mathscr{O}$ yields germs of functions as we defined them above.

► **Exercise 5.6.** Suppose M is a Hausdorff space and G_p is a skyscraper sheaf as defined in Example 5.2(j). Show that the stalk $(G_p)_p$ is isomorphic to G, while for $q \ne p$, $(G_p)_q = 0$. Give an example of a skyscraper sheaf on a non-Hausdorff space for which this is not true.

Suppose $\mathscr{S}$ and $\mathscr{T}$ are presheaves of abelian groups on M. Every presheaf morphism $F\colon \mathscr{S} \to \mathscr{T}$ induces natural homomorphisms on stalks as follows. Given $p \in M$, we define the ***stalk homomorphism*** $F_p\colon \mathscr{S}_p \to \mathscr{T}_p$ by $F_p([s]_p) = [F(s)]_p$, where $s \in \mathscr{S}(U)$ is any representative of the germ $[s]_p$. To see that this is well defined, suppose $s \in \mathscr{S}(U)$ and $s' \in \mathscr{S}(U')$ represent the same germ at p. Then there is an open set $W \subseteq U \cap U'$ containing p such that $s|_W = s'|_W$, and (5.5) implies $F(s)|_W = F(s|_W) = F(s'|_W) = F(s')|_W$, so $F(s)$ and $F(s')$ represent the same germ at p. Also, because F and the restriction maps are homomorphisms on each set $\mathscr{S}(U)$, it follows that F_p is a homomorphism; and if $\mathscr{S}$ is a sheaf of rings or vector spaces, then F_p is a homomorphism in the same category.

Using the concept of stalks, we will define a topological space canonically associated with each presheaf, which will turn out to provide an important mechanism for turning presheaves into sheaves. An ***étalé space***[1] over a topological space M is a topological space E together with a local homeomorphism $\pi : E \to M$. The preimage $E_p = \pi^{-1}(p)$ of a point $p \in M$ is called the ***stalk of E over p***. An étalé space is called an ***étalé space of abelian groups*** if each stalk has an abelian group structure and the group operations are continuous in the following sense: negation is a continuous map from E to E, while group addition is continuous as a map from $E \times_M E$ to E, where $E \times_M E \subseteq E \times E$ is the fiber product of E with E over M (that is, the subset $\{(x_1, x_2) \in E \times E : \pi(x_1) = \pi(x_2)\}$ with the subspace topology). ***Étalé spaces of vector spaces or rings*** are defined analogously, with addition and multiplication continuous as above, and scalar multiplication continuous as a map from $\mathbb{R} \times E$ or $\mathbb{C} \times E$ to E when $\mathbb{R}$ or $\mathbb{C}$ is given the discrete topology.

Theorem 5.7 (The Étalé Space of a Presheaf). *Let $\mathscr{S}$ be a presheaf over a topological space M, and let $\operatorname{Et}(\mathscr{S})$ be the disjoint union of the stalks $\mathscr{S}_p$ for all $p \in M$, with projection $\pi : \operatorname{Et}(\mathscr{S}) \to M$ defined by $\pi([s]_p) = p$. For each open set $U \subseteq M$ and each $s \in \mathscr{S}(U)$, define a map $s^+ : U \to \operatorname{Et}(\mathscr{S})$ by*

$$s^+(p) = [s]_p. \tag{5.6}$$

Then $\operatorname{Et}(\mathscr{S})$ has a unique topology such that π is a local homeomorphism and each s^+ is continuous. If $\mathscr{S}$ is a presheaf of abelian groups, rings, or vector spaces, then $\operatorname{Et}(\mathscr{S})$ is an étalé space of objects in the same category.

Proof. We wish to take the collection of all subsets of the form $s^+(U) = \{[s]_p : p \in U\}$, for open sets $U \subseteq M$ and sections $s \in \mathscr{S}(U)$, as a basis for a topology on $\operatorname{Et}(\mathscr{S})$. To see that it is a basis, we need to verify two things: (i) every point of $\operatorname{Et}(\mathscr{S})$ is in some $s^+(U)$; and (ii) if two basis sets $s^+(U)$ and $t^+(V)$ intersect at a point $[w]_p$, then there is a basis set $v^+(W)$ such that $[w]_p \in v^+(W) \subseteq s^+(U) \cap t^+(V)$. Property (i) is immediate: every germ $[s]_p$ is represented by some section $s \in \mathscr{S}(U)$, and then $[s]_p$ is an element of the basis set $s^+(U)$. For (ii), assume $[w]_p \in s^+(U) \cap t^+(V)$; this means $p \in U \cap V$ and the germs of w, s, and t at p are all equal. In other words, there is some neighborhood $W \subseteq U \cap V$ of p such that $w|_W = s|_W = t|_W$, and then $[w]_p \in w^+(W) \subseteq s^+(U) \cap t^+(V)$ as required.

[1] The French word *étalé* is pronounced "ay-tah-LAY" and means "spread out." There is another closely related French word *étale* ("ay-TAHL") without the second accent mark, meaning "slack," which has a variety of definitions in algebra and algebraic geometry; but *étalé* seems to have only this one mathematical meaning.

To see that each map $s^+ : U \to \mathrm{Et}(\mathscr{S})$ is continuous, let $t^+(V) \subseteq \mathrm{Et}(\mathscr{S})$ be a basis open set and observe that

$$\begin{aligned}(s^+)^{-1}(t^+(V)) &= \left\{p \in U \cap V : [s]_p = [t]_p\right\} \\ &= \left\{p \in U \cap V : s|_W = t|_W \text{ for some neighborhood } W \text{ of } p\right\},\end{aligned}$$

which is open in U.

Next we show that π is a local homeomorphism. It suffices to show that the restriction of π to each basis open set $s^+(U)$ is a homeomorphism onto U. To see that it is continuous, let $V \subseteq U$ be open. Then $\pi^{-1}(V) \cap s^+(U)$ is the set of germs of s at points of V, or in other words exactly the basis set $(s|_V)^+(V)$. On the other hand, $\pi|_{s^+(U)} : s^+(U) \to U$ has a continuous inverse given by the local section $s^+ : U \to s^+(U)$, so it is a homeomorphism onto its image.

To show that the topology is unique, suppose $\widetilde{E}$ is the same set with another topology such that π is a local homeomorphism and each s^+ is continuous. First we will show that each set $s^+(U)$ is open in $\widetilde{E}$. Thus let $\xi \in s^+(U)$ be arbitrary, and set $p = \pi(\xi)$. Let $\widetilde{W}$ be a neighborhood of ξ in $\widetilde{E}$ such that $\pi|_{\widetilde{W}}$ is a homeomorphism from $\widetilde{W}$ to an open set $W \subseteq M$, and let $W_0 = (s^+)^{-1}(\widetilde{W})$; because s^+ is continuous as a map to $\widetilde{E}$, W_0 is an open neighborhood of p contained in $U \cap W$. Because $\pi|_{\widetilde{W}}$ is a homeomorphism, $\widetilde{W}_0 = (\pi|_{\widetilde{W}})^{-1}(W_0)$ is an open neighborhood of ξ contained in $\widetilde{W}$. Also, the fact that $\pi \circ s^+ = \mathrm{Id}_U$ implies $\pi(\widetilde{W}_0) = \pi \circ s^+ \circ \pi(\widetilde{W}_0)$, and injectivity of π on $\widetilde{W}$ then implies $\widetilde{W}_0 = s^+ \circ \pi(\widetilde{W}_0)$. Thus $\widetilde{W}_0$ is also contained in $s^+(U)$. Since each point of $s^+(U)$ has a neighborhood contained in $s^+(U)$, we see that $s^+(U)$ is open in $\widetilde{E}$.

Now both identity maps $\mathrm{Et}(\mathscr{S}) \to \widetilde{E}$ and $\widetilde{E} \to \mathrm{Et}(\mathscr{S})$ are continuous, because each is equal to the composition $s^+ \circ \pi$ when restricted to one of the open sets $s^+(U)$. Thus the two topologies are the same.

If $\mathscr{S}$ is a presheaf of abelian groups, rings, or vector spaces, then each stalk $\mathscr{S}_p$ inherits the appropriate algebraic structure from the direct limit operation as described above. Let $a\colon \mathrm{Et}(\mathscr{S}) \times_M \mathrm{Et}(\mathscr{S}) \to \mathrm{Et}(\mathscr{S})$ be addition. To see that it is continuous, suppose $s^+(U) \subseteq \mathrm{Et}(\mathscr{S})$ is a basis open set. Then $a^{-1}(s^+(U))$ is the set of all pairs of the form $([v]_p, [w]_p)$ where $p \in U$, $v, w \in \mathscr{S}(Y)$ for some neighborhood Y of p contained in U, and $v + w = s|_Y$. That is to say,

$$a^{-1}(s^+(U)) = \bigcup_{\substack{p\in Y\subseteq U \\ v,w\in \mathscr{S}(Y) \\ v+w=s|_Y}} v^+(Y) \times w^+(Y).$$

As a union of open sets, this is open. Similar arguments show that the other algebraic operations are continuous in appropriate cases. □

The next lemma gives a useful characterization of continuity of sections of $\mathrm{Et}(\mathscr{S})$ in terms of sections of $\mathscr{S}$.

Lemma 5.8. *Let $\mathscr{S}$ be a presheaf on a topological space M and $\pi: \operatorname{Et}(\mathscr{S}) \to M$ its étalé space. For any subset $B \subseteq M$, a rough section $\varphi: B \to \operatorname{Et}(\mathscr{S})$ (i.e., a map such that $\pi \circ \varphi = \operatorname{Id}_B$) is continuous if and only if each point in B has a neighborhood V in M and a section $s \in \mathscr{S}(V)$ such that $\varphi|_{V \cap B} = s^+|_{V \cap B}$.*

Proof. Suppose first that $\varphi: B \to \operatorname{Et}(\mathscr{S})$ is a continuous section. Given $p \in B$, the definition of $\operatorname{Et}(\mathscr{S})$ shows that $\varphi(p) = [s]_p$ for some section $s \in \mathscr{S}(W)$ on some neighborhood W of p. Then $s^+(W)$ is a neighborhood of $[s]_p$ in $\operatorname{Et}(\mathscr{S})$, and since φ is continuous, $V = \varphi^{-1}(s^+(W))$ is a neighborhood of p in U. This means $\varphi(q) = [s]_q$ for all $q \in V \cap B$, or in other words φ agrees with the restriction of s^+ to $V \cap B$.

Conversely, if φ satisfies the condition in statement of the lemma, then it is continuous because it agrees with a continuous section s^+ in a neighborhood of each point. □

Theorem 5.9 (The Sheafification Functor). *Suppose M is a topological space and $\mathscr{S}$ is a presheaf on M. Then there is a sheaf $\mathscr{S}^+$, called the* ***sheafification of $\mathscr{S}$****, together with a canonical presheaf morphism $\theta_{\mathscr{S}}: \mathscr{S} \to \mathscr{S}^+$, satisfying the following properties.*

(a) *If $\mathscr{S}$ is a presheaf of abelian groups, rings, or vector spaces, then $\mathscr{S}^+$ is a sheaf of objects in the same category.*

(b) *$\mathscr{S}^+$ satisfies the following universal property: If $\mathscr{T}$ is any sheaf and $F: \mathscr{S} \to \mathscr{T}$ is any presheaf morphism, there is a unique sheaf morphism $\overline{F}: \mathscr{S}^+ \to \mathscr{T}$ satisfying $\overline{F} \circ \theta_{\mathscr{S}} = F$:*

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{\ F\ } & \mathscr{T} \\ {\scriptstyle \theta_{\mathscr{S}}}\big\downarrow & \nearrow_{\overline{F}} & \\ \mathscr{S}^+ & & \end{array} \tag{5.7}$$

(c) *$\theta_{\mathscr{S}}$ is an isomorphism if and only if $\mathscr{S}$ is a sheaf.*

(d) *If $\mathscr{T}$ is any presheaf and $F: \mathscr{S} \to \mathscr{T}$ is a presheaf morphism, there is a unique sheaf morphism $F^+: \mathscr{S}^+ \to \mathscr{T}^+$ satisfying $\theta_{\mathscr{T}} \circ F = F^+ \circ \theta_{\mathscr{S}}$:*

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{\ F\ } & \mathscr{T} \\ {\scriptstyle \theta_{\mathscr{S}}}\big\downarrow & & \big\downarrow{\scriptstyle \theta_{\mathscr{T}}} \\ \mathscr{S}^+ & \underset{F^+}{\dashrightarrow} & \mathscr{T}^+. \end{array} \tag{5.8}$$

It also satisfies $(\operatorname{Id}_{\mathscr{S}})^+ = \operatorname{Id}_{\mathscr{S}^+}$ *and* $(F \circ G)^+ = F^+ \circ G^+$.

Proof. Define the sheaf $\mathcal{S}^+$ on M by letting $\mathcal{S}^+(U)$ be the set of local sections of $\operatorname{Et}(\mathcal{S})$ over the open set $U \subseteq M$, that is, continuous maps $\varphi\colon U \to \operatorname{Et}(\mathcal{S})$ such that $\pi \circ \varphi = \operatorname{Id}_U$; and letting the restriction maps be actual restrictions of sections. This is a sheaf by the result of Exercise 5.1(b). The algebraic operations on $\mathcal{S}^+$ are defined pointwise: for example, if $\mathcal{S}$ is a presheaf of abelian groups, we define a group structure on each set $\mathcal{S}^+(U)$ by

$$(s_1 + s_2)(p) = s_1(p) + s_2(p).$$

(If $U = \varnothing$, we just interpret $\mathcal{S}^+(U) = \{\varnothing\}$ to be the trivial group.) The fact that the group operations are continuous guarantees that the result is a continuous section of $\operatorname{Et}(\mathcal{S})$. The same argument applies to negation and to the other algebraic operations in the case of a presheaf of rings or vector spaces. With these pointwise operations, it is immediate that each restriction map is a homomorphism. This completes the proof of (a).

The presheaf morphism $\theta_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}^+$ is defined by $\theta_{\mathcal{S}}(s) = s^+$ for $s \in S(U)$. To prove (b), suppose $\mathcal{T}$ is a sheaf and $F : \mathcal{S} \to \mathcal{T}$ is a presheaf morphism. We define the sheaf morphism $\overline{F} : \mathcal{S}^+ \to \mathcal{T}$ first by defining $\overline{F}(s^+) = F(s) \in \mathcal{T}(U)$ for any open set $U \in M$ and $s \in \mathcal{S}(U)$; this ensures that $\overline{F} \circ \theta_{\mathcal{S}} = F$. Then we extend $\overline{F}$ to act on an arbitrary section $\varphi \in \mathcal{S}^+(U)$ as follows: Lemma 5.8 shows that there is an open cover $\{U_\alpha\}$ of U and sections $s_\alpha \in \mathcal{S}(U_\alpha)$ such that $\varphi|_{U_\alpha} = s_\alpha^+$ for each α. Let $\overline{F}(\varphi)$ be the element $\tau \in \mathcal{T}(U)$ such that $\tau|_{U_\alpha} = F(s_\alpha)$ for all α. Such a section exists because $\mathcal{T}$ satisfies the gluing property, and it is well defined independently of the choice of open cover because $\mathcal{T}$ satisfies the locality property. Moreover, any other such map $\overline{F}'$ would have to agree with this one because the restriction of $\overline{F}'(\varphi)$ to sufficiently small open sets has to agree with sections of the form $\overline{F}(s^+) = \overline{F} \circ \theta_{\mathcal{S}}(s)$. This completes the proof of (b).

To prove (c), note first that if $\theta_{\mathcal{S}}$ is a presheaf isomorphism, then $\mathcal{S}$ is certainly a sheaf because $\mathcal{S}^+$ is. Conversely, assume $\mathcal{S}$ is a sheaf. Applying (b) to $\operatorname{Id}_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$, we conclude there is a sheaf morphism $\overline{\operatorname{Id}_{\mathcal{S}}} : \mathcal{S}^+ \to \mathcal{S}$ such that $\overline{\operatorname{Id}_{\mathcal{S}}} \circ \theta_{\mathcal{S}} = \operatorname{Id}_{\mathcal{S}}$. On the other hand, the same result applied to $\theta_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}^+$ shows that there is a unique morphism $g : \mathcal{S}^+ \to \mathcal{S}^+$ satisfying $g \circ \theta_{\mathcal{S}} = \theta_{\mathcal{S}}$; since both $g = \operatorname{Id}_{\mathcal{S}^+}$ and $g = \theta_{\mathcal{S}} \circ \overline{\operatorname{Id}_{\mathcal{S}}}$ are such morphisms, we conclude that $\theta_{\mathcal{S}} \circ \overline{\operatorname{Id}_{\mathcal{S}}} = \operatorname{Id}_{\mathcal{S}^+}$. Thus $\overline{\operatorname{Id}_{\mathcal{S}}}$ is an inverse for $\theta_{\mathcal{S}}$.

Finally, to prove (d), suppose $F : \mathcal{S} \to \mathcal{T}$ is a presheaf morphism. Define $F^+ : \mathcal{S}^+ \to \mathcal{T}^+$ by $F^+ = \overline{\theta_{\mathcal{T}} \circ F}$, the morphism whose existence and uniqueness are guaranteed by the universal property of part (b); then (5.8) follows from (5.7). The argument in the preceding paragraph showed that $(\operatorname{Id}_{\mathcal{S}})^+ = \overline{\theta_{\mathcal{S}}} = \operatorname{Id}_{\mathcal{S}^+}$. If $F : \mathcal{S} \to \mathcal{T}$ and $G : \mathcal{T} \to \mathcal{U}$ are presheaf morphisms, then both $h_1 = G^+ \circ F^+$ and $h_2 = (G \circ F)^+$ satisfy $h_j \circ \theta_{\mathcal{S}} = \theta_{\mathcal{U}} \circ G \circ F$, so by the uniqueness part of statement (b), they are equal. □

If $\pi: E \to M$ and $\pi': E' \to M$ are étalé spaces over M, an ***étalé space morphism*** from E to E' is a continuous map $F: E \to E'$ such that $\pi' \circ F = \pi$, and such that F restricts to a homomorphism on each stalk with respect to whatever algebraic structure the stalks are endowed with. For each topological space M, we can define a category $\mathsf{Sh}(M)$ of sheaves (of abelian groups, say) on M and sheaf morphisms, and a category $\mathsf{Et}(M)$ of étalé spaces of abelian groups on M and étalé space morphisms. The construction of the étalé space of a sheaf defines a functor $\mathrm{Et}: \mathsf{Sh}(M) \to \mathsf{Et}(M)$.

The next proposition shows, essentially, that sheaves and étalé spaces contain exactly the same information. If C and D are categories, a functor $\mathscr{F}: \mathsf{C} \to \mathsf{D}$ is called an ***equivalence of categories*** if every object of D is isomorphic to $\mathscr{F}(c)$ for some object c of C, and for every pair of objects c_1, c_2 of C, the map $\mathscr{F}: \mathrm{Hom}_{\mathsf{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathsf{D}}(\mathscr{F}(c_1), \mathscr{F}(c_2))$ is bijective.

Proposition 5.10. *Let M be a topological space,* $\mathsf{Sh}(M)$ *the category of sheaves of abelian groups on M, and* $\mathsf{Et}(M)$ *the category of étalé spaces of abelian groups over M. The étalé space functor* $\mathrm{Et}: \mathsf{Sh}(M) \to \mathsf{Et}(M)$ *is an equivalence of categories.*

Proof. Problem 5-5. □

You will find that some authors *define* sheaves to be étalé spaces, and sheaf morphisms to be étalé space morphisms. The preceding proposition shows that one can translate back and forth between that definition and the one we have given without any loss of information.

The sheafification functor has some important applications to defining natural combinations of sheaves. We begin with quotient sheaves. Suppose $\mathscr{S}$ is a sheaf of abelian groups on M and $\mathscr{R}$ is a subsheaf of $\mathscr{S}$. In general, the presheaf $U \mapsto \mathscr{S}(U)/\mathscr{R}(U)$ need not be a sheaf, as the next example illustrates.

Example 5.11. Suppose M is a compact, connected complex manifold and p, q are distinct points in M. Let $\mathscr{O}$ be the sheaf of holomorphic functions on M, and let $\mathscr{I} \hookrightarrow \mathscr{O}$ be the subsheaf of holomorphic functions that vanish at both p and q (the ideal sheaf of $\{p, q\}$). Let $U_1 = M \smallsetminus \{p\}$ and $U_2 = M \smallsetminus \{q\}$. The constant function 1 on U_1 determines a nontrivial element of the quotient space $\mathscr{O}(U_1)/\mathscr{I}(U_1)$, and the constant function 2 determines a nontrivial element of $\mathscr{O}(U_2)/\mathscr{I}(U_2)$. Since $\mathscr{O}(U_1 \cap U_2)/\mathscr{I}(U_1 \cap U_2)$ is the zero vector space, both elements restrict to zero in this quotient space. But there is no element of $\mathscr{O}(M)/\mathscr{I}(M)$ that restricts to the equivalence class of 1 on U_1 and the equivalence class of 2 on U_2, because every element of $\mathscr{O}(M)$ is constant. Thus the presheaf $U \mapsto \mathscr{O}(U)/\mathscr{I}(U)$ does not satisfy the gluing property. //

To get around this problem, we define the ***quotient sheaf*** $\mathscr{S}/\mathscr{R}$ to be the sheafification of the presheaf $U \mapsto \mathscr{S}(U)/\mathscr{R}(U)$.

Another use of sheafification is to define tensor products of sheaves. Suppose $\mathscr{R}$ is a sheaf of commutative rings on M, and $\mathscr{S}$ and $\mathscr{T}$ are both sheaves of $\mathscr{R}$-modules. We can form the presheaf $U \mapsto \mathscr{S}(U) \otimes_{\mathscr{R}(U)} \mathscr{T}(U)$, but once again this might not be a sheaf. (See Problem 5-9 for a counterexample.) We define the ***tensor product of $\mathscr{S}$ and $\mathscr{T}$ over $\mathscr{R}$***, denoted by $\mathscr{S} \otimes_{\mathscr{R}} \mathscr{T}$, to be the sheafification of the presheaf $U \mapsto \mathscr{S}(U) \otimes_{\mathscr{R}(U)} \mathscr{T}(U)$. The next proposition illustrates the naturalness of this concept.

Proposition 5.12. *Suppose M is a complex manifold and $E, E' \to M$ are holomorphic vector bundles over M. Then the tensor product sheaf $\mathscr{O}(E) \otimes_{\mathscr{O}} \mathscr{O}(E')$ is isomorphic to $\mathscr{O}(E \otimes E')$.*

Proof. Let $U \subseteq M$ be an open subset. An element $\sigma \in \mathscr{O}(U; E) \otimes_{\mathscr{O}(U)} \mathscr{O}(U; E')$ can be represented as a finite sum of abstract tensor products, $\sum_j \sigma_j \otimes \sigma_j'$, where $\sigma_j \in \mathscr{O}(U; E)$ and $\sigma_j' \in \mathscr{O}(U; E')$. We let $F_U(\sigma)$ be the holomorphic section of the tensor product bundle $E \otimes E'$ over U given by the same formula: $x \mapsto \sum_j \sigma_j(x) \otimes \sigma_j'(x)$. Because this formula is bilinear over $\mathscr{O}(U)$, this gives a well-defined homomorphism

$$F_U \colon \mathscr{O}(U; E) \otimes_{\mathscr{O}(U)} \mathscr{O}(U; E') \to \mathscr{O}(U; E \otimes E'),$$

and because F_V is the restriction of F_U whenever $V \subseteq U$, this defines a presheaf morphism. By Theorem 5.9(b), it induces a sheaf morphism $\overline{F}$ from $\mathscr{O}(E) \otimes_{\mathscr{O}} \mathscr{O}(E')$ to $\mathscr{O}(E \otimes E')$.

We will prove the proposition by showing that $\overline{F}$ is bijective on stalks (see Problem 5-2). It suffices to show that F_U is bijective whenever $U \subseteq M$ is an open subset over which both E and E' are trivial.

Suppose U is such a set, and (s_j), (s_k') are holomorphic local frames on U for E and E', respectively. Suppose first that $\sigma = \sum_j \sigma_j \otimes \sigma_j'$ is an element of $\mathscr{O}(U; E) \otimes_{\mathscr{O}(U)} \mathscr{O}(U; E')$ such that $F_U(\sigma) = 0$. Writing $\sigma_j = \sum_k f_j^k s_k$ and $\sigma_j' = \sum_l f_j'^l s_l'$ for some holomorphic functions $f_j^k, f_j'^l$ and using the properties of the tensor product, we see that the following holds for all $x \in U$:

$$0 = \sum_j \sigma_j(x) \otimes \sigma_j'(x) = \sum_{j,k,l} f_j^k(x) f_j'^l(x) s_k(x) \otimes s_l'(x).$$

Since this is true for all $x \in U$ and the elements $s_k(x) \otimes s_l'(x)$ are linearly independent, it follows that $\sum_j f_j^k f_j'^l \equiv 0$ on U for each k and l. Thus

$$\sum_j \sigma_j \otimes \sigma_j' = \sum_k s_k \otimes \sum_l \Big(\sum_j f_j^k f_j'^l \Big) s_l' = 0,$$

so F_U is injective.

On the other hand, if $\gamma = \sum_{k,l} g^{kl} s_k \otimes s'_l$ is an arbitrary holomorphic section of $E \otimes E'$ over U, then

$$\gamma = F_U\Big(\sum_k s_k \otimes \Big(\sum_l g^{kl} s'_l\Big)\Big),$$

so F_U is surjective as well. □

► **Exercise 5.13.** Given a sheaf $\mathscr{R}$ of commutative rings, show that $\mathscr{R}$ acts as an identity under tensor product, in the sense that if $\mathscr{S}$ is any sheaf of $\mathscr{R}$-modules, then $\mathscr{R} \otimes_{\mathscr{R}} \mathscr{S} \cong \mathscr{S} \otimes_{\mathscr{R}} \mathscr{R} \cong \mathscr{S}$.

Suppose $\mathscr{R}$ is a sheaf of commutative rings on a topological space M. A sheaf $\mathscr{S}$ of $\mathscr{R}$-modules is called a ***free sheaf of rank k*** if it is $\mathscr{R}$-module isomorphic to the k-fold direct sum $\mathscr{R}^k = \mathscr{R} \oplus \cdots \oplus \mathscr{R}$. It is called a ***locally free sheaf of rank k*** if every point of M has a neighborhood U on which the restricted sheaf $\mathscr{S}|_U$ is isomorphic to $\mathscr{R}^k|_U$.

The next proposition shows that locally free sheaves of $\mathscr{O}$-modules of fixed finite rank on a complex manifold correspond to holomorphic vector bundles.

Proposition 5.14 (Locally Free Sheaves and Vector Bundles). *Let M be a complex manifold. If $E \to M$ is a holomorphic vector bundle of rank k, then $\mathscr{O}(E)$ is a locally free sheaf of $\mathscr{O}$-modules of rank k. Conversely, if $\mathscr{S}$ is a locally free sheaf of $\mathscr{O}$-modules of rank k on M, then there is a rank-k holomorphic vector bundle $E \to M$, unique up to isomorphism, such that $\mathscr{S} \cong \mathscr{O}(E)$ (as sheaves of $\mathscr{O}$-modules).*

Proof. First let $E \to M$ be a holomorphic vector bundle of rank k. On any open subset $U \subseteq M$ over which E is trivial, we can choose a holomorphic local frame $(s_1, \dots, s_k)$. Define a morphism $\Lambda\colon \mathscr{O}^k|_U \to \mathscr{O}(E)|_U$ by

$$\Lambda_V(f_1, \dots, f_k) = f_1 s_1 + \cdots + f_k s_k \in \mathscr{O}(V; E)$$

for each open subset $V \subseteq U$ and holomorphic functions $f_1, \dots, f_k \in \mathscr{O}(V)$. This is easily verified to be an $\mathscr{O}$-module isomorphism.

Conversely, suppose $\mathscr{S}$ is a locally free sheaf of $\mathscr{O}$-modules of rank k on M. There is an open cover $\{U_\alpha\}_{\alpha \in A}$ of M such that for each $\alpha \in A$ there exists an isomorphism $\Psi_\alpha\colon \mathscr{S}|_{U_\alpha} \cong \mathscr{O}^k|_{U_\alpha}$. Where two such open sets U_α, U_β intersect, we have two isomorphisms of $\mathscr{O}(U_\alpha \cap U_\beta)$-modules:

$$\mathscr{O}(U_\alpha \cap U_\beta)^k \xleftarrow{(\Psi_\beta)_{U_\alpha \cap U_\beta}} \mathscr{S}(U_\alpha \cap U_\beta) \xrightarrow{(\Psi_\alpha)_{U_\alpha \cap U_\beta}} \mathscr{O}(U_\alpha \cap U_\beta)^k.$$

Because the composite map $(\Psi_\alpha)_{U_\alpha \cap U_\beta} \circ (\Psi_\beta)^{-1}_{U_\alpha \cap U_\beta}$ is an $\mathscr{O}(U_\alpha \cap U_\beta)$-module isomorphism, it has the form

$$(f_1, \dots, f_k) \mapsto \big(t_1^j f_j, \dots, t_k^j f_j\big) \tag{5.9}$$

for some invertible $k \times k$ matrix of functions $t_i^j \in \mathcal{O}(U_\alpha \cap U_\beta)$. Let us denote this matrix-valued holomorphic function by $\tau_{\alpha\beta}$. A straightforward computation shows that $\tau_{\alpha\beta}\tau_{\beta\gamma} = \tau_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ whenever the intersection is nonempty. Thus by the vector bundle construction theorem, the collection of functions $\{\tau_{\alpha\beta} : \alpha, \beta \in A\}$ forms the data for a holomorphic vector bundle $E \to M$ that is trivial over each set U_α.

To see that the sheaf $\mathcal{O}(E)$ is isomorphic to $\mathcal{S}$, note that each local trivialization $\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^k$ induces a sheaf isomorphism $\widetilde{\Phi}_\alpha : \mathcal{O}(E)|_{U_\alpha} \to \mathcal{O}^k|_{U_\alpha}$ (the inverse of the isomorphism constructed in the first paragraph of this proof). Where U_α and U_β intersect, these isomorphisms satisfy

$$(5.10) \qquad (\widetilde{\Phi}_\alpha)_{U_\alpha \cap U_\beta} \circ (\widetilde{\Phi}_\beta)^{-1}_{U_\alpha \cap U_\beta}(f_1, \dots, f_k) = \big((\tau_{\alpha\beta})^j_1 f_j, \dots, (\tau_{\alpha\beta})^j_k f_j\big).$$

Thus restricted to $U_\alpha \cap U_\beta$, it follows from (5.9) and (5.10) that

$$(5.11) \qquad (\Psi_\alpha)_{U_\alpha \cap U_\beta} \circ (\Psi_\beta)^{-1}_{U_\alpha \cap U_\beta} = (\widetilde{\Phi}_\alpha)_{U_\alpha \cap U_\beta} \circ (\widetilde{\Phi}_\beta)^{-1}_{U_\alpha \cap U_\beta}.$$

For each α, define a sheaf isomorphism $F_\alpha : \mathcal{O}(E)|_{U_\alpha} \to \mathcal{S}|_{U_\alpha}$ by

$$F_\alpha = \Psi_\alpha^{-1} \circ \widetilde{\Phi}_\alpha.$$

It follows from (5.11) that the restrictions of F_α and F_β agree on $U_\alpha \cap U_\beta$. Thus we can define a sheaf isomorphism $F : \mathcal{O}(E) \to \mathcal{S}$ as follows: for each open set $V \subseteq M$ and section $\sigma \in \mathcal{O}(V; E)$, let $F_V(\sigma) \in \mathcal{S}(V)$ be the section satisfying

$$F_V(\sigma)\big|_{V \cap U_\alpha} = (F_\alpha)_{V \cap U_\alpha}\big(\sigma\big|_{V \cap U_\alpha}\big) \quad \text{for each } \alpha \in A.$$

The gluing and locality properties of $\mathcal{S}$ guarantee that this is well defined.

To show that the bundle E is unique up to isomorphism, it suffices to show that if E and E' are rank-k holomorphic vector bundles over M such that $\mathcal{O}(E) \cong \mathcal{O}(E')$, then $E \cong E'$. Given such bundles E and E', suppose $F : \mathcal{O}(E) \to \mathcal{O}(E')$ is a sheaf isomorphism. We can choose an open cover $\{U_\alpha\}_{\alpha \in A}$ of M such that E and E' are both trivial over each U_α, with local trivializations Φ_α and Φ'_α and transition functions $\tau_{\alpha\beta}$ and $\tau'_{\alpha\beta}$. As above, these isomorphisms lead to sheaf isomorphisms $\widetilde{\Phi}_\alpha : \mathcal{O}(E)|_{U_\alpha} \to \mathcal{O}^k|_{U_\alpha}$ and $\widetilde{\Phi}'_\alpha : \mathcal{O}(E')|_{U_\alpha} \to \mathcal{O}^k|_{U_\alpha}$. For each $\alpha \in A$, we have a composition of $\mathcal{O}(U_\alpha)$-module isomorphisms

$$\mathcal{O}(U_\alpha)^k \xrightarrow{(\widetilde{\Phi}_\alpha)^{-1}_{U_\alpha}} \mathcal{O}(U_\alpha; E) \xrightarrow{(F)_{U_\alpha}} \mathcal{O}(U_\alpha; E') \xrightarrow{(\widetilde{\Phi}'_\alpha)_{U_\alpha}} \mathcal{O}(U_\alpha)^k.$$

As before, each such composition is represented by a matrix-valued holomorphic function $\psi_\alpha : U_\alpha \to \mathrm{GL}(k,\mathbb{C})$, and a straightforward computation shows that these functions satisfy $\tau_{\alpha\beta} = \psi_\alpha^{-1}\tau'_{\alpha\beta}\psi_\beta$ on $U_\alpha \cap U_\beta$, so Proposition 3.7 shows that the corresponding bundles are isomorphic. □

► **Exercise 5.15.** Verify that the same proof shows there are analogous correspondences between locally free sheaves of $\mathscr{E}$-modules and smooth vector bundles on a smooth manifold, and between locally free sheaves of $\mathscr{C}$-modules and topological vector bundles on a topological space.

A locally free sheaf (of $\mathscr{O}$-modules, say) of rank 1 is called an ***invertible sheaf***, reflecting the fact that every such sheaf has an inverse under tensor product: if L is a holomorphic line bundle, then it follows from Proposition 5.12 and Lemma 3.28(d) that $\mathscr{O}(L) \otimes_{\mathscr{O}} \mathscr{O}(L^*) \cong \mathscr{O}(L \otimes L^*) \cong \mathscr{O}$.

In the algebraic geometry literature, it is common to focus primarily on locally free sheaves of $\mathscr{O}$-modules, with vector bundles relegated to the background. If vector bundles are mentioned at all, they are often not distinguished from the corresponding sheaves. In particular, on $\mathbb{CP}^n$, the sheaf $\mathscr{O}(H^d)$ of holomorphic sections of the d-fold tensor power of the hyperplane bundle is usually abbreviated by $\mathscr{O}(d)$, and that shorthand notation is used both for the invertible sheaf of sections and the line bundle H^d itself. From a differential-geometric point of view, however, vector bundles and sheaves are very different objects, and we will continue to use different notations for them.

The next proposition expresses an extremely important relationship between ideal sheaves and invertible sheaves.

Proposition 5.16. *Suppose M is a complex manifold, $S \subseteq M$ is a closed complex hypersurface, $L_S \to M$ is the holomorphic line bundle associated with S, and $E \to M$ is any holomorphic vector bundle.*

(a) *The ideal sheaf $\mathscr{I}_S$ is isomorphic to the sheaf $\mathscr{O}(L_S^*)$ of holomorphic sections of the dual bundle L_S^*.*

(b) *More generally, the sheaf $\mathscr{I}_S(E)$ of holomorphic sections of E vanishing on S is isomorphic to $\mathscr{I}_S \otimes_{\mathscr{O}} \mathscr{O}(E) \cong \mathscr{O}(L_S^* \otimes E)$.*

(c) *The sheaf $\mathscr{I}_S^2(E)$ of holomorphic sections of E that vanish to second order on S is isomorphic to $\mathscr{O}(L_S^* \otimes L_S^* \otimes E)$.*

Proof. Problem 5-10. □

Exact Sequences of Sheaves

Most of the applications of sheaves are based on the notion of *exact sequences*. For abelian groups, rings, or vector spaces, the definition of an ***exact sequence*** is straightforward—it is a sequence of homomorphisms such that the image of each homomorphism is equal to the kernel of the next. But that simple characterization fails to make sense for sheaves.

If $F: \mathscr{S} \to \mathscr{T}$ is a morphism between sheaves of abelian groups, we can define a subsheaf $\operatorname{Ker} F \subseteq \mathscr{S}$, called the ***kernel of F***, by

$$(\operatorname{Ker} F)(U) = \operatorname{Ker} F_U \subseteq \mathscr{S}(U).$$

► **Exercise 5.17.** In the situation described above, show that $\operatorname{Ker} F$ satisfies the gluing and locality properties, and thus is a subsheaf of $\mathscr{S}$.

However, for the *image* of a sheaf morphism, things are not so easy, because the presheaf $U \mapsto \operatorname{Im} F_U$ may fail to be a sheaf, as the following example demonstrates.

Example 5.18. Let M be a smooth n-manifold, and for each nonnegative integer k, let $\mathscr{E}^k$ be the sheaf of smooth k-forms on M. Let $d: \mathscr{E}^k \to \mathscr{E}^{k+1}$ be the sheaf morphism defined by exterior differentiation. The image presheaf of d is the presheaf for which $(\operatorname{Im} d)(U)$ is the space of exact $(k+1)$-forms on U. However, the gluing property might fail for this presheaf: a form that is locally exact need not be globally exact. //

To remedy this problem, we need a special definition of exactness for sequences of sheaf morphisms. Consider sheaves $\mathscr{R}, \mathscr{S}, \mathscr{T}$ of abelian groups on a topological space M together with sheaf morphisms

$$\mathscr{R} \xrightarrow{F} \mathscr{S} \xrightarrow{G} \mathscr{T}. \tag{5.12}$$

The sequence is said to be ***exact*** if it is exact on each stalk: for each $p \in M$, the sequence of induced stalk homomorphisms

$$\mathscr{R}_p \xrightarrow{F_p} \mathscr{S}_p \xrightarrow{G_p} \mathscr{T}_p \tag{5.13}$$

is exact in the usual sense that the image of F_p is equal to the kernel of G_p.

The next exercise shows that exactness on the presheaf level implies exactness on the sheaf level.

► **Exercise 5.19.** Suppose $\mathscr{R}$, $\mathscr{S}$, and $\mathscr{T}$ are presheaves of abelian groups on a topological space M, and $F: \mathscr{R} \to \mathscr{S}$ and $G: \mathscr{S} \to \mathscr{T}$ are morphisms such that the following sequence is exact for each open subset $U \subseteq M$:

$$\mathscr{R}(U) \xrightarrow{F_U} \mathscr{S}(U) \xrightarrow{G_U} \mathscr{T}(U).$$

Prove that the associated sheaf sequence

$$\mathscr{R}^+ \xrightarrow{F^+} \mathscr{S}^+ \xrightarrow{G^+} \mathscr{T}^+$$

is exact.

But sheaf sequences that are not exact on the presheaf level may still be exact as sheaf sequences. It is worth unpacking the definition to see what this means explicitly in practice.

Lemma 5.20. *The sheaf sequence* (5.12) *is exact if and only if both of the following conditions are satisfied:*

(i) $G_U \circ F_U = 0$ *for every open set* $U \subseteq M$*, and*

(ii) *Given* $s \in \mathscr{S}(U)$ *such that* $G_U(s) = 0$*, for each* $p \in U$ *there are an open set* $V \subseteq U$ *containing* p *and a section* $r \in \mathscr{R}(V)$ *such that* $F_V(r) = s|_V$.

Proof. First assume that (i) and (ii) hold. Condition (i) implies that the image of the stalk homomorphism F_p is contained in $\operatorname{Ker} G_p$ for each p. To prove the reverse inclusion, let $[s]_p \in \operatorname{Ker} G_p$ be arbitrary. Then there is some neighborhood U of p and a section s representing this germ such that $G_U(s) = 0 \in \mathscr{T}(U)$. Condition (ii) guarantees the existence of a smaller neighborhood V of p such that $s|_V = F_V(r)$ for some section $r \in \mathscr{R}(V)$. It follows that $F_p([r]_p) = [F_V(r)]_p = [s]_p$, so $\operatorname{Ker} G_p \subseteq \operatorname{Im} F_p$.

Conversely, assume the sequence (5.12) is exact. Suppose $r \in \mathscr{R}(U)$ for some open set U. Then for each $p \in U$, the germ of $G_U \circ F_U(r)$ at p represents an element of $G_p \circ F_p(\mathscr{R}_p)$, which is zero. This means $G_U \circ F_U(r)$ restricts to zero in a neighborhood of each point, so by the locality property it is zero. This shows that (i) holds. The proof of (ii) is left as an exercise. □

► **Exercise 5.21.** Complete the proof of this lemma by showing that if the sequence (5.12) is exact, then condition (ii) holds.

An important special case is a ***short exact sequence*** of sheaves: this is a five-term exact sequence of the form

$$0 \to \mathscr{R} \xrightarrow{F} \mathscr{S} \xrightarrow{G} \mathscr{T} \to 0, \tag{5.14}$$

where the zeros on the ends represent the ***trivial sheaf***, whose spaces of sections and restriction maps are all zero.

Lemma 5.20 shows that exactness at $\mathscr{R}$ means that whenever $r \in \mathscr{R}(U)$ is in the kernel of F_U, it restricts to zero in a neighborhood of each point; but in this case, since there is only one zero section, it follows that $F_U : \mathscr{R}(U) \to \mathscr{S}(U)$ is injective for each U. If this is the case, we say the sheaf morphism F is ***injective***. On the other hand, exactness at $\mathscr{T}$ means that given $t \in \mathscr{T}(U)$, for each $p \in U$ there is a neighborhood V of p contained in U and a section $s \in \mathscr{S}(V)$ such that $G_V(s) = t|_V$; in this case, we say the sheaf morphism G is ***surjective***.

Proposition 5.22. *Suppose M is a topological space.*

(a) *If $\mathscr{S}$ is a sheaf of abelian groups on M and $\mathscr{R}$ is a subsheaf of $\mathscr{S}$, there is a canonical sheaf morphism $\Pi : \mathscr{S} \to \mathscr{S}/\mathscr{R}$ such that the following sheaf sequence is exact:*

$$0 \to \mathscr{R} \overset{i}{\hookrightarrow} \mathscr{S} \overset{\Pi}{\longrightarrow} \mathscr{S}/\mathscr{R} \to 0, \tag{5.15}$$

where i is the inclusion of $\mathscr{R}$ into $\mathscr{S}$.

(b) *For any short exact sequence of sheaf morphisms* (5.14) *between sheaves of abelian groups on M, F is an isomorphism onto* $\operatorname{Ker} G$, *and there is a sheaf isomorphism $\overline{G} : \mathscr{S}/\operatorname{Ker} G \to \mathscr{T}$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{R} & \overset{F}{\longrightarrow} & \mathscr{S} & \overset{G}{\longrightarrow} & \mathscr{T} & \longrightarrow & 0. \\ & & & & & {\scriptstyle \Pi}\searrow & \uparrow {\scriptstyle \overline{G}} & & \\ & & & & & & \mathscr{S}/\operatorname{Ker} G & & \end{array} \tag{5.16}$$

Proof. Given a subsheaf $\mathscr{R} \hookrightarrow \mathscr{S}$, let $\mathscr{P}$ be the presheaf $\mathscr{P}(U) = \mathscr{S}(U)/\mathscr{R}(U)$, whose sheafification is the quotient sheaf $\mathscr{S}/\mathscr{R}$; and let $\pi : \mathscr{S} \to \mathscr{P}$ be the presheaf morphism given by the quotient maps $\pi_U : \mathscr{S}(U) \to \mathscr{S}(U)/\mathscr{R}(U)$. We define the sheaf morphism $\Pi : \mathscr{S} \to \mathscr{S}/\mathscr{R}$ by $\Pi = \theta_{\mathscr{P}} \circ \pi$, where $\theta_{\mathscr{P}} : \mathscr{P} \to \mathscr{S}/\mathscr{R}$ is the canonical presheaf morphism given by Theorem 5.9.

The sequence (5.15) is exact at $\mathscr{R}$ because i is injective. For exactness at $\mathscr{S}$, we will use Lemma 5.20. Note first that $\pi \circ i = 0$, which implies $\Pi \circ i = 0$, so condition (i) of the lemma is satisfied. On the other hand, given $s \in \mathscr{S}(U)$ such that $\Pi_U(s) = 0$, this means that for each $p \in U$ there is a neighborhood V of p such that $s|_V \in \mathscr{R}(V)$, so condition (ii) is satisfied.

Finally, exactness at $\mathscr{S}/\mathscr{R}$ is just the fact that each element ξ of the stalk $(\mathscr{S}/\mathscr{R})_p$ is represented on some neighborhood V of p by some section $s \in \mathscr{S}(V) \bmod \mathscr{R}(V)$, and it follows that $\Pi_p([s]_p) = \xi$. This completes the proof of (a).

To prove the first claim in (b), let $\mathscr{K}$ denote the subsheaf $\operatorname{Ker} G \subseteq \mathscr{S}$. For each open set $U \subseteq M$, Lemma 5.20 shows that $F_U(\mathscr{R}(U)) \subseteq \mathscr{K}(U)$, and the discussion in the paragraph preceding the statement of the proposition showed that F_U is injective. On the other hand, given $k \in \mathscr{K}(U)$, each $p \in U$ has a neighborhood $V \subseteq U$ on which $k|_V = F_V(r)$ for some $r \in \mathscr{R}(V)$. The collection of all such neighborhoods is an open cover of U, and wherever two such sets V, V' overlap and r, r' are the corresponding sections, we have

$$\begin{aligned} F_{V \cap V'}\big(r|_{V \cap V'}\big) &= F_V(r)\big|_{V \cap V'} = k\big|_{V \cap V'} = F_{V'}(r')\big|_{V \cap V'} \\ &= F_{V \cap V'}\big(r'|_{V \cap V'}\big), \end{aligned}$$

and injectivity of $F_{V\cap V'}$ shows that $r|_{V\cap V'} = r'|_{V\cap V'}$. Thus these sections glue together to form a section $\tilde{r} \in \mathscr{R}(U)$ satisfying $F_U(\tilde{r}) = k$, showing that the homomorphism $F_U : \mathscr{R}(U) \to \mathscr{K}(U)$ is also surjective.

To prove the second claim in (b), note first that the existence of the morphism $\overline{G}$ making (5.16) commute follows from the result of Problem 5-8(b) (identifying $\mathscr{T}$ with the quotient sheaf $\mathscr{T}/0$). We will show that for each open subset $U \subseteq M$, the group homomorphism $\overline{G}_U : (\mathscr{S}/\mathscr{K})(U) \to \mathscr{T}(U)$ is bijective. From the definition of the sheafification functor, for any sections $\sigma \in (\mathscr{S}/\mathscr{K})(U)$ and $\tau \in \mathscr{T}(U)$, the equation $\overline{G}_U(\sigma) = \tau$ means that for each $p \in U$ there exist a neighborhood V_p and a section $s_p \in \mathscr{S}(V_p)/\mathscr{K}(V_p)$ such that $\overline{G}_{V_p}(s_p) = \tau|_{V_p}$. To see that $\overline{G}_U$ is injective, if $\overline{G}_U(\sigma) = 0$, then because each $\overline{G}_{V_p}$ is injective on the quotient group $\mathscr{S}(V_p)/\mathscr{K}(V_p)$, we see that σ restricts to zero on each V_p and thus is zero itself. To see that $\overline{G}_U$ is surjective, given $\tau \in \mathscr{T}(U)$, the surjectivity of G means that for each $p \in U$ we can choose a neighborhood V_p small enough that there exists $s_p \in \mathscr{S}(V_p)$ with $G_{V_p}(s_p) = \tau|_{V_p}$, and any two such sections agree modulo $\mathscr{K}(V_p)$; thus the local sections can be glued together to form a section $\sigma \in (\mathscr{S}/\mathscr{K})(U)$ satisfying $\overline{G}_U(\sigma) = \tau$. □

Example 5.23 (Exact Sheaf Sequences).

(a) On an n-dimensional smooth manifold M, we have a sequence of sheaves of complex vector spaces

$$0 \to \underline{\mathbb{C}} \hookrightarrow \mathscr{E}^0 \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{E}^n \to 0. \tag{5.17}$$

It is an exact sheaf sequence because every closed form is locally in the image of d.

(b) For each $k \geq 0$, we can extract a short exact sheaf sequence from the one above:

$$0 \to \mathscr{Z}^k \hookrightarrow \mathscr{E}^k \xrightarrow{d} \mathscr{Z}^{k+1} \to 0, \tag{5.18}$$

where $\mathscr{Z}^k$ and $\mathscr{Z}^{k+1}$ are the sheaves of closed forms.

(c) On a complex n-manifold M, thanks to the $\bar{\partial}$-Poincaré lemma, for each p we have an exact sheaf sequence associated with the Dolbeault complex:

$$0 \to \Omega^p \hookrightarrow \mathscr{E}^{p,0} \xrightarrow{\bar{\partial}} \mathscr{E}^{p,1} \xrightarrow{\bar{\partial}} \mathscr{E}^{p,2} \to \cdots \to \mathscr{E}^{p,n} \to 0, \tag{5.19}$$

where Ω^p is the sheaf of holomorphic p-forms (i.e., $\bar{\partial}$-closed $(p,0)$-forms). As before, we can extract short exact sequences:

$$0 \to \mathscr{Z}^{p,q} \hookrightarrow \mathscr{E}^{p,q} \xrightarrow{\bar{\partial}} \mathscr{Z}^{p,q+1} \to 0.$$

More generally, if $E \to M$ is a holomorphic vector bundle, then we have an exact sheaf sequence of bundle-valued (p,q)-forms:

$$0 \to \Omega^p(E) \hookrightarrow \mathscr{E}^{p,0}(E) \xrightarrow{\bar\partial_E} \mathscr{E}^{p,1}(E) \xrightarrow{\bar\partial_E} \mathscr{E}^{p,2}(E) \to \cdots \to \mathscr{E}^{p,n}(E) \to 0.$$

(d) On a complex manifold M, we have the following sequence of sheaves of abelian groups:

(5.20) $$0 \to \underline{\mathbb{Z}} \overset{\iota}{\hookrightarrow} \mathscr{O} \xrightarrow{\varepsilon} \mathscr{O}^* \to 0,$$

where $\underline{\mathbb{Z}}$ is the constant sheaf, ι is inclusion of integer-valued functions as holomorphic functions, and $\varepsilon(f) = e^{2\pi i f}$. This is called the ***exponential sheaf sequence***. It is immediate that ι is injective and $\varepsilon \circ \iota(f) \equiv 1$. (By custom, we denote the right-hand trivial sheaf by 0, even though $\mathscr{O}^*$ is a sheaf of groups under multiplication whose identity section is more naturally written as 1.) To check exactness at $\mathscr{O}$, just note that if $e^{2\pi i f} = 1$ on an open set U, then f must be constant and integer-valued on each connected component of U. Surjectivity at $\mathscr{O}^*$ follows from the fact that each point of M has a neighborhood on which every nonvanishing holomorphic function has a holomorphic logarithm (see Problem 5-11).

(e) On a smooth manifold, there is a sheaf sequence analogous to the one above, the ***smooth exponential sheaf sequence***:

$$0 \to \underline{\mathbb{Z}} \overset{\iota}{\hookrightarrow} \mathscr{E} \xrightarrow{\varepsilon} \mathscr{E}^* \to 0,$$

where $\mathscr{E}$ and $\mathscr{E}^*$ are the sheaves of smooth complex-valued functions and nonvanishing smooth complex-valued functions, respectively. It is exact by the same argument.

(f) Suppose M is a complex manifold and $p \in M$. We have a short exact sheaf sequence

(5.21) $$0 \to \mathscr{I}_{\{p\}} \hookrightarrow \mathscr{O} \xrightarrow{e} \mathbb{C}_p \to 0,$$

where $\mathscr{I}_{\{p\}}$ is the ideal sheaf of $\{p\}$ (the sheaf of holomorphic functions that vanish at p), $\mathbb{C}_p$ is the skyscraper sheaf whose stalk at p is $\mathbb{C}$ and all other stalks are zero, and e is the evaluation map $e_U(f) = f(p)$ if $p \in U$ and 0 otherwise. //

The basic fact about sheaves is that even if a sheaf sequence is exact, the local nature of sheaf exactness means that exactness does not necessarily hold on spaces of sections. For example, in the sheaf sequence (5.17), if we look at the spaces of global sections, we find a sequence of vector spaces and linear maps

(5.22) $$0 \to \mathscr{E}^0(M) \xrightarrow{d} \mathscr{E}^1(M) \xrightarrow{d} \mathscr{E}^2(M) \to \cdots \to \mathscr{E}^n(M) \to 0.$$

This sequence is not exact, but it is a cochain complex. Its cohomology groups are the ***de Rham cohomology groups with complex coefficients***, which we denote by $H^k_{\mathrm{dR}}(M;\mathbb{C})$. To avoid confusion, we will denote ordinary (real) de Rham cohomology by $H^k_{\mathrm{dR}}(M;\mathbb{R})$. Because conjugation of complex differential forms commutes with the exterior derivative operator, conjugation descends to a conjugate-linear automorphism of each complex de Rham cohomology group. Thus we can view $H^k_{\mathrm{dR}}(M;\mathbb{R})$ as a real-linear subspace of $H^k_{\mathrm{dR}}(M;\mathbb{C})$, namely the space of cohomology classes that are invariant under conjugation, and $H^k_{\mathrm{dR}}(M;\mathbb{C})$ is isomorphic to the complexification of $H^k_{\mathrm{dR}}(M;\mathbb{R})$. It follows that the complex dimension of $H^k_{\mathrm{dR}}(M;\mathbb{C})$ is equal to the real dimension of $H^k_{\mathrm{dR}}(M;\mathbb{R})$, namely the kth Betti number $b^k(M)$.

We can also look at the global section sequence associated with the short exact sequence (5.18):

$$0 \to \mathscr{Z}^k(M) \hookrightarrow \mathscr{E}^k(M) \xrightarrow{d} \mathscr{Z}^{k+1}(M).$$

This global section sequence is exact as far as it goes, as you can check (or see Prop. 5.24 below), but the last homomorphism need not be surjective, which is why we have left off the last zero. In fact, the question of when d is surjective onto the space of closed forms, and if not, how to quantify the failure of exactness, is exactly the question addressed by de Rham cohomology.

For another example, on a complex manifold M, let p, q be distinct points of M and consider the following variation on the sequence (5.21):

$$0 \to \mathscr{I}_{\{p,q\}} \hookrightarrow \mathscr{O} \xrightarrow{e} \mathbb{C}_p \oplus \mathbb{C}_q \to 0,$$

where now e evaluates a function at the two points p and q. (The sheaf $\mathbb{C}_p \oplus \mathbb{C}_q$ is a "double skyscraper sheaf": it has two nontrivial stalks and all the rest are zero.) When we examine the global section maps, we find

$$0 \to \mathscr{I}_{\{p,q\}}(M) \hookrightarrow \mathscr{O}(M) \xrightarrow{e_M} \mathbb{C} \oplus \mathbb{C},$$

where $e_M(f) = (f(p), f(q))$. Again, you can check that this sequence is exact as far as it goes, but the question of whether e_M is surjective for all p and q is exactly the question of whether $\mathscr{O}(M)$ separates points, a key requirement for M to be a Stein manifold. A variant of this sheaf sequence will play a central role in our proof of the Kodaira embedding theorem in Chapter 10.

In the last two examples, global section sequences associated with short exact sheaf sequences failed to be exact, but it was only surjectivity at the last term that failed. The next proposition shows that this pattern is quite general.

Proposition 5.24. *On a topological space M, suppose the following is an exact sequence of sheaves of abelian groups*

$$0 \to \mathscr{R} \xrightarrow{F} \mathscr{S} \xrightarrow{G} \mathscr{T} \to 0. \tag{5.23}$$

Then for each open subset $U \subseteq M$, the section sequence

$$0 \to \mathscr{R}(U) \xrightarrow{F_U} \mathscr{S}(U) \xrightarrow{G_U} \mathscr{T}(U) \tag{5.24}$$

is exact.

Proof. As we noted above, exactness of (5.23) at $\mathscr{R}$ implies that $F_U : \mathscr{R}(U) \to \mathscr{S}(U)$ is injective for all U, so (5.24) is exact at $\mathscr{R}(U)$. To prove exactness at $\mathscr{S}(U)$, note that Proposition 5.22 showed that F is an isomorphism from $\mathscr{R}$ onto the subsheaf $\operatorname{Ker} G \subseteq \mathscr{S}$; in particular, this means that $F_U(\mathscr{R}(U)) = \operatorname{Ker}(G_U)$. □

In the next chapter, we will develop some powerful tools to help determine when the global section sequence associated with a short exact sheaf sequence is exact.

Most of the sheaves that arise in complex geometry are of three types: constant sheaves, which carry topological information (see, for example, Thm. 6.18 below); sheaves of $\mathscr{O}$-modules (called ***analytic sheaves***), which carry information about the holomorphic structure (e.g., Thm. 6.19); and sheaves of $\mathscr{E}$-modules, which provide a crucial tool for deriving properties of the other two types (e.g., Thm. 6.11). The analytic sheaves that we will be able to say the most about are the locally free ones, that is, the sheaves of sections of holomorphic vector bundles.

It is worth remarking that there is a generalization of locally free sheaves that turns out to be extremely important in algebraic geometry and analysis of several complex variables, because sheaves of this type have many of the same properties as locally free sheaves. An analytic sheaf $\mathscr{S}$ on a complex manifold M is said to be ***coherent*** if it is ***locally finitely generated***, meaning each point of M has a neighborhood U and finitely many sections $s_1, \dots, s_m \in \mathscr{S}(U)$ that generate each stalk $\mathscr{S}_x$ as an $\mathscr{O}_x$-module for $x \in U$; and given any such sections, the kernel of the sheaf morphism $\mathscr{O}^m|_U \to \mathscr{S}|_U$ given by $(f_1, \dots, f_m) \mapsto \sum_i f_i s_i$ is also locally finitely generated (where, as before, $\mathscr{O}^m = \mathscr{O} \oplus \cdots \oplus \mathscr{O}$). Locally free analytic sheaves are coherent, as are ideal sheaves associated with analytic or algebraic varieties (see [**GH94**, pp. 695–704]). We will not have occasion to use coherent analytic sheaves in this book.

Problems

5-1. Suppose $\mathscr{S}$ is a sheaf of abelian groups and $\mathscr{R} \hookrightarrow \mathscr{S}$ is a subsheaf. Show that each stalk homomorphism $\mathscr{R}_p \to \mathscr{S}_p$ is injective, so it makes sense to identify $\mathscr{R}_p$ with a subgroup of $\mathscr{S}_p$.

5-2. Suppose $F : \mathscr{S} \to \mathscr{T}$ is a morphism between sheaves of abelian groups over a topological space M. Show that F is an isomorphism if and only if each stalk homomorphism F_p is bijective.

5-3. Let M be a topological space and G an abelian group. Define the ***constant presheaf on M with coefficients in G*** to be the presheaf $\mathscr{G}$ for which $\mathscr{G}(U) = G$ for every open set U, and all restriction maps are the identity. Show that the sheafification of $\mathscr{G}$ is isomorphic to the constant sheaf $\underline{G}$.

5-4. Suppose $\mathscr{S} \to M$ is an étalé space, $U \subseteq M$ is open, and $\sigma, \tau : U \to \mathscr{S}$ are continuous sections. Show that the set $\{x \in U : \sigma(x) = \tau(x)\}$ is open. If U is connected and $\sigma(p) = \tau(p)$ for some $p \in M$, does this imply that $\sigma \equiv \tau$ on U?

5-5. Prove Proposition 5.10 (the étalé space functor is an equivalence of categories).

5-6. (a) Let $\mathscr{C}$ be the sheaf of continuous real-valued functions on $\mathbb{R}$. Show that the étalé space $\operatorname{Et}(\mathscr{C})$ is not Hausdorff.
(b) Let $\mathscr{O}$ be the sheaf of holomorphic functions on a complex manifold. Show that $\operatorname{Et}(\mathscr{O})$ is Hausdorff.

5-7. Suppose $\mathscr{S}$ and $\mathscr{T}$ are sheaves of abelian groups on a topological space M. Define a presheaf $\mathscr{H}om(\mathscr{S}, \mathscr{T})$ by letting $\mathscr{H}om(\mathscr{S}, \mathscr{T})(U)$ be the group of sheaf morphisms from $\mathscr{S}|_U$ to $\mathscr{T}|_U$. Show that $\mathscr{H}om(\mathscr{S}, \mathscr{T})$ is a sheaf of abelian groups on M.

5-8. Suppose $\mathscr{S}$, $\mathscr{S}'$ are sheaves of abelian groups on a topological space M and $\mathscr{R} \hookrightarrow \mathscr{S}$, $\mathscr{R}' \hookrightarrow \mathscr{S}'$ are subsheaves.
(a) Show that each stalk $(\mathscr{S}/\mathscr{R})_p$ is canonically isomorphic to the quotient group $\mathscr{S}_p/\mathscr{R}_p$, where $\mathscr{R}_p$ is considered as a subgroup of $\mathscr{S}_p$ as in Exercise 5.6.
(b) Show that if $F : \mathscr{S} \to \mathscr{S}'$ is a morphism that takes $\mathscr{R}$ to $\mathscr{R}'$, then F passes to the quotient to yield a morphism $\overline{F} : \mathscr{S}/\mathscr{R} \to \mathscr{S}'/\mathscr{R}'$.

5-9. Let M be a complex manifold and let $L \to M$ be a holomorphic line bundle that admits no nontrivial global holomorphic sections (such as the tautological bundle on $\mathbb{CP}^n$). Show that the presheaf $\mathscr{T}$ defined by $\mathscr{T}(U) = \mathscr{O}(U; L) \otimes_{\mathscr{O}(U)} \mathscr{O}(U; L^*)$ is not a sheaf because it does not satisfy the gluing property.

5-10. Prove Proposition 5.16 (relating ideal sheaves and sheaves of sections of line bundles).

5-11. Suppose $U \subseteq \mathbb{C}^n$ is a simply connected open set and $f : U \to \mathbb{C}$ is a nonvanishing smooth function. Prove that there is a smooth function $L : U \to \mathbb{C}$ such that $f(z) = e^{L(z)}$ for all $z \in U$; and show that L is holomorphic if f is. [Hint: Consider the closed 1-form $\omega = df/f$.]

5-12. Let X and Y be topological spaces and $f : X \to Y$ a continuous map. For any sheaf $\mathscr{S}$ on X (of abelian groups, say), define a presheaf $f_*\mathscr{S}$ on Y by $f_*\mathscr{S}(U) = \mathscr{S}(f^{-1}(U))$; and define restriction maps $r_{*V}^{\;U} : f_*\mathscr{S}(U) \to f_*\mathscr{S}(V)$ by $r_{*V}^{\;U} = r_{f^{-1}(V)}^{f^{-1}(U)}$.

(a) Show that $f_*\mathscr{S}$ is a sheaf of abelian groups on Y, called a ***direct image sheaf***.

(b) Show that for each sheaf morphism $A : \mathscr{S} \to \mathscr{S}'$ between sheaves on X, there is a morphism $A_* : f_*\mathscr{S} \to f_*\mathscr{S}'$, so that the assignment $\mathscr{S} \mapsto f_*\mathscr{S}$, $A \mapsto A_*$ is a covariant functor from the category of sheaves of abelian groups on X to the analogous category on Y.

(c) Show that if f is a constant map, then $f_*\mathscr{S}$ is a skyscraper sheaf whose nontrivial stalk is the space $\mathscr{S}(X)$ of global sections.

5-13. Let X, Y, and f be as in Problem 5-12. In addition to the operation taking sheaves on X to direct image sheaves on Y, there is a reverse operation taking sheaves on Y to sheaves on X, but it is a little less straightforward to define. Given a sheaf $\mathscr{T}$ of abelian groups on Y, let $\pi : \operatorname{Et}(\mathscr{T}) \to Y$ be its étalé space, let $\mathscr{T}^\# \subseteq X \times \operatorname{Et}(\mathscr{T})$ be the following fiber product:

$$\mathscr{T}^\# = \{(x,\tau) \in X \times \operatorname{Et}(\mathscr{T}) : f(x) = \pi(\tau)\}.$$

(a) Show that with the subspace topology and the projection $\pi_1 : \mathscr{T}^\# \to X$ given by $\pi_1(x,\tau) = x$, $\mathscr{T}^\#$ is an étalé space of abelian groups on X. Its sheaf of sections, denoted by $f^{-1}\mathscr{T}$, is called an ***inverse image sheaf***.

(b) Show that for each sheaf morphism $A : \mathscr{T} \to \mathscr{T}'$ between sheaves on Y, it is possible to assign a morphism $f^{-1}A : f^{-1}\mathscr{T} \to f^{-1}\mathscr{T}'$ in such a way as to define a covariant functor from sheaves on Y to sheaves on X.

(c) Show that if $f : U \to Y$ is inclusion of an open subset, then $f^{-1}\mathscr{T}$ is isomorphic to the restriction $\mathscr{T}|_U$ defined by (5.3).

(d) Show that if $f : X \to Y$ is a constant map with value $c \in Y$, then $f^{-1}\mathscr{T}$ is the constant sheaf $\underline{\mathscr{T}_c}$, where $\mathscr{T}_c$ is the stalk of $\mathscr{T}$ at c.

10.1090/gsm/244/06

Chapter 6

Sheaf Cohomology

As the discussion in the previous chapter suggests, an important question in the theory of sheaves is ascertaining when the global section sequence associated with an exact sheaf sequence is exact, and if not, how to characterize the failure of exactness. In this chapter, we introduce a powerful machine called *sheaf cohomology* that can answer questions like these and many more. There is a considerable amount of technical work that has to be done to establish the necessary results; but the work will pay off richly when we start seeing applications of the theory to complex manifolds.

Definitions

As with many aspects of this subject, there are various definitions of sheaf cohomology available. The construction we will give is called *Čech cohomology* after the early twentieth century Czech mathematician Eduard Čech, who introduced the main ideas behind it in 1932 [**Čec32**]. It is not the most general construction, because it only behaves well on paracompact Hausdorff spaces; but that topological restriction is not an issue for us because we only need to consider sheaves on manifolds, and the Čech construction is supremely well suited to our purposes. At the end of the chapter, we will explain its relationship with other sheaf cohomology theories.

Suppose M is a topological space and $\mathscr{S}$ is a sheaf of abelian groups on M. Given an indexed open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of M and a nonnegative integer p, a ***p-cochain on $\mathcal{U}$ with coefficients in $\mathscr{S}$*** (sometimes called a ***Čech cochain*** to distinguish it from the singular cochains used in algebraic topology) is an operator c that assigns to every multi-index $(\alpha_0, \dots, \alpha_p)$ of length $(p+1)$ a section $c_{\alpha_0\dots\alpha_p} \in \mathscr{S}(U_{\alpha_0}\cap\dots\cap U_{\alpha_p})$. (Note that this means $c_{\alpha_0\dots\alpha_p} = 0$ whenever $U_{\alpha_0}\cap\dots\cap U_{\alpha_p} = \varnothing$.) The ***pth cochain group on $\mathcal{U}$ with coefficients in $\mathscr{S}$*** is the set $C^p(\mathcal{U};\mathscr{S})$ of all such cochains, which is an abelian group under the addition operation defined by

$$(c+c')_{\alpha_0\dots\alpha_p} = c_{\alpha_0\dots\alpha_p} + c'_{\alpha_0\dots\alpha_p}.$$

If $\mathscr{S}$ is a sheaf of real or complex vector spaces, then $C^p(\mathscr{U};\mathscr{S})$ is also a vector space with the obvious scalar multiplication. (This notation $C^p(\mathscr{U};\mathscr{S})$, with arguments consisting of an open cover and a sheaf, should not be confused with notations like $C^p(M,N)$ for spaces of maps of class C^p between manifolds.)

The ***coboundary operator*** is the homomorphism $\delta\colon C^p(\mathscr{U};\mathscr{S}) \to C^{p+1}(\mathscr{U};\mathscr{S})$ defined by

$$(\delta c)_{\alpha_0\dots\alpha_{p+1}} = \sum_{j=0}^{p+1}(-1)^j c_{\alpha_0\dots\widehat{\alpha_j}\dots\alpha_{p+1}}\Big|_{U_0\cap\dots\cap U_{p+1}}.$$

Lemma 6.1. *With δ defined as above, $\delta\circ\delta = 0$.*

Proof. We compute

$$\begin{aligned}(\delta\delta c)_{\alpha_0\dots\alpha_{p+2}} &= \sum_{\substack{j\\ 0\le j\le p+1}} (-1)^j(\delta c)_{\alpha_0\dots\widehat{\alpha_j}\dots\alpha_{p+2}}\Big|_{U_0\cap\dots\cap U_{p+2}}\\ &= \sum_{\substack{j,k\\ 0\le k<j\le p+1}} (-1)^{j+k} c_{\alpha_0\dots\widehat{\alpha_k}\dots\widehat{\alpha_j}\dots\alpha_{p+1}}\Big|_{U_0\cap\dots\cap U_{p+2}}\\ &\quad+ \sum_{\substack{j,k\\ 0\le j<k\le p+1}} (-1)^{j+k-1} c_{\alpha_0\dots\widehat{\alpha_j}\dots\widehat{\alpha_k}\dots\alpha_{p+1}}\Big|_{U_0\cap\dots\cap U_{p+2}},\end{aligned}$$

where the sign in the last sum reflects the fact that when $k > j$, the index α_k is in position $k-1$ in $(\delta c)_{\alpha_0\dots\widehat{\alpha_j}\dots\alpha_{p+2}}$. After interchanging the dummy indices j and k in the last sum, we see that these two sums exactly cancel each other. □

(Those who have studied simplicial cohomology might recognize these formulas as being close kin to the formula for the coboundary operator in simplicial cohomology. This connection is explained in Problem 6-10.)

Lemma 6.1 shows that the cochain groups fit together in a cochain complex:

$$0 \to C^0(\mathscr{U};\mathscr{S}) \xrightarrow{\delta} C^1(\mathscr{U};\mathscr{S}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^p(\mathscr{U};\mathscr{S}) \xrightarrow{\delta} C^{p+1}(\mathscr{U};\mathscr{S}) \to \cdots.$$

The cohomology groups of this complex are called the ***Čech cohomology groups on $\mathscr{U}$ with coefficients in $\mathscr{S}$***:

$$H^p(\mathscr{U};\mathscr{S}) = \frac{\operatorname{Ker}\big(\delta\colon C^p(\mathscr{U};\mathscr{S}) \to C^{p+1}(\mathscr{U};\mathscr{S})\big)}{\operatorname{Im}\big(\delta\colon C^{p-1}(\mathscr{U};\mathscr{S}) \to C^p(\mathscr{U};\mathscr{S})\big)}$$

(where we interpret $C^p(\mathcal{U};\mathcal{S})$ to be zero when $p < 0$). In general these are abelian groups; if $\mathcal{S}$ is a sheaf of (real or complex) vector spaces, then each $H^p(\mathcal{U};\mathcal{S})$ is a vector space.

A cochain $c \in C^p(\mathcal{U};\mathcal{S})$ satisfying $\delta c = 0$ is called a ***(Čech) cocycle***, and one satisfying $c = \delta b$ for some $b \in C^{p-1}(\mathcal{U};\mathcal{S})$ is called a ***(Čech) coboundary***. We denote the group of cocycles on $\mathcal{U}$ by $Z^p(\mathcal{U};\mathcal{S})$, and the group of coboundaries by $B^p(\mathcal{U};\mathcal{S})$, so we can also write $H^p(\mathcal{U};\mathcal{S}) = Z^p(\mathcal{U};\mathcal{S})/B^p(\mathcal{U};\mathcal{S})$. The equivalence class of a cocycle c in this quotient is denoted by $[c]$ and is called a ***cohomology class***; two cocycles that determine the same cohomology class are said to be ***cohomologous***.

This is already rather abstract, so let us look at a few examples before proceeding.

Example 6.2 (Čech Cocycles in Degree 0). Suppose $\mathcal{S}$ is a sheaf of abelian groups on a topological space M and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ is an indexed open cover of M. A 0-cochain c on $\mathcal{U}$ just assigns a section $c_\alpha \in \mathcal{S}(U_\alpha)$ for each α. The coboundary of such a cochain is

$$(\delta c)_{\alpha\beta} = c_\beta\big|_{U_\alpha\cap U_\beta} - c_\alpha\big|_{U_\alpha\cap U_\beta}.$$

Since there are no (-1)-cochains, there are no nontrivial 0-coboundaries, and thus $H^0(\mathcal{U};\mathcal{S})$ is equal to the space $Z^0(\mathcal{U};\mathcal{S})$ of cocycles.

One way to obtain such a cocycle is to start with a global section $\sigma \in \mathcal{S}(M)$, and define a 0-cochain c by $c_\alpha = \sigma|_{U_\alpha}$ for each α. It follows automatically from the properties of the sheaf restriction maps that c is a cocycle, so this defines a map $I_{\mathcal{U},\mathcal{S}} : \mathcal{S}(M) \to H^0(\mathcal{U};\mathcal{S})$, which is a homomorphism of whatever algebraic structures $\mathcal{S}(M)$ and $H^0(\mathcal{U};\mathcal{S})$ are endowed with. Injectivity of $I_{\mathcal{U},\mathcal{S}}$ follows from the locality property of sheaves, and surjectivity from the gluing property, so $I_{\mathcal{U},\mathcal{S}}$ is an isomorphism. Thus $H^0(\mathcal{U};\mathcal{S})$ is always isomorphic to the space $\mathcal{S}(M)$ of global sections of $\mathcal{S}$. //

Example 6.3 (Čech Cocycles in Degree 1). Let M, $\mathcal{S}$, and $\mathcal{U}$ be as in the preceding example. A 1-cochain on $\mathcal{U}$ is a choice of a section $c_{\alpha\beta} \in \mathcal{S}(U_\alpha \cap U_\beta)$ for each pair of indices α, β. The coboundary operator is

$$(\delta c)_{\alpha\beta\gamma} = c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

(where the phrase "on $U_\alpha\cap U_\beta\cap U_\gamma$" should be interpreted to mean that each section is restricted to that set before performing the addition and subtraction). Thus c is a cocycle if and only if $c_{\alpha\gamma} = c_{\alpha\beta} + c_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for all α, β, γ. Two such cocycles c and c' are cohomologous if and only if there is a 0-cochain b such that

$$c_{\alpha\beta} - c'_{\alpha\beta} = b_\beta - b_\alpha \quad \text{on } U_\alpha \cap U_\beta.$$

If $\mathcal{S}$ is a sheaf of abelian groups written multiplicatively, we would write

$$(\delta c)_{\alpha\beta\gamma} = c_{\beta\gamma}c_{\alpha\gamma}^{-1}c_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma,$$

so the formula for c to be a cocycle would be

$$c_{\alpha\beta}c_{\beta\gamma} = c_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

Two 1-cocycles c and c' are cohomologous if and only if there is a 0-cochain b such that

$$c_{\alpha\beta} = c'_{\alpha\beta} b_\beta b_\alpha^{-1} \quad \text{on } U_\alpha \cap U_\beta.$$

This might look remarkably similar to the isomorphism criterion for line bundles in terms of their transition functions; that is not an accident, as we will see later in this chapter. //

Example 6.4 (1-Cocycles of Closed Forms). Suppose M is a smooth manifold and ω is a smooth closed k-form on M. By the Poincaré lemma, we can find an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ such that on each U_α there is a smooth $(k-1)$-form η_α satisfying $\omega|_{U_\alpha} = d\eta_\alpha$. These $(k-1)$-forms might not agree where they overlap, so for each α and β we can define a $(k-1)$-form $\gamma_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ by

$$\gamma_{\alpha\beta} = \eta_\beta\big|_{U_\alpha\cap U_\beta} - \eta_\alpha\big|_{U_\alpha\cap U_\beta}.$$

Since $d\eta_\beta = \omega = d\eta_\alpha$ on the intersection of their domains, $\gamma_{\alpha\beta}$ is a closed form, and thus this assignment defines a 1-cochain $\gamma \in C^1(\mathcal{U};\mathscr{Z}^{k-1})$, where as before $\mathscr{Z}^{k-1}$ is the sheaf of closed $(k-1)$-forms on M. It follows immediately from the definition of γ that $\delta\gamma = 0$, so γ is actually a 1-cocycle with coefficients in $\mathscr{Z}^{k-1}$. It is a coboundary if and only if there is a collection of *closed* $(k-1)$-forms $\sigma_\alpha \in \mathscr{Z}^{k-1}(U_\alpha)$ such that

$$\gamma_{\alpha\beta} = \sigma_\beta - \sigma_\alpha \quad \text{on } U_\alpha \cap U_\beta,$$

which is to say

$$\sigma_\beta - \sigma_\alpha = \eta_\beta - \eta_\alpha \quad \text{on } U_\alpha \cap U_\beta.$$

If this is the case, then we see that $\eta_\alpha - \sigma_\alpha$ and $\eta_\beta - \sigma_\beta$ restrict to the same form on the intersection of their domains, so they piece together to produce a global $(k-1)$-form θ such that $d\theta = \omega$ (because $d\theta = d\eta_\alpha - d\sigma_\alpha = \omega - 0$ on U_α). Conversely, if ω is exact, we can choose the η_α's all to be restrictions of a global form η, so the cocycle γ is zero. Thus starting with a closed k-form ω, we have produced a 1-cocycle on $\mathcal{U}$ with coefficients in $\mathscr{Z}^{k-1}$, which is a coboundary if and only if ω is exact. This is a special case of a deep connection between de Rham cohomology and sheaf cohomology, which we will explore later in the chapter. //

Dependence on the Open Cover

Next we have to examine how the Čech cohomology groups depend on the choice of open cover. Given an indexed open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$, recall that another such open cover $\mathcal{V} = \{V_\beta\}_{\beta\in B}$ is a ***refinement of*** $\boldsymbol{\mathcal{U}}$ if for each $\beta \in B$, there is some $\alpha \in A$ such that $V_\beta \subseteq U_\alpha$. If this is the case, then we can choose a ***refining map*** $\rho\colon \mathcal{V} \to \mathcal{U}$, by which we mean a map $\rho\colon B \to A$ such that $V_\beta \subseteq U_{\rho(\beta)}$ for every β. Using this, we define a homomorphism $\rho^\# : C^p(\mathscr{U};\mathscr{S}) \to C^p(\mathscr{V};\mathscr{S})$ for each p by

$$(\rho^\# c)_{\beta_0\dots\beta_p} = c_{\rho(\beta_0)\dots\rho(\beta_p)}\big|_{V_{\beta_0}\cap\dots\cap V_{\beta_p}}.$$

Then a straightforward computation shows that $\delta(\rho^\# c) = \rho^\#(\delta c)$, so $\rho^\#$ maps cocycles to cocycles and coboundaries to coboundaries, and thus descends to a homomorphism $\rho^* : H^p(\mathscr{U};\mathscr{S}) \to H^p(\mathscr{V};\mathscr{S})$. If $\mathscr{S}$ is a sheaf of real or complex vector spaces, then ρ^* is linear.

Lemma 6.5. *The homomorphism ρ^* is independent of the choice of refining map.*

Proof. Given $\mathscr{U}$ and $\mathscr{V}$ as above, suppose $\rho, \tilde\rho : \mathscr{V} \to \mathscr{U}$ are two different refining maps. For each p, define a homomorphism $\theta : C^p(\mathscr{U};\mathscr{S}) \to C^{p-1}(\mathscr{V};\mathscr{S})$ by

$$(\theta c)_{\beta_0\dots\beta_{p-1}} = \sum_{j=0}^{p-1}(-1)^j c_{\rho(\beta_0)\dots\rho(\beta_j)\tilde\rho(\beta_j)\dots\tilde\rho(\beta_{p-1})}\big|_{V_{\beta_0}\cap\dots\cap V_{\beta_{p-1}}}.$$

We will show that θ satisfies the following ***cochain homotopy formula***:

$$\theta\delta + \delta\theta = \tilde\rho^\# - \rho^\#. \tag{6.1}$$

(The terminology comes from algebraic topology, where such formulas are commonly used for showing that two maps between chain complexes or cochain complexes induce the same map on homology or cohomology; a classic example is to show that homotopic maps between topological spaces induce the same homology and cohomology homomorphisms.)

Granting (6.1) for now, given a cohomology class $[c] \in H^p(\mathscr{U};\mathscr{S})$ represented by a cocycle $c \in Z^p(\mathscr{U};\mathscr{S})$, we see that $\tilde\rho^\# c - \rho^\# c = 0 + \delta\theta c$, so $\tilde\rho^*[c] = [\tilde\rho^\# c] = [\rho^\# c] = \rho^*[c]$, which proves the lemma.

To complete the proof, we need to verify the cochain homotopy formula (6.1). This is just a messy computation. Here it is. (We suppress the restrictions from the notation for brevity; all of the sections on the right-hand side are understood to be restricted to $V_{\beta_0} \cap \dots \cap V_{\beta_p}$.)

$$\begin{aligned}(\theta\delta c)_{\beta_0\dots\beta_p} &= \sum_{j=0}^{p}(-1)^j(\delta c)_{\rho(\beta_0)\dots\rho(\beta_j)\tilde\rho(\beta_j)\dots\tilde\rho(\beta_p)}\\ &= \sum_{\substack{j,k\\0\le k\le j\le p}} (-1)^{j+k} c_{\rho(\beta_0)\dots\widehat{\rho(\beta_k)}\dots\rho(\beta_j)\tilde\rho(\beta_j)\dots\tilde\rho(\beta_p)}\\ &\quad + \sum_{\substack{j,k\\0\le j\le k\le p}} (-1)^{j+k+1} c_{\rho(\beta_0)\dots\rho(\beta_j)\tilde\rho(\beta_j)\dots\widehat{\tilde\rho(\beta_k)}\dots\tilde\rho(\beta_p)};\end{aligned}$$

$$
\begin{aligned}
(\delta\theta c)_{\beta_0\dots\beta_p} &= \sum_{k=0}^{p}(-1)^k(\theta c)_{\beta_0\dots\widehat{\beta_k}\dots\beta_p}\\
&= \sum_{\substack{j,k\\0\le j<k\le p}} (-1)^{k+j} c_{\rho(\beta_0)\dots\rho(\beta_j)\tilde\rho(\beta_j)\dots\widehat{\tilde\rho(\beta_k)}\dots\tilde\rho(\beta_p)}\\
&\quad+ \sum_{\substack{j,k\\0\le k<j\le p}} (-1)^{k+j-1} c_{\rho(\beta_0)\dots\widehat{\rho(\beta_k)}\dots\rho(\beta_j)\tilde\rho(\beta_j)\dots\tilde\rho(\beta_p)}.
\end{aligned}
$$

When these expressions are added together, each term in the expression for $\theta\delta c$ is canceled by one term in the expression for $\delta\theta c$, except the terms in $\theta\delta c$ where $j = k$. Those terms give

$$
\begin{aligned}
(\theta\delta c + \delta\theta c)_{\beta_0\dots\beta_p} = &\sum_{j=0}^{p} c_{\rho(\beta_0)\dots\rho(\beta_{j-1})\tilde\rho(\beta_j)\dots\tilde\rho(\beta_p)}\\
&+ \sum_{j=0}^{p}(-1)c_{\rho(\beta_0)\dots\rho(\beta_j)\tilde\rho(\beta_{j+1})\dots\tilde\rho(\beta_p)}.
\end{aligned}
$$

This is a telescoping sum in which all the terms cancel except the $j = 0$ term in the first sum and the $j = p$ term in the second sum; what is left is exactly $(\tilde\rho^{\#}c)_{\beta_0\dots\beta_p} - (\rho^{\#}c)_{\beta_0\dots\beta_p}$. □

The result of this lemma is that whenever $\mathscr{V}$ is a refinement of $\mathscr{U}$, there is a canonical homomorphism $\rho^*_{\mathscr{U}\mathscr{V}} : H^p(\mathscr{U};\mathscr{S}) \to H^p(\mathscr{V};\mathscr{S})$. Moreover, if in addition $\mathscr{W}$ is a refinement of $\mathscr{V}$, we can choose our refining maps so that the refining map from $\mathscr{W}$ to $\mathscr{U}$ is the composition of the maps from $\mathscr{W}$ to $\mathscr{V}$ and $\mathscr{V}$ to $\mathscr{U}$, so that $\rho^*_{\mathscr{V}\mathscr{W}} \circ \rho^*_{\mathscr{U}\mathscr{V}} = \rho^*_{\mathscr{U}\mathscr{W}}$. It follows that the collection of all open covers of M with the homomorphisms $\rho^*_{\mathscr{U}\mathscr{V}}$ is a direct system—given any two open covers, the collection of all of their intersections is a refinement of both. (The set of open covers with the relation $\mathscr{U} \le \mathscr{V}$ if $\mathscr{V}$ is a refinement of $\mathscr{U}$ is an example of a directed set that is not partially ordered: two open covers can be refinements of each other without being equal.)

We define the ***pth sheaf cohomology group of M with coefficients in $\mathscr{S}$*** as the direct limit of this system:

$$H^p(M;\mathscr{S}) = \varinjlim H^p(\mathscr{U};\mathscr{S}).$$

If $\mathscr{S}$ is a sheaf of real or complex vector spaces, then $H^p(M;\mathscr{S})$ is a vector space. These groups are sometimes called the ***Čech cohomology groups with coefficients in $\mathscr{S}$***, to distinguish them from other constructions of sheaf cohomology groups; see the last section of this chapter for the relationships among the various constructions.

For each open cover $\mathscr{U}$, there is a canonical homomorphism from $H^p(\mathscr{U};\mathscr{S})$ to $H^p(M;\mathscr{S})$, which sends a cohomology class $[c]$ to its equivalence class in the direct limit. Let us denote this equivalence class by $[[c]] \in H^p(M;\mathscr{S})$. Unwinding the definitions, we see that for a given cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$, two cocycles $c, c' \in C^p(\mathscr{U};\mathscr{S})$ satisfy $[[c]] = [[c']]$ if and only if there is a cover $\mathscr{V} = \{V_\beta\}_{\beta\in B}$ refining $\mathscr{U}$ and a refining map $\rho\colon \mathscr{V} \to \mathscr{U}$ such that $\rho^\#(c-c')$ is a coboundary in $C^p(\mathscr{V};\mathscr{S})$.

Theorem 6.6 (Functoriality of Sheaf Cohomology). *Suppose $\mathscr{S}$, $\mathscr{T}$ are sheaves of abelian groups on a topological space M and $F\colon \mathscr{S} \to \mathscr{T}$ is a sheaf morphism. For each p, there is a homomorphism $F_*\colon H^p(M;\mathscr{S}) \to H^p(M;\mathscr{T})$, and these homomorphisms satisfy*

$$(\mathrm{Id}_{\mathscr{S}})_* = \mathrm{Id}_{H^p(M;\mathscr{S})} \qquad \text{and} \qquad (F\circ G)_* = F_*\circ G_*.$$

If F is a morphism between sheaves of real or complex vector spaces, then F_* is linear.

Proof. For a specific open cover $\mathscr{U} = \{U_\alpha\}$, define $F_\#\colon C^p(\mathscr{U};\mathscr{S}) \to C^p(\mathscr{U};\mathscr{T})$ by

$$(F_\# c)_{\alpha_0\dots\alpha_p} = F(c_{\alpha_0\dots\alpha_p}). \tag{6.2}$$

It clearly satisfies $\mathrm{Id}_\# = \mathrm{Id}$ and $(F\circ G)_\# = F_\#\circ G_\#$. Moreover, a straightforward computation shows that

$$F_\#\circ\delta = \delta\circ F_\#, \tag{6.3}$$

which implies that $F_\#$ descends to a homomorphism (still denoted by $F_\#$) from $H^p(\mathscr{U};\mathscr{S})$ to $H^p(\mathscr{U};\mathscr{T})$. Another straightforward computation shows that for any pair of covers $\mathscr{U}$ and $\mathscr{V}$ equipped with a refining map $\rho\colon \mathscr{V} \to \mathscr{U}$, we have $F_\#\circ\rho^\# = \rho^\#\circ F_\#$, so $F_\#$ descends to a homomorphism $F_*\colon H^p(M;\mathscr{S}) \to H^p(M;\mathscr{T})$ satisfying the conclusions of the proposition. For sheaves of vector spaces, one need only check that all of the homomorphisms above are linear maps. □

Theorem 6.7 (Degree 0 Sheaf Cohomology). *For every sheaf $\mathscr{S}$ of abelian groups on a topological space M, there is an isomorphism $I_{\mathscr{S}}\colon \mathscr{S}(M) \cong H^0(M;\mathscr{S})$ such that for every sheaf morphism $F\colon \mathscr{S} \to \mathscr{T}$, the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{S}(M) & \xrightarrow{\ I_{\mathscr{S}}\ } & H^0(M;\mathscr{S}) \\ {\scriptstyle F_M}\downarrow & & \downarrow{\scriptstyle F_*} \\ \mathscr{T}(M) & \xrightarrow{\ I_{\mathscr{T}}\ } & H^0(M;\mathscr{T}). \end{array} \tag{6.4}$$

If $\mathscr{S}$ is a sheaf of real or complex vector spaces, then $I_{\mathscr{S}}$ is linear.

Proof. Suppose $\mathscr{S}$ is a sheaf of abelian groups on M. Example 6.2 showed that for each open cover $\mathscr{U}$ of M, there is an isomorphism $I_{\mathscr{U},\mathscr{S}}\colon \mathscr{S}(M) \to H^0(\mathscr{U};\mathscr{S})$.

For any open cover $\mathscr{V}$ refining $\mathscr{U}$, it follows from the definitions that these isomorphisms commute with refining maps: $\rho^*_{\mathscr{U}\mathscr{V}} \circ I_{\mathscr{U},\mathscr{S}} = I_{\mathscr{V},\mathscr{S}}$. Thus they pass to the direct limit to yield a canonical isomorphism $I_{\mathscr{S}} : \mathscr{S}(M) \to H^0(M;\mathscr{S})$, which is linear if $\mathscr{S}$ is a sheaf of vector spaces.

If $F : \mathscr{S} \to \mathscr{T}$ is a sheaf morphism, recall that for each open cover $\mathscr{U}$, the map $F_\# : H^0(\mathscr{U};\mathscr{S}) \to H^0(\mathscr{U};\mathscr{T})$ is defined by $(F_\# c)_\alpha = F(c_\alpha)$. It is immediate that $F_\# \circ I_{\mathscr{U},\mathscr{S}} = I_{\mathscr{U},\mathscr{T}} \circ F_M$, and then commutativity of (6.4) follows by passing to the direct limit. □

The Long Exact Cohomology Sequence

By far the most important property of the sheaf cohomology groups is expressed in the next theorem. It relies on the following standard result in homological algebra. If A^*, B^*, and C^* are cochain complexes, a sequence of cochain maps

(6.5)
$$0 \to A^* \xrightarrow{\alpha} B^* \xrightarrow{\beta} C^* \to 0$$

is called an ***exact sequence of cochain complexes*** if each sequence of homomorphisms $0 \to A^p \to B^p \to C^p \to 0$ is exact.

Lemma 6.8 (The Zigzag Lemma). *Suppose we are given an exact sequence of cochain complexes of the form* (6.5)*. Then for each p there is a* ***connecting homomorphism*** $\delta_* : H^p(C^*) \to H^{p+1}(A^*)$ *such that the following sequence is exact:*

$$\cdots \xrightarrow{\delta_*} H^p(A^*) \xrightarrow{\alpha_*} H^p(B^*) \xrightarrow{\beta_*} H^p(C^*) \xrightarrow{\delta_*} H^{p+1}(A^*) \xrightarrow{\alpha_*} \cdots .$$

For each p, the homomorphism $\delta_* : H^p(C^*) \to H^{p+1}(A^*)$ is characterized as follows:

(6.6) $\delta_*([c]) = [a]$ *if and only if there exists some $b \in B^p$ such that* $\beta(b) = c$ *and* $\alpha(a) = \delta b$.

If all of the groups and homomorphisms are real or complex vector spaces, then the connecting homomorphisms are linear maps.

See [**LeeTM**, Lemma 13.17] or [**Hat02**, Thm. 2.16] for a proof. (The zigzag lemma is stated and proved there for *chain complexes*, in which the arrows go in the direction of decreasing indices, but the proof for cochain complexes works exactly the same way.)

Theorem 6.9 (The Long Exact Sequence in Sheaf Cohomology). *Suppose $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are sheaves of abelian groups on a paracompact Hausdorff space M, and the following sequence of sheaf morphisms is exact:*

(6.7)
$$0 \to \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \to 0.$$

Then for each $p \geq 0$, there exists a connecting homomorphism

$$\delta_*: H^p(M;\mathscr{C}) \to H^{p+1}(M;\mathscr{A})$$

such that the following sequence is exact:

$$(6.8)\quad 0 \to H^0(M;\mathscr{A}) \xrightarrow{\alpha_*} H^0(M;\mathscr{B}) \xrightarrow{\beta_*} H^0(M;\mathscr{C}) \xrightarrow{\delta_*} H^1(M;\mathscr{A}) \xrightarrow{\alpha_*} \cdots$$
$$\cdots \xrightarrow{\alpha_*} H^p(M;\mathscr{B}) \xrightarrow{\beta_*} H^p(M;\mathscr{C}) \xrightarrow{\delta_*} H^{p+1}(M;\mathscr{A}) \xrightarrow{\alpha_*} \cdots.$$

If $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are all sheaves of real or complex vector spaces, then δ_* is linear. It satisfies the following naturality property: given a commutative diagram of sheaves and sheaf morphisms

$$(6.9)\quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{A} & \xrightarrow{\alpha} & \mathscr{B} & \xrightarrow{\beta} & \mathscr{C} & \longrightarrow & 0 \\ & & \downarrow A & & \downarrow B & & \downarrow C & & \\ 0 & \longrightarrow & \mathscr{A}' & \xrightarrow[\alpha']{} & \mathscr{B}' & \xrightarrow[\beta']{} & \mathscr{C}' & \longrightarrow & 0 \end{array}$$

in which the horizontal rows are exact, the two connecting homomorphisms δ_* and δ'_* satisfy $A_* \circ \delta_* = \delta'_* \circ C_*$ for each p:

$$(6.10)\quad \begin{array}{ccc} H^p(M;\mathscr{C}) & \xrightarrow{\delta_*} & H^{p+1}(M;\mathscr{A}) \\ C_* \downarrow & & \downarrow A_* \\ H^p(M;\mathscr{C}') & \xrightarrow[\delta'_*]{} & H^{p+1}(M;\mathscr{A}'). \end{array}$$

Proof. Begin with an arbitrary open cover $\mathscr{U}$ for M. For each $p \geq 0$, consider the following sequence of cochain groups:

$$0 \to C^p(\mathscr{U};\mathscr{A}) \xrightarrow{\alpha_\#} C^p(\mathscr{U};\mathscr{B}) \xrightarrow{\beta_\#} C^p(\mathscr{U};\mathscr{C}).$$

Proposition 5.24 applied on each intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ shows that this sequence is exact. However, $\beta_\#$ might not be surjective. So we cheat: define the subgroup $C^p_\beta(\mathscr{U};\mathscr{C}) \subseteq C^p(\mathscr{U};\mathscr{C})$ to be the image of $\beta_\#: C^p(\mathscr{U};\mathscr{B}) \to C^p(\mathscr{U};\mathscr{C})$, so we have a short exact sequence

$$0 \to C^p(\mathscr{U};\mathscr{A}) \xrightarrow{\alpha_\#} C^p(\mathscr{U};\mathscr{B}) \xrightarrow{\beta_\#} C^p_\beta(\mathscr{U};\mathscr{C}) \to 0.$$

Now consider the following diagram of group homomorphisms (where we consider cochain groups to be zero in degrees less than 0):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^{p-1}(\mathscr{U};\mathscr{A}) & \xrightarrow{\alpha_\#} & C^{p-1}(\mathscr{U};\mathscr{B}) & \xrightarrow{\beta_\#} & C^{p-1}_\beta(\mathscr{U};\mathscr{C}) & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \longrightarrow & C^{p}(\mathscr{U};\mathscr{A}) & \xrightarrow{\alpha_\#} & C^{p}(\mathscr{U};\mathscr{B}) & \xrightarrow{\beta_\#} & C^{p}_\beta(\mathscr{U};\mathscr{C}) & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \longrightarrow & C^{p+1}(\mathscr{U};\mathscr{A}) & \xrightarrow{\alpha_\#} & C^{p+1}(\mathscr{U};\mathscr{B}) & \xrightarrow{\beta_\#} & C^{p+1}_\beta(\mathscr{U};\mathscr{C}) & \longrightarrow & 0.
\end{array}$$

Since δ commutes with $\beta_\#$ by (6.3), it follows that δ takes $C^{p-1}_\beta(\mathscr{U};\mathscr{C})$ to $C^{p}_\beta(\mathscr{U};\mathscr{C})$ and $C^{p}_\beta(\mathscr{U};\mathscr{C})$ to $C^{p+1}_\beta(\mathscr{U};\mathscr{C})$. The horizontal rows of this diagram are exact, and the columns are cochain complexes; and it commutes by (6.3). Let $H^{*}_\beta(\mathscr{U};\mathscr{C})$ denote the cohomology of the cochain complex $C^{*}_\beta(\mathscr{U};\mathscr{C})$.

The zigzag lemma shows that for each p there is a connecting homomorphism $\delta_*\colon H^{p}_\beta(\mathscr{U};\mathscr{C}) \to H^{p+1}(\mathscr{U};\mathscr{A})$ such that the following sequence is exact:

$$\cdots \to H^p(\mathscr{U};\mathscr{A}) \xrightarrow{\alpha_*} H^p(\mathscr{U};\mathscr{B}) \xrightarrow{\beta_*} H^p_\beta(\mathscr{U};\mathscr{C}) \xrightarrow{\delta_*} H^{p+1}(\mathscr{U};\mathscr{A}) \to \cdots.$$

Next we need to examine what happens when we pass to the direct limit. Given a cover $\mathscr{V}$ refining $\mathscr{U}$ and a refining map $\rho\colon \mathscr{V} \to \mathscr{U}$, we get a homomorphism $\rho^\#\colon H^p(\mathscr{U};\mathscr{A}) \to H^p(\mathscr{V};\mathscr{A})$ that commutes with all of the above maps, as you can check, with similar homomorphisms for $\mathscr{A}$ and $\mathscr{B}$. Thus in the limit we obtain a sequence

(6.11)
$$\cdots \to H^p(M;\mathscr{A}) \xrightarrow{\alpha_*} H^p(M;\mathscr{B}) \xrightarrow{\beta_*} H^p_\beta(M;\mathscr{C}) \xrightarrow{\delta_*} H^{p+1}(M;\mathscr{A}) \to \cdots,$$

where $H^p_\beta(M;\mathscr{C}) = \varinjlim H^p_\beta(\mathscr{U};\mathscr{C})$ for each p. It is straightforward to check that this sequence is still exact.

To complete the proof, we will show that the map

$$\mathscr{I}\colon H^p_\beta(M;\mathscr{C}) \to H^p(M;\mathscr{C})$$

induced by inclusion $C^p_\beta(\mathscr{U};\mathscr{C}) \hookrightarrow C^p(\mathscr{U};\mathscr{C})$ is an isomorphism for each p. The proof will be based on the following fact, whose verification we postpone until the end of the proof:

(6.12) *Given a cochain $c \in C^p(\mathscr{U};\mathscr{C})$, there exist a refinement $\mathscr{V}$ of $\mathscr{U}$ and a refining map $\rho\colon \mathscr{V} \to \mathscr{U}$ such that $\rho^\# c \in C^p_\beta(\mathscr{V};\mathscr{C})$.*

Granting this for the moment, we prove that $\mathscr{I}$ is bijective. To show that it is injective, suppose $\mathscr{I}([[c]]) = 0$. We can choose a representative cochain $c \in C^p_\beta(\mathscr{U};\mathscr{C})$ for some open cover $\mathscr{U}$, and the hypothesis implies there is a refining map $\rho\colon \mathscr{V} \to \mathscr{U}$ and a cochain $\gamma \in C^{p-1}(\mathscr{V};\mathscr{C})$ such that $\rho^\# c = \delta\gamma$. By (6.12), there is another refining map $\tilde\rho\colon \mathscr{W} \to \mathscr{V}$ such that $\tilde\rho^\#\gamma = \beta_\# b$ for some $b \in C^{p-1}(\mathscr{W};\mathscr{C})$. Then

$$\tilde\rho^\#\rho^\# c = \tilde\rho^\#\delta\gamma = \delta\tilde\rho^\#\gamma = \delta\beta_\# b,$$

so, after refinement, c maps to a coboundary in $C^p_\beta(\mathscr{W};\mathscr{C})$ and thus $[[c]] = 0 \in H^p_\beta(M;\mathscr{C})$.

To show that $\mathscr{I}$ is surjective, let $[[c]] \in H^p(M;\mathscr{C})$ be arbitrary, represented by a cochain $c \in C^p(\mathscr{U};\mathscr{C})$ with $\delta c = 0$. By (6.12), there exists a refining map $\rho\colon \mathscr{V} \to \mathscr{U}$ such that $\rho^\# c = \beta_\# b$ for some $b \in C^p(\mathscr{V};\mathscr{B})$. Now

$$\delta\beta_\# b = \delta\rho^\# c = \rho^\#\delta c = 0,$$

so $\beta_\# b$ represents a cohomology class in $H^p_\beta(\mathscr{U};\mathscr{C})$, and passing to the limit, we see that $[[c]] = \mathscr{I}([[\beta_\# b]])$.

Thus we can replace $H^p_\beta(\mathscr{U};\mathscr{C})$ with $H^p(\mathscr{U};\mathscr{C})$ in (6.11), replace β_* by $\beta_* \circ \mathscr{I}$ (which is just $\beta_*\colon H^p(M;\mathscr{B}) \to H^p(M;\mathscr{C})$), and replace δ_* by $\delta_* \circ \mathscr{I}^{-1}$, and we obtain the long exact sequence (6.8).

Now we prove the naturality claim. Suppose we have a commutative diagram (6.9) of sheaf morphisms with exact rows. Let $\gamma \in H^p(M;\mathscr{C})$ be arbitrary. By (6.12), we can choose an open cover $\mathscr{U}$ such that γ is represented by a cocycle $c \in C^p_\beta(\mathscr{U};\mathscr{C})$. By (6.6), $\delta_*(\gamma)$ is represented by a cocycle $a \in C^{p+1}(\mathscr{U};\mathscr{A})$ such that there exists $b \in C^p(\mathscr{U};\mathscr{B})$ with

$$\beta_\# b = c \quad\text{and}\quad \alpha_\# a = \delta b. \tag{6.13}$$

Set $b' = B_\# b \in C^p(\mathscr{U};\mathscr{B}')$ and $a' = A_\# a \in C^{p+1}(\mathscr{U};\mathscr{A}')$. Note that (6.3) shows that $B_\#$ commutes with δ, and commutativity of (6.9) implies that $\alpha'_\# \circ A_\# = B_\# \circ \alpha_\#$ and $\beta'_\# \circ B_\# = C_\# \circ \beta_\#$. These identities together with (6.13) yield

$$\begin{aligned}\beta_\# b' &= \beta'_\# B_\# b = C_\#\beta_\# b = C_\# c,\\ \alpha'_\# a' &= \alpha'_\# A_\# a = B_\#\alpha_\# a = B_\#\delta b = \delta B_\# b = \delta b',\end{aligned}$$

so it follows from (6.3) that $\delta'_* C_*\gamma = \delta'_*[[C_\# c]]$ is represented by a'. Therefore, $A_*\delta_*\gamma = A_*[[a]] = [[A_\# a]] = [[a']] = \delta'_* C_*\gamma$, which shows that diagram (6.10) commutes as claimed.

It remains only to prove statement (6.12). (This is the only part of the proof where we use the fact that M is a paracompact Hausdorff space.) Suppose $c \in C^p(\mathscr{U};\mathscr{C})$ is arbitrary. After refining the cover if necessary, we can assume that $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ is locally finite. By [**LeeTM**, Lemma 4.84], for each $\alpha \in A$ there exists an open set W_α such that $\overline{W}_\alpha \subseteq U_\alpha$ and the collection $\{W_\alpha\}_{\alpha\in A}$ still covers

M. Because the cover $\mathscr{U}$ is locally finite and the sheaf sequence (6.7) is exact, for each $x \in M$ we can choose a neighborhood V_x small enough that the following properties are satisfied:

(i) If $x \in U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$, then $V_x \subseteq U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ and there exists a section $b^{(x)}_{\alpha_0 \ldots \alpha_p} \in \mathscr{B}(V_x)$ such that $\beta\big(b^{(x)}_{\alpha_0 \ldots \alpha_p}\big) = c_{\alpha_0 \ldots \alpha_p}|_{V_x}$.

(ii) If $x \in W_\alpha$, then $V_x \subseteq W_\alpha$.

(iii) If $V_x \cap W_\alpha \neq \varnothing$, then $V_x \subseteq U_\alpha$.

Let $\mathscr{V}$ be the indexed open cover $\{V_x\}_{x \in M}$. By (i), it is a refinement of $\mathscr{U}$. Choose a refining map $\rho\colon \mathscr{V} \to \mathscr{U}$ such that for each $x \in M$, we have $x \in W_{\rho(x)} \subseteq U_{\rho(x)}$. For any $(p+1)$-tuple $x_0, \ldots, x_p$, define

$$\tilde{b}_{x_0 \ldots x_p} = b^{(x_0)}_{\rho(x_0) \ldots \rho(x_p)}.$$

Then it is straightforward to check that the cochain $b \in C^p(\mathscr{V}; \mathscr{B})$ satisfies $\beta_\# b = \rho^\# c$. □

For many sheaves, we can define an important numerical invariant as follows. Suppose $\mathscr{S}$ is a sheaf of real or complex vector spaces on a topological space M. If the spaces $H^k(M; \mathscr{S})$ are all finite-dimensional and are nonzero for only finitely many values of k, we define the ***Euler characteristic of $\mathscr{S}$*** to be the integer

$$\chi(\mathscr{S}) = \sum_k (-1)^k \dim H^k(M; \mathscr{S}).$$

(The name reflects the analogy with the Euler characteristic of a topological space, which can be computed as the alternating sum of the ranks of the singular homology groups [**LeeTM**, Thm. 13.36].)

Proposition 6.10. *Suppose $0 \to \mathscr{R} \to \mathscr{S} \to \mathscr{T} \to 0$ is a short exact sequence of sheaves of vector spaces on a locally compact Hausdorff space M. If the Euler characteristics $\chi(\mathscr{R})$, $\chi(\mathscr{S})$, and $\chi(\mathscr{T})$ are all defined, then*

$$\chi(\mathscr{S}) = \chi(\mathscr{R}) + \chi(\mathscr{T}).$$

Proof. By Theorem 6.9, we have a long exact sequence

$$\cdots \to H^k(M; \mathscr{R}) \to H^k(M; \mathscr{S}) \to H^k(M; \mathscr{T}) \to H^{k+1}(M; \mathscr{R}) \to \cdots.$$

For each k, let $Z^k(M; \mathscr{S})$ denote the kernel of the homomorphism $H^k(M; \mathscr{S}) \to H^k(M; \mathscr{T})$, which is also the image of $H^k(M; \mathscr{R}) \to H^k(M; \mathscr{S})$; and define $Z^k(M; \mathscr{R})$ and $Z^k(M; \mathscr{T})$ similarly. It follows from the rank-nullity law of linear algebra (which says that for a linear map $F\colon V \to W$ between finite-dimensional vector spaces, the dimension of V is equal to the sum of the dimensions of the kernel and the image of F) that

$$\begin{aligned}\dim H^k(M;\mathscr{R}) &= \dim Z^k(M;\mathscr{R}) + \dim Z^k(M;\mathscr{S}),\\ \dim H^k(M;\mathscr{S}) &= \dim Z^k(M;\mathscr{S}) + \dim Z^k(M;\mathscr{T}),\\ \dim H^k(M;\mathscr{T}) &= \dim Z^k(M;\mathscr{T}) + \dim Z^{k+1}(M;\mathscr{R}).\end{aligned}$$

Multiplying each of these equations by $(-1)^k$ and summing over k gives formulas for the respective Euler characteristics. When we add the first and last of these equations and subtract the second, we find that all the terms on the right-hand side cancel, yielding $\chi(\mathscr{R}) - \chi(\mathscr{S}) + \chi(\mathscr{T}) = 0$. □

Acyclic Resolutions

The definition of the sheaf cohomology groups is too abstract to be useful for computations in most circumstances. In this section, we introduce an important tool that often makes computations straightforward.

Suppose $\mathscr{S}$ is a sheaf of abelian groups on a topological space M. A ***resolution of*** $\mathscr{S}$ is a (possibly infinite) exact sequence of sheaves of abelian groups and sheaf morphisms of the form

$$0 \to \mathscr{S} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \mathscr{A}^2 \to \cdots. \tag{6.14}$$

If $\mathscr{S}$ is a sheaf of real or complex vector spaces, then we require sheaves and morphisms in the same category. Thus, for example, the de Rham sheaf sequence (5.17) is a resolution of the constant sheaf $\underline{\mathbb{C}}$.

A sheaf $\mathscr{A}$ of abelian groups on M is said to be ***acyclic*** if $H^q(M;\mathscr{A}) = 0$ for all $q \geq 1$. (Of course, $H^0(M;\mathscr{A}) \cong \mathscr{A}(M)$, so we cannot expect that to be zero in most cases.) A resolution (6.14) is called an ***acyclic resolution*** if each of the sheaves $\mathscr{A}^k$ is acyclic for $k \geq 0$.

The main tool for computing sheaf cohomology groups is the following theorem. It is named the *de Rham–Weil theorem* because André Weil [**Wei52**] introduced this technique as a way of proving the de Rham theorem (see Theorem 6.20 below).

Theorem 6.11 (De Rham–Weil). *Suppose $\mathscr{S}$ is a sheaf of abelian groups on a paracompact Hausdorff space M and*

$$0 \to \mathscr{S} \xrightarrow{\iota} \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \to \cdots \tag{6.15}$$

is an acyclic resolution of $\mathscr{S}$. Then the sequence of global sections

$$0 \to \mathscr{A}^0(M) \xrightarrow{d} \mathscr{A}^1(M) \xrightarrow{d} \mathscr{A}^2(M) \to \cdots$$

is a cochain complex, and for each q, the sheaf cohomology group $H^q(M;\mathscr{S})$ is isomorphic to the cohomology group $H^q(\mathscr{A}^*(M))$ of this complex; more precisely,

$$H^0(M;\mathscr{S}) \cong \operatorname{Ker}\big(d : \mathscr{A}^0(M) \to \mathscr{A}^1(M)\big), \quad \textit{and}$$
$$H^q(M;\mathscr{S}) \cong \frac{\operatorname{Ker}\big(d : \mathscr{A}^q(M) \to \mathscr{A}^{q+1}(M)\big)}{\operatorname{Im}\big(d : \mathscr{A}^{q-1}(M) \to \mathscr{A}^q(M)\big)} \quad \textit{for } q \geq 1.$$

If (6.15) *is a resolution of sheaves of real or complex vector spaces, then the isomorphisms are linear. The isomorphisms satisfy the following naturality property: Suppose $F : \mathscr{S} \to \mathscr{T}$ is a morphism between sheaves of abelian groups on M and we have a commutative diagram of sheaf homomorphisms in which both horizontal rows are acyclic resolutions:*

(6.16)
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{S} & \xrightarrow{\iota} & \mathscr{A}^0 & \xrightarrow{d} & \mathscr{A}^1 & \xrightarrow{d} & \mathscr{A}^2 & \xrightarrow{d} & \cdots \\ & & F\big\downarrow & & \varphi^0\big\downarrow & & \varphi^1\big\downarrow & & \varphi^2\big\downarrow & & \\ 0 & \longrightarrow & \mathscr{T} & \xrightarrow[\iota']{} & \mathscr{B}^0 & \xrightarrow[d']{} & \mathscr{B}^1 & \xrightarrow[d']{} & \mathscr{B}^2 & \xrightarrow[d']{} & \cdots. \end{array}$$

Then for each $q \geq 0$, the following diagram commutes:

(6.17)
$$\begin{array}{ccc} H^q(M;\mathscr{S}) & \xrightarrow{\cong} & H^q(\mathscr{A}^*(M)) \\ F_*\big\downarrow & & \big\downarrow\varphi^q_* \\ H^q(M;\mathscr{T}) & \xrightarrow{\cong} & H^q(\mathscr{B}^*(M)), \end{array}$$

where φ^q_* is the cohomology homomorphism induced by the global section homomorphisms $\varphi^q : \mathscr{A}^q(M) \to \mathscr{B}^q(M)$.

Proof. Given an acyclic resolution (6.15), the fact that the global section sequence is a cochain complex follows from Lemma 5.20, and the claim about $H^0(M;\mathscr{S})$ follows from Proposition 5.24. From now on, we assume $q \geq 1$.

For each $k \geq 0$, define a sheaf $\mathscr{Z}^k$ by

$$\mathscr{Z}^k = \operatorname{Ker}\big(d : \mathscr{A}^k \to \mathscr{A}^{k+1}\big).$$

This is a subsheaf of $\mathscr{A}^k$ by Exercise 5.17, but not necessarily acyclic. In particular, $\mathscr{Z}^0$ is isomorphic to $\mathscr{S}$ by Proposition 5.22.

For each $k \geq 0$, we get a short exact sequence

$$0 \to \mathscr{Z}^k \hookrightarrow \mathscr{A}^k \xrightarrow{d} \mathscr{Z}^{k+1} \to 0.$$

The associated long exact sequence reads in part

$$H^{q-1}(M;\mathscr{A}^k) \to H^{q-1}(M;\mathscr{Z}^{k+1}) \to H^q(M;\mathscr{Z}^k) \to H^q(M;\mathscr{A}^k).$$

When $q > 1$, the groups on both ends are zero because $\mathscr{A}^k$ is acyclic; therefore we have isomorphisms

$$H^{q-1}(M;\mathscr{Z}^{k+1}) \cong H^q(M;\mathscr{Z}^k) \text{ for } q > 1. \tag{6.18}$$

To prove the theorem, we proceed as follows. Assuming $q > 1$, we apply (6.18) repeatedly to conclude

$$\begin{aligned} H^q(M;\mathscr{S}) &\cong H^q(M;\mathscr{Z}^0) \\ &\cong H^{q-1}(M;\mathscr{Z}^1) \\ &\cong H^{q-2}(M;\mathscr{Z}^2) \\ &\ \ \vdots \\ &\cong H^1(M;\mathscr{Z}^{q-1}), \end{aligned}$$

and the same conclusion holds trivially when $q = 1$. At this point, we can no longer use (6.18). Instead, we have the exact sequence

$$H^0(M;\mathscr{A}^{q-1}) \xrightarrow{d} H^0(M;\mathscr{Z}^q) \xrightarrow{\delta_*} H^1(M;\mathscr{Z}^{q-1}) \to H^1(M;\mathscr{A}^{q-1}).$$

Here the right-hand group is zero, which means that δ_* is surjective, and exactness implies

$$H^1(M;\mathscr{Z}^{q-1}) \cong \frac{H^0(M;\mathscr{Z}^q)}{\operatorname{Ker}\,\delta_*} \cong \frac{\mathscr{Z}^q(M)}{\operatorname{Im}\,d} \cong \frac{\operatorname{Ker}\big(d : \mathscr{A}^q(M) \to \mathscr{A}^{q+1}(M)\big)}{\operatorname{Im}\big(d : \mathscr{A}^{q-1}(M) \to \mathscr{A}^q(M)\big)},$$

which is what we needed to prove. In the case of sheaves of vector spaces, all the maps above are linear.

To prove the naturality claim, suppose we have a commutative diagram of the form (6.16). For each k, let $\mathscr{W}^k \subseteq \mathscr{B}^k$ be the kernel of $d' : \mathscr{B}^k \to \mathscr{B}^{k+1}$. Since (6.16) commutes, φ^k maps $\mathscr{Z}^k$ to $\mathscr{W}^k$ for each k, and we have commutative diagrams of sheaf morphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{Z}^k & \hookrightarrow & \mathscr{A}^k & \longrightarrow & \mathscr{Z}^{k+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^k} & & \downarrow{\scriptstyle \varphi^k} & & \downarrow{\scriptstyle \varphi^{k+1}} & & \\ 0 & \longrightarrow & \mathscr{W}^k & \hookrightarrow & \mathscr{B}^k & \longrightarrow & \mathscr{W}^{k+1} & \longrightarrow & 0, \end{array} \tag{6.19}$$

in which the first and third vertical maps are understood to be restricted to the appropriate subsheaves. Because the isomorphisms $H^q(M;\mathscr{S}) \cong H^q(\mathscr{A}^*(M))$ and $H^q(M;\mathscr{T}) \cong H^q(\mathscr{B}^*(M))$ are obtained as compositions of (restrictions of) connecting homomorphisms in the long exact sequences associated with the rows of (6.19), the commutativity of (6.17) follows by repeatedly applying the naturality result of Theorem 6.9. □

Fine Sheaves

To apply the de Rham–Weil theorem, we need to have a good supply of acyclic sheaves. We start with an important example.

Example 6.12 (Sheaves of Smooth Forms Are Acyclic). Let M be a smooth manifold. We will show that each sheaf $\mathscr{E}^k$ of smooth complex-valued k-forms is acyclic.

To prove this claim, it suffices to show that $H^p(\mathcal{U};\mathscr{E}^k) = 0$ for every open cover $\mathcal{U}$ of M. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an arbitrary open cover, and choose a smooth partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ subordinate to $\mathcal{U}$. Define a homomorphism $\theta\colon C^p(\mathcal{U};\mathscr{E}^k) \to C^{p-1}(\mathcal{U};\mathscr{E}^k)$ by

$$(\theta c)_{\alpha_0\dots\alpha_{p-1}} = \sum_{\beta\in A} \varphi_\beta c_{\beta\alpha_0\dots\alpha_{p-1}}, \tag{6.20}$$

where the k-form $\varphi_\beta c_{\beta\alpha_0\dots\alpha_{p-1}}$ is extended to all of $U_{\alpha_0}\cap\dots\cap U_{\alpha_{p-1}}$ by defining it to be zero outside the support of φ_β. Because this is a finite sum of smooth k-forms in a neighborhood of each point, it defines a smooth form on $U_{\alpha_0}\cap\dots\cap U_{\alpha_{p-1}}$ and thus a $(p-1)$-cochain. We will show that θ satisfies a cochain homotopy formula $\delta\theta+\theta\delta = \mathrm{Id}_{C^p(\mathcal{U};\mathscr{E}^k)}$, from which it follows that every cocycle in $C^p(\mathcal{U};\mathscr{E}^k)$ is a coboundary, thus proving the claim.

To verify the cochain homotopy formula, we compute

$$\begin{aligned}
(\delta\theta c)_{\alpha_0\dots\alpha_p} &= \sum_{j=0}^{p}(-1)^j(\theta c)_{\alpha_0\dots\widehat{\alpha_j}\dots\alpha_p}\\
&= \sum_{j=0}^{p}\sum_{\beta}(-1)^j\varphi_\beta c_{\beta\alpha_0\dots\widehat{\alpha_j}\dots\alpha_p},\\
(\theta\delta c)_{\alpha_0\dots\alpha_p} &= \sum_{\beta}\varphi_\beta(\delta c)_{\beta\alpha_0\dots\alpha_p}\\
&= \sum_{\beta}\varphi_\beta c_{\widehat{\beta}\alpha_0\dots\alpha_p} + \sum_{\beta}\sum_{j=0}^{p}(-1)^{j+1}\varphi_\beta c_{\beta\alpha_0\dots\widehat{\alpha_j}\dots\alpha_p}.
\end{aligned}$$

When these expressions are added together, everything cancels except the first sum in the last line, which is equal to $c_{\alpha_0\dots\alpha_p}$ because $\sum_\beta \varphi_\beta \equiv 1$. //

The same argument generalizes in an obvious way to any sheaf of smooth sections of a smooth vector bundle on a smooth manifold, and to any sheaf of continuous sections of a topological vector bundle on a paracompact Hausdorff space (which is the type of space on which partitions of unity can be constructed; see [**LeeTM**, Thm. 4.85]). Since the purpose of sheaf cohomology is to determine when local objects can be patched together to form global ones, you should think of these phenomena as expressions of the fact that there are no obstructions to patching things together when partitions of unity are available.

The preceding example can be generalized significantly. Given a sheaf $\mathscr{S}$ of abelian groups on a topological space M and a locally finite indexed open cover $\mathcal{U} = \{U_\beta\}_{\beta\in B}$, we define a ***sheaf partition of unity subordinate to $\mathcal{U}$*** to be an indexed collection of sheaf morphisms $\eta_\beta : \mathscr{S} \to \mathscr{S}$ satisfying the following two conditions:

(i) For each $\beta \in B$, the support of η_β is contained in U_β (where the ***support*** of a sheaf morphism $F : \mathscr{S} \to \mathscr{T}$ is the closure of the set of points $x \in M$ such that the stalk homomorphism F_x is nonzero).

(ii) For each x, $\sum_{\beta\in B}(\eta_\beta)_x = \mathrm{Id}_{\mathscr{S}_x}$.

(The fact that the open cover is locally finite ensures that each point has a neighborhood on which $\eta_\beta = 0$ for all but finitely many β, so the sum in (ii) has only finitely many nonzero terms.) A sheaf $\mathscr{S}$ is said to be ***fine*** if for every locally finite open cover there exists a sheaf partition of unity subordinate to it. To avoid confusion, we will refer to a partition of unity in the usual topological sense (a collection of continuous functions taking values in $[0, 1]$ whose sum is 1 and whose supports are locally finite) as a ***topological partition of unity***; if the underlying space is a smooth manifold and each of the functions is smooth, it will be called a ***smooth partition of unity***.

Example 6.13 (Fine Sheaves).

(a) On a smooth manifold M, the sheaf $\mathscr{E}$ of smooth complex-valued functions is fine: given a locally finite open cover $\mathcal{U} = \{U_\beta\}_{\beta\in B}$, let $\{\psi_\beta\}_{\beta\in B}$ be a subordinate smooth partition of unity, and define $\eta_\beta : \mathscr{E} \to \mathscr{E}$ by sending $f \in \mathscr{E}(U)$ to $\psi_\beta f$. By the same argument, every sheaf of $\mathscr{E}$-modules on M is fine, as is every sheaf of modules over the sheaf $\mathscr{E}_{\mathbb{R}}$ of real-valued smooth functions. The most important examples are sheaves of sections of smooth vector bundles.

(b) If M is a paracompact Hausdorff space, the sheaf $\mathscr{C}$ of continuous complex-valued functions is fine by essentially the same argument, as is any sheaf of $\mathscr{C}$-modules on M.

(c) Every skyscraper sheaf is fine. Let G_p be a skyscraper sheaf on a topological space M supported at $p \in M$. Given a locally finite open cover $\{U_\alpha\}_{\alpha\in A}$ of M, choose one index $\beta \in A$ such that $p \in U_\beta$, and define η_β to be the identity morphism and $\eta_\alpha = 0$ for all $\alpha \neq \beta$. //

On the other hand, constant sheaves and sheaves of $\mathscr{O}$-modules are almost never fine (see Problem 6-1).

Proposition 6.14. *If $\mathscr{S}$ is a fine sheaf on a paracompact Hausdorff space M, then $\mathscr{S}$ is acyclic.*

Proof. Let $\mathcal{S}$ be such a sheaf. Since every open cover of M has a locally finite open refinement, to prove the proposition it suffices to show that $H^k(\mathcal{U};\mathcal{S}) = 0$ for every $k \geq 1$ and every locally finite open cover $\mathcal{U}$. Given a locally finite open cover $\mathcal{U} = \{U_\beta\}_{\beta\in B}$, let $\{\eta_\beta\}_{\beta\in B}$ be a sheaf partition of unity subordinate to it. For each of the sheaf morphisms η_β and any open subset $V \subseteq M$, the homomorphism $\eta_\beta \colon \mathcal{S}(V\cap U_\beta) \to \mathcal{S}(V\cap U_\beta)$ extends to a homomorphism $\tilde{\eta}_\beta \colon \mathcal{S}(V\cap U_\beta) \to \mathcal{S}(V)$ by requiring that

$$\tilde{\eta}_\beta(c)\big|_{V\cap U_\beta} = \eta_\beta(c) \quad \text{and} \quad \tilde{\eta}_\beta(c)\big|_{V\smallsetminus \operatorname{supp}\eta_\beta} = 0;$$

the gluing and locality properties of sheaves guarantee that this is uniquely defined.

For each $p \geq 1$, define a map $\theta \colon C^p(\mathcal{U};\mathcal{S}) \to C^{p-1}(\mathcal{U};\mathcal{S})$ by analogy with (6.20):

$$(\theta c)_{\alpha_0\dots\alpha_{p-1}} = \sum_{\beta\in B} \tilde{\eta}_\beta\big(c_{\beta\alpha_0\dots\alpha_{p-1}}\big),$$

where we interpret the sum on the right-hand side by noting that $U_{\alpha_0} \cap \dots \cap U_{\alpha_{p-1}}$ has an open cover such that only finitely many terms of this sum are nonzero on each open set of the cover, and the gluing property ensures that these finite sums patch together to determine an element of $\mathcal{S}(U_{\alpha_0}\cap\dots\cap U_{\alpha_{p-1}})$. The rest of the proof of the proposition is exactly like the argument in Example 6.12. □

In complex manifold theory, the acyclic sheaves that will concern us are sheaves of $\mathcal{E}$-modules and skyscraper sheaves.

Sheaf Cohomology and Singular Cohomology

For the next result about sheaf cohomology, we need a few more definitions. We begin with a very brief review of singular homology and cohomology theory. For much more detail on the subject, consult any algebraic topology text such as [**Hat02**] or [**Mun84**].

Singular Homology

For a nonnegative integer k, the ***standard k-simplex*** Δ_k is the convex hull of the $k+1$ points $\{e_0, e_1, \dots, e_k\}$ in $\mathbb{R}^k$, where $e_0 = 0$ and e_j is the jth standard basis vector for $j \geq 1$. Since every point in the convex hull can be written as a linear combination $\sum_{j=0}^k t^j e_j$ with $\sum_{j=0}^k t^j = 1$, we can express Δ_k more explicitly as

$$\Delta_k = \left\{(t^1,\dots,t^k) \in \mathbb{R}^k : 0 \leq t^j \leq 1 \text{ and } \sum_{j=1}^k t^j \leq 1\right\}.$$

Let M be a topological space. A ***singular k-simplex*** in M is a continuous map $\sigma \colon \Delta_k \to M$. The free abelian group on the set of all singular k-simplices is called the ***singular chain group*** in dimension k; we will denote it by $\mathrm{Sing}_k(M)$. An element of that group, called a ***singular k-chain***, is a finite formal linear combination of singular k-simplices with integer coefficients.

The ***boundary*** of a singular k-simplex σ is the singular $(k-1)$-chain $\partial\sigma$ defined by

$$\partial\sigma = \sum_{i=0}^{k}(-1)^i \sigma \circ F_{i,k},$$

where $F_{i,k} : \Delta_{k-1} \to \Delta_k$, called the ith ***face map*** in dimension k, is the restriction of the unique affine map from $\mathbb{R}^{k-1}$ to $\mathbb{R}^k$ that sends the vertices $e_0, \dots, e_{k-1}$ to $e_0, \dots, \widehat{e_i}, \dots, e_k$, respectively, so it maps Δ_{k-1} homeomorphically onto the face of Δ_k opposite e_i. The boundary operator extends by linearity to a group homomorphism $\partial : \mathrm{Sing}_k(M) \to \mathrm{Sing}_{k-1}(M)$. A computation shows that $\partial \circ \partial = 0$, and therefore it makes sense to define the ***kth singular homology group of M*** by

$$H_k(M) = \frac{\mathrm{Ker}\left(\partial : \mathrm{Sing}_k(M) \to \mathrm{Sing}_{k-1}(M)\right)}{\mathrm{Im}\left(\partial : \mathrm{Sing}_{k+1}(M) \to \mathrm{Sing}_k(M)\right)}.$$

A singular k-chain c is called a ***cycle*** if it satisfies $\partial c = 0$, and a ***boundary*** if $c = \partial b$ for some $(k-1)$-chain b. Every cycle c determines an element of $H_k(M)$, called its ***homology class*** and denoted by $[c]$; any other cycle differing from c by a boundary represents the same homology class. Two cycles that differ by a boundary are said to be ***homologous***.

We need the following basic fact about singular homology of manifolds. Suppose M is a compact, connected, orientable smooth n-manifold. Every such manifold admits a ***smooth triangulation***, which is a singular n-cycle $c = \sum_j \sigma_j$ with the property that each $\sigma_j : \Delta_n \to M$ is a smooth orientation-preserving embedding, and the images of two distinct σ_j's intersect only along their boundaries if at all. (See [**Mun66**, Thm. 10.6] for a proof.) The cohomology class of c depends only on the orientation of M; it is denoted by $[M]$ and called the ***fundamental class of M***. For a proof of the following proposition, see [**Mun84**, Cor. 65.3] or [**Hat02**, Thm. 3.26]).

Proposition 6.15. *Suppose M is a compact, connected, oriented smooth n-dimensional manifold, and $[M]$ is its fundamental class. Then $H_n(M)$ is an infinite cyclic group generated by $[M]$.*

Singular Cohomology

If G is an abelian group, a ***singular k-cochain in M with coefficients in G*** is a group homomorphism $\varphi : \mathrm{Sing}_k(M) \to G$. The set of all such cochains, denoted by $\mathrm{Sing}^k(M;G) = \mathrm{Hom}\left(\mathrm{Sing}_k(M), G\right)$, is a group under pointwise addition: $(\varphi + \varphi')(c) = \varphi(c) + \varphi'(c)$. If in addition G is a real or complex vector space, then $\mathrm{Sing}^k(M;G)$ is a vector space under pointwise scalar multiplication. A singular cochain is uniquely determined by its action on each singular simplex.

The dual of the boundary map is a group homomorphism $\delta\colon \mathrm{Sing}^k(M;G) \to \mathrm{Sing}^{k+1}(M;G)$ called the ***coboundary operator***; it is defined by

$$(\delta\varphi)(c) = \varphi(\partial c).$$

A singular cochain φ satisfying $\delta\varphi = 0$ is called a ***(singular) cocycle***, and one satisfying $\varphi = \delta\psi$ for some cochain ψ is a ***(singular) coboundary***. The fact that $\partial \circ \partial = 0$ implies immediately that $\delta \circ \delta = 0$, so every coboundary is a cocycle. Thus we can form the following quotient space, called the ***kth singular cohomology group of M with coefficients in G***:

$$H^k_{\mathrm{Sing}}(M;G) = \frac{\mathrm{Ker}\left(\delta\colon \mathrm{Sing}^k(M;G) \to \mathrm{Sing}^{k+1}(M;G)\right)}{\mathrm{Im}\left(\delta\colon \mathrm{Sing}^{k-1}(M;G) \to \mathrm{Sing}^{k}(M;G)\right)}.$$

If G is a real or complex vector space, then so is $H^k_{\mathrm{Sing}}(M;G)$.

Every homomorphism $F\colon G \to H$ between abelian groups induces a homomorphism $F_\#\colon \mathrm{Sing}^k(M;G) \to \mathrm{Sing}^k(M;H)$ by $F_\#(\varphi)(c) = F(\varphi(c))$ for each singular cochain φ and singular chain c. This homomorphism commutes with the coboundary operators, so it defines a cochain map and thus descends to a homomorphism $F_*\colon H^k_{\mathrm{Sing}}(M;G) \to H^k_{\mathrm{Sing}}(M;H)$, called a ***coefficient homomorphism***. For a fixed space M, the coefficient homomorphisms satisfy

$$(\mathrm{Id}_G)_* = \mathrm{Id}_{H^k_{\mathrm{Sing}}(M;G)} \quad \text{and} \quad (F \circ F')_* = F_* \circ F'_*,$$

so the assignment $G \mapsto H^k_{\mathrm{Sing}}(M;G)$, $F \mapsto F_*$ is a covariant functor from the category of abelian groups to itself, with a similar statement for the category of real or complex vector spaces.

It is a standard result in algebraic topology that the singular homology and cohomology groups are homotopy invariants. One important consequence of this fact is that if M is a contractible space, then its singular cohomology groups agree with those of a one-point space, namely $H^0_{\mathrm{Sing}}(M;G) \cong G$ and $H^k_{\mathrm{Sing}}(M;G) = 0$ for $k > 0$.

The natural action of singular cochains on singular chains defines a bilinear map from $\mathrm{Sing}^k(M;G) \times \mathrm{Sing}_k(M)$ to G, carrying (φ, c) to $\varphi(c)$. This descends to a bilinear map

$$H^k_{\mathrm{Sing}}(M;G) \times H_k(M) \to G,$$

called the ***Kronecker pairing*** and denoted by $\langle[\varphi],[c]\rangle = \varphi(c)$. To see that this is well-defined, note that because c is a cycle, if $\varphi = \delta\psi$ is a coboundary we have

$\langle[\varphi],[c]\rangle = \langle[\delta\psi],[c]\rangle = (\delta\psi)(c) = \psi(\partial c) = 0$; and similarly if c is a boundary then $\langle[\varphi],[c]\rangle = 0$. Using this pairing, we define the ***Kronecker homomorphism***

$$\kappa\colon\ H^k_{\mathrm{Sing}}(M;G) \to \mathrm{Hom}(H_k(M),G)$$

by

$$\kappa([\varphi])([c]) = \langle[\varphi],[c]\rangle.$$

The singular cohomology groups of a space contain exactly the same information as the homology groups, but arranged in a different way. The precise statement of this fact, called the *universal coefficient theorem* ([**Hat02**, Chap. 3] or [**Mun84**, §53]), gives explicit formulas for the cohomology groups in terms of the homology groups. We do not need the full strength of that theorem, but we will need the following two consequences.

Proposition 6.16 (Universal Coefficient Theorem, Special Case). *Let M be a topological space and G be an abelian group, and let $\kappa\colon H^k_{\mathrm{Sing}}(M;G) \to \mathrm{Hom}(H_k(M);G)$ be the Kronecker homomorphism in degree k.*

(a) *κ is surjective.*

(b) *If G is a field, then κ is a vector space isomorphism.*

(c) *If $H_{k-1}(M)$ is a free abelian group, then κ is a group isomorphism.*

The following consequence of the universal coefficient theorem is proved in [**Hat02**, Cor. 3.4] or [**Mun84**, Thm. 45.5].

Proposition 6.17 (Homology Isomorphisms Yield Cohomology Isomorphisms). *Suppose A_* and B_* are chain complexes of free abelian groups and $\varphi\colon A_* \to B_*$ is a chain map, meaning that the following diagram commutes:*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & A_{k+1} & \xrightarrow{\partial} & A_k & \xrightarrow{\partial} & A_{k-1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \\
\cdots & \xrightarrow[\partial]{} & B_{k+1} & \xrightarrow[\partial]{} & B_k & \xrightarrow[\partial]{} & B_{k-1} & \xrightarrow[\partial]{} & \cdots .
\end{array}$$

Let G be an abelian group or a real or complex vector space, and define $A^k = \mathrm{Hom}(A_k,G)$ and $B^k = \mathrm{Hom}(B_k,G)$. Consider the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta} & A^{k-1} & \xrightarrow{\delta} & A^k & \xrightarrow{\delta} & A^{k+1} & \xrightarrow{\delta} & \cdots \\
 & & {\scriptstyle\varphi^\#}\uparrow & & {\scriptstyle\varphi^\#}\uparrow & & {\scriptstyle\varphi^\#}\uparrow & & \\
\cdots & \xrightarrow[\delta]{} & B^{k-1} & \xrightarrow[\delta]{} & B^k & \xrightarrow[\delta]{} & B^{k+1} & \xrightarrow[\delta]{} & \cdots ,
\end{array}$$

where $(\varphi^\# \gamma)(c) = \gamma(\varphi c)$, and $(\delta\gamma)(c) = \gamma(\partial c)$. If the induced homology homomorphisms $\varphi_* : H_k(A_*) \to H_k(B_*)$ are all isomorphisms, then so are the induced cohomology homomorphisms $\varphi^* : H^k(B^*) \to H^k(A^*)$.

Cohomology of Constant Sheaves

The next theorem shows that for a class of spaces that includes all manifolds, the sheaf cohomology groups of a constant sheaf agree with the singular cohomology groups. We need one last definition before introducing the theorem: a topological space is said to be ***locally contractible*** if it has a basis of contractible open subsets. For example, every topological manifold is locally contractible.

Theorem 6.18 (Cohomology of Constant Sheaves). *Suppose M is a locally contractible paracompact Hausdorff space, G is an abelian group, and $\underline{G}$ is the corresponding constant sheaf on M. Then for each $k \geq 0$, $H^k(M;\underline{G})$ is isomorphic to $H^k_{\mathrm{Sing}}(M;G)$. If G is a real or complex vector space, then the isomorphism is linear. The isomorphisms are natural in the following sense: if $F : G \to H$ is a homomorphism of abelian groups* (*or a linear map between real or complex vector spaces*) *and $\underline{F} : \underline{G} \to \underline{H}$ is the corresponding morphism between the constant sheaves as in Example* 5.5(d), *then the following diagram commutes:*

$$\begin{array}{ccc} H^k(M;\underline{G}) & \xrightarrow{\cong} & H^k_{\mathrm{Sing}}(M;G) \\ {\scriptstyle \underline{F}_*}\big\downarrow & & \big\downarrow{\scriptstyle F_*} \\ H^k(M;\underline{H}) & \xrightarrow{\cong} & H^k_{\mathrm{Sing}}(M;H). \end{array} \tag{6.21}$$

Proof. Let G be fixed. For each $k \geq 0$, we define a presheaf Sing^k on M by $U \mapsto \mathrm{Sing}^k(U;G)$, the group of singular k-cochains on U, with restriction maps r^U_V given by restricting cochains to singular chains in V. Let $\mathrm{Sing}^{k,+}$ be its sheafification. Because the singular coboundary operator δ commutes with restrictions, it induces a presheaf morphism $\delta : \mathrm{Sing}^k \to \mathrm{Sing}^{k+1}$ for each k, and corresponding sheaf morphisms $\delta^+ : \mathrm{Sing}^{k,+} \to \mathrm{Sing}^{k+1,+}$.

Consider the following sequence of sheaves on M:

$$0 \to \underline{G} \xrightarrow{\iota} \mathrm{Sing}^{0,+} \xrightarrow{\delta^+} \mathrm{Sing}^{1,+} \xrightarrow{\delta^+} \cdots, \tag{6.22}$$

where ι is obtained from the presheaf morphism that maps a locally constant function $f : U \to G$ to the 0-cochain that assigns the value $f(\sigma(0))$ to each singular 0-simplex $\sigma : \Delta_0 \to U$. We will show that this sequence is an acyclic resolution of $\underline{G}$.

Exactness of the sequence at $\underline{G}$ (i.e., injectivity of ι) is immediate. To prove exactness at $\mathrm{Sing}^{0,+}$, note that a singular 0-chain $c \in \mathrm{Sing}^0(U;G)$ is an arbitrary function from points in U to G, and it satisfies $\delta c = 0$ if and only if $c(x) = c(y)$ for all x and y in the same path component of U. Since we are assuming M is locally contractible, it is in particular locally path-connected, so this is the same as saying c is locally constant. This means that a cochain $\varphi \in \mathrm{Sing}^0(U;G)$ satisfies $\delta\varphi = 0$ if and only if it is locally constant, so exactness already holds at Sing^0 on the presheaf level.

To see that the sequence is exact at $\mathrm{Sing}^{k,+}$ for $k \geq 1$, observe first that $\delta^+ \circ \delta^+ = 0$ because $\delta \circ \delta = 0$. Suppose $\Phi \in \mathrm{Sing}^{k,+}(U;G)$ satisfies $\delta^+\Phi = 0$. For each $x \in U$, $\Phi(x)$ is the germ of a k-cochain φ on a neighborhood V of x satisfying $\delta\varphi = 0$. Since M is locally contractible, we can shrink V if necessary so that it is contractible. Since $H^k_{\mathrm{Sing}}(V;G) = 0$, it follows that φ is equal to $\delta\beta$ for some $\beta \in \mathrm{Sing}^{k-1}(V;G)$; so the sheaf sequence is exact by Lemma 5.20.

To see that the sheaves $\mathrm{Sing}^{k,+}$ are acyclic, we will show they are fine. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be any locally finite open cover of M, and let $\{\psi_\alpha\}$ be a subordinate topological partition of unity. Define another (discontinuous) partition of unity $\{\Psi_\alpha\}$ subordinate to $\mathcal{U}$ by choosing a well-ordering of the index set A and letting $\Psi_\alpha(x) = 1$ if α is the least index for which $\psi_\alpha(x) > 0$, and otherwise $\Psi_\alpha(x) = 0$. Note that the support of Ψ_α is contained in that of ψ_α and thus in U_α, and $\sum_\alpha \Psi_\alpha(x) = 1$ for all x because exactly one term in the sum is equal to 1 and the rest are zero. Then we define a collection of presheaf morphisms $\eta_\alpha : \mathrm{Sing}^k \to \mathrm{Sing}^k$ by

$$\eta_\alpha(\varphi)(\sigma) = \Psi_\alpha(\sigma(0))\varphi(\sigma)$$

for any $\varphi \in \mathrm{Sing}^k(U;G)$ and any singular k-simplex $\sigma : \Delta_k \to U$; here $\sigma(0)$ is the image of $0 \in \Delta_k$ under the map σ. The associated sheaf morphisms $\eta_\alpha^+ : \mathrm{Sing}^{k,+} \to \mathrm{Sing}^{k,+}$ form a sheaf partition of unity subordinate to $\mathcal{U}$, so $\mathrm{Sing}^{k,+}$ is fine.

It follows from the de Rham–Weil theorem that for each $k \geq 1$,

$$H^k(M;\underline{G}) \cong \frac{\mathrm{Ker}\left(\delta^+ : \mathrm{Sing}^{k,+}(M;G) \to \mathrm{Sing}^{k+1,+}(M;G)\right)}{\mathrm{Im}\left(\delta^+ : \mathrm{Sing}^{k-1,+}(M;G) \to \mathrm{Sing}^{k,+}(M;G)\right)}.$$

Let $H^k_{\mathrm{Sing}+}(M;G)$ denote the quotient group on the right-hand side of this equation. The last step of the proof is to show that $H^k_{\mathrm{Sing}+}M;G)$ is isomorphic to $H^k_{\mathrm{Sing}}(M;G)$, which is given by the same formula but with Sing^* and δ in place of $\mathrm{Sing}^{*,+}$ and δ^+.

Consider the following commutative diagram of group homomorphisms:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Sing}^0_0(M;G) & \hookrightarrow & \mathrm{Sing}^0(M;G) & \xrightarrow{\theta} & \mathrm{Sing}^{0,+}(M;G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta^+} & & \\
0 & \longrightarrow & \mathrm{Sing}^1_0(M;G) & \hookrightarrow & \mathrm{Sing}^1(M;G) & \xrightarrow{\theta} & \mathrm{Sing}^{1,+}(M;G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta^+} & & \\
0 & \longrightarrow & \mathrm{Sing}^2_0(M;G) & \hookrightarrow & \mathrm{Sing}^2(M;G) & \xrightarrow{\theta} & \mathrm{Sing}^{2,+}(M;G) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\tag{6.23}
$$

where for each k, $\mathrm{Sing}^k_0(M;G)$ is the subgroup of cochains $\varphi \in \mathrm{Sing}^k(M;G)$ that are ***locally zero***, meaning there is some open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ such that $\varphi|_{U_\alpha} = 0$ for every α; and $\theta = \theta_{\mathrm{Sing}^k}$ is the global section map associated with the canonical presheaf morphism $\theta_{\mathrm{Sing}^k} : \mathrm{Sing}^k \to \mathrm{Sing}^{k,+}$ given by Theorem 5.9. We begin by showing that each horizontal row is exact.

Exactness at $\mathrm{Sing}^k_0(M;G)$ and $\mathrm{Sing}^k(M;G)$ follows easily from the definitions. To see that θ is surjective, suppose $\Phi \in \mathrm{Sing}^{k,+}(M;G)$ is arbitrary. By Lemma 5.8, there is an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of M and for each α a cochain $\varphi_\alpha \in \mathrm{Sing}^k(U_\alpha;G)$ such that $\Phi|_{U_\alpha} = \varphi_\alpha^+$. By passing to a refinement, we may assume the cover is locally finite. It is important to note that these cochains might not agree on overlaps: since the presheaf Sing^k does not satisfy the locality property, cochains with the same germs at each point need not agree. Thus we need a more delicate argument.

Let $\{\psi_\alpha\}_{\alpha\in A}$ be a topological partition of unity subordinate to the cover $\mathcal{U}$. For each $x \in M$, we wish to choose a neighborhood V_x of x satisfying the following properties:

(i) The set $A(x) = \{\alpha \in A : V_x \cap \operatorname{supp}\psi_\alpha \neq \varnothing\}$ is finite.

(ii) $V_x \subseteq U_\alpha$ for each $\alpha \in A(x)$.

(iii) The restrictions $\varphi_\alpha|_{V_x}$ are all equal for $\alpha \in A(x)$.

(iv) $V_x \cap \operatorname{supp}\psi_\beta = \varnothing$ if $\beta \notin A(x)$.

To see that this is possible, note that the existence of a neighborhood V_x satisfying (i) is just a restatement of the local finiteness of the cover $\{\operatorname{supp}\psi_\alpha\}$. Then we can shrink V_x successively to satisfy the other three conditions: (ii) because $\bigcap_{\alpha\in A(x)} U_\alpha$ is an open set containing x; (iii) because the cochains φ_α all have the same germ at x for $\alpha \in A(x)$; and (iv) because $\bigcup_{\beta\notin A(x)} \operatorname{supp}\psi_\beta$ is closed in M by local finiteness

[**LeeTM**, Lemma 4.75]. For each $x \in M$, let $\widetilde{\varphi}_x \in \operatorname{Sing}^k(V_x; G)$ be the cochain that is the common value of $\varphi_\alpha|_{V_x}$ for all $\alpha \in A(x)$.

Now suppose x and y are points of M such that $V_x \cap V_y \neq \varnothing$, and let z be a point in $V_x \cap V_y$. There is an index $\alpha \in A$ such that $z \in \operatorname{supp} \psi_\alpha$, and then property (iv) ensures that α lies in both $A(x)$ and $A(y)$. This implies that

$$\widetilde{\varphi}_x|_{V_x \cap V_y} = \varphi_\alpha|_{V_x \cap V_y} = \widetilde{\varphi}_y|_{V_x \cap V_y}. \tag{6.24}$$

Thus we can define a global cochain $\widetilde{\varphi} \in \operatorname{Sing}^k(M; G)$ as follows: for each singular k-simplex σ in M, we set

$$\widetilde{\varphi}(\sigma) = \begin{cases} \widetilde{\varphi}_x(\sigma) & \text{if } \sigma(\Delta_k) \subseteq V_x \text{ for some } x \in M, \\ 0 & \text{if } \sigma(\Delta_k) \text{ is not contained in any } V_x, \end{cases}$$

and (6.24) ensures that this is well defined. It follows that $\theta(\widetilde{\varphi}) = \Phi$, showing that $\theta\colon \operatorname{Sing}^k(M; G) \to \operatorname{Sing}^{k,+}(M; G)$ is surjective for each k.

The zigzag lemma applied to (6.23) yields a long exact sequence which reads in part

$$H_0^k(M; G) \to H^k_{\mathrm{Sing}}(M; G) \xrightarrow{\theta_*} H^k_{\mathrm{Sing}+}(M; G) \to H_0^{k+1}(M; G),$$

where $H_0^k(M; G)$ denotes the kth cohomology group of the leftmost column of (6.23). Thus to complete the proof, it suffices to show that $H_0^k(M; G) = 0$ for all k.

To that end, suppose $\varphi \in \operatorname{Sing}_0^k(M; G)$ satisfies $\delta\varphi = 0$. By definition of Sing_0^k, there is an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of M such that $\varphi|_{U_\alpha} = 0$ for each α. A singular simplex in M is said to be ***$\mathcal{U}$-small*** if its image is contained in some U_α, and a chain $c \in \operatorname{Sing}_k(M)$ is $\mathcal{U}$-small if it can be written as a formal linear combination of $\mathcal{U}$-small simplices. Let $\operatorname{Sing}_k^{\mathcal{U}}(M) \subseteq \operatorname{Sing}_k(M)$ denote the subgroup consisting of $\mathcal{U}$-small chains. The singular boundary operator takes $\operatorname{Sing}_k^{\mathcal{U}}(M)$ to $\operatorname{Sing}_{k-1}^{\mathcal{U}}(M)$, so we have a chain complex, which we denote by $\operatorname{Sing}_*^{\mathcal{U}}(M)$. The inclusion maps $\iota\colon \operatorname{Sing}_k^{\mathcal{U}}(M) \to \operatorname{Sing}_k(M)$ commute with the boundary operators and thus define a chain map from $\operatorname{Sing}_*^{\mathcal{U}}(M)$ to $\operatorname{Sing}_*(M)$ (where $\operatorname{Sing}_*(M)$ denotes the full singular chain complex), and a subdivision argument [**LeeTM**, Prop. 13.19] shows that this chain map induces isomorphisms on all the homology groups. Let $\operatorname{Sing}^{k,\mathcal{U}}(M; G) = \operatorname{Hom}\big(\operatorname{Sing}_k^{\mathcal{U}}(M), G\big)$, and let $\iota^*\colon \operatorname{Sing}^k(M; G) \to \operatorname{Sing}^{k,\mathcal{U}}(M; G)$ be the map dual to ι; concretely, $\iota^*(\varphi)$ is just the restriction of a cochain φ to $\mathcal{U}$-small chains. Because ι induces isomorphisms on all homology groups, it follows from Proposition 6.17 that ι^* induces isomorphisms on cohomology.

Let $\operatorname{Sing}_0^{k,\mathcal{U}}(M;G) \subseteq \operatorname{Sing}^k(M;G)$ denote the kernel of ι^*; it is the group of cochains that assign the value zero to every $\mathcal{U}$-small simplex. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Sing}_0^{0,\mathcal{U}}(M;G) & \hookrightarrow & \operatorname{Sing}^0(M;G) & \xrightarrow{\iota^*} & \operatorname{Sing}^{0,\mathcal{U}}(M;G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & \operatorname{Sing}_0^{1,\mathcal{U}}(M;G) & \hookrightarrow & \operatorname{Sing}^1(M;G) & \xrightarrow{\iota^*} & \operatorname{Sing}^{1,\mathcal{U}}(M;G) & \longrightarrow & 0, \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

in which the horizontal rows are exact. Thus the zigzag lemma yields a long exact cohomology sequence, which reads in part

$$\begin{aligned}
H^{k-1}_{\text{Sing}}(M;G) \xrightarrow{\cong} H^{k-1}_{\text{Sing},\mathcal{U}}(M;G) \to H^k_{\text{Sing},\mathcal{U},0}(M;G) \\
\to H^k_{\text{Sing}}(M;G) \xrightarrow{\cong} H^k_{\text{Sing},\mathcal{U}}(M;G),
\end{aligned}$$

where the notations $H^k_{\text{Sing},\mathcal{U},0}$ and $H^k_{\text{Sing},\mathcal{U}}$ have the obvious meanings. The fact that the first and last maps are isomorphisms implies that $H^k_{\text{Sing},\mathcal{U},0}(M;G) = 0$. Therefore, there is a cochain $\beta \in \operatorname{Sing}_0^{k-1,\mathcal{U}}(M;G) \subseteq \operatorname{Sing}_0^{k-1}(M;G)$ such that $\delta\beta = \varphi$. This completes the proof that $H^k(M;\underline{G}) \cong H^k_{\text{Sing}}(M;G)$.

To prove the naturality statement, we note that the isomorphism $H^k(M;\underline{G}) \cong H^k_{\text{Sing}}(M;G)$ is the composition of two isomorphisms:

$$H^k(M;\underline{G}) \cong H^k_{\text{Sing}+}(M;G) \cong H^k_{\text{Sing}}(M;G).$$

The first isomorphism was obtained by applying the de Rham–Weil theorem to the sequence (6.22), so it is natural with respect to group homomorphisms by the naturality statement of the de Rham–Weil theorem. The second isomorphism is induced by the map $\theta\colon \operatorname{Sing}^k(M;G) \to \operatorname{Sing}^{k,+}(M;G)$, which just maps a cochain φ to the section φ^+ of the sheafification defined by (5.6); and it is straightforward to check that this map is also natural with respect to group homomorphisms. □

We have proved the preceding theorem for locally contractible paracompact Hausdorff spaces, which includes all topological manifolds with or without boundary and is more than adequate for our purposes. But the conclusion is true for somewhat more general spaces: an unpublished 2016 paper by Yehonatan Sella [**Sel16**] (see also [**Pet22**]) shows that it is true for constant sheaves over any space M that is ***semilocally contractible***, meaning that each open subset $U \subseteq M$ has an open cover by sets whose inclusions into U are homotopic to constant maps, without any paracompactness requirement. Some topological restriction is necessary, however: for example, if M is a paracompact Hausdorff space that is connected but not path-connected, then $H^0(M;\underline{\mathbb{Z}}) \cong \mathbb{Z}$, while $H^0_{\text{Sing}}(M;\mathbb{Z})$ is a direct product of copies of $\mathbb{Z}$, one for each path component.

Applications of Sheaf Cohomology

In this section, we describe several applications of sheaf cohomology to complex manifolds.

The Dolbeault Theorem

Our first application is central to complex manifold theory. This theorem was first proved in the 1950s by Pierre Dolbeault [**Dol53**, **Dol56**].

Theorem 6.19 (The Dolbeault Theorem). *Suppose M is a complex manifold and Ω^p is its sheaf of holomorphic p-forms. For each $q \geq 0$,*

$$H^{p,q}(M) \cong H^q(M;\Omega^p). \tag{6.25}$$

More generally, if $E \to M$ is a holomorphic vector bundle, then

$$H^{p,q}(M;E) \cong H^q(M;\Omega^p(E)). \tag{6.26}$$

Proof. The isomorphism (6.25) is just the special case of (6.26) in which E is the trivial line bundle $M \times \mathbb{C} \to M$, so we just need to prove (6.26). Example 5.23(c) showed that the sheaves of smooth E-valued forms $\mathscr{E}^{p,q}(E)$ give a resolution of $\Omega^p(E)$. Because each sheaf $\mathscr{E}^{p,q}(E)$ is a sheaf of $\mathscr{E}$-modules, it is fine and therefore acyclic. The theorem follows from the de Rham–Weil theorem. □

The de Rham Theorem

As we mentioned above, the de Rham–Weil theorem can be used to give a quick sheaf-theoretic proof of the following version of the de Rham theorem.

Theorem 6.20 (De Rham Theorem, Sheaf-Theoretic Version). *Suppose M is a smooth manifold. For each $k \geq 0$,*

$$H^k_{\text{dR}}(M;\mathbb{C}) \cong H^k(M;\underline{\mathbb{C}}). \tag{6.27}$$

The analogous statement is true for real de Rham cohomology.

Proof. For each nonnegative integer k, let $\mathscr{E}^k$ be the sheaf of smooth complex-valued k-forms on M. Consider the following sheaf sequence:

$$0 \to \underline{\mathbb{C}} \hookrightarrow \mathscr{E}^0 \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \xrightarrow{d} \cdots. \tag{6.28}$$

We observed in Example 5.23(a) that this sequence is exact, and since the sheaves $\mathscr{E}^k$ are fine, it is an acyclic resolution of the constant sheaf $\underline{\mathbb{C}}$. Thus the cohomology groups of its global section sequence (which are the complex de Rham cohomology groups) are isomorphic to the sheaf cohomology groups of $\underline{\mathbb{C}}$ by the de Rham–Weil theorem. The same argument applies with real coefficients. □

The naturality statement of the de Rham–Weil theorem leads to the following relationship between the de Rham and Dolbeault cohomology groups on a complex manifold.

Proposition 6.21. *Let M be a complex manifold. For each integer $q \geq 0$, the projection $\pi^{0,q} : \mathscr{E}^q(M) \to \mathscr{E}^{0,q}(M)$ descends to a linear map $\pi_*^{0,q} : H^q_{\mathrm{dR}}(M;\mathbb{C}) \to H^{0,q}(M)$, such that the following diagram commutes:*

$$\begin{array}{ccc} H^q_{\mathrm{dR}}(M;\mathbb{C}) & \xrightarrow{\mathscr{R}} & H^q(M;\underline{\mathbb{C}}) \\ {\scriptstyle \pi_*^{0,q}}\downarrow & & \downarrow{\scriptstyle i_*} \\ H^{0,q}(M) & \xrightarrow[\mathscr{D}]{} & H^q(M;\mathscr{O}), \end{array} \tag{6.29}$$

where $\mathscr{R}$ and $\mathscr{D}$ are the isomorphisms given by Theorems 6.20 and 6.19, respectively, and i_* is induced by the sheaf inclusion $i : \underline{\mathbb{C}} \hookrightarrow \mathscr{O}$.

Proof. Consider the following diagram of sheaf morphisms:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \underline{\mathbb{C}} & \longrightarrow & \mathscr{E}^0 & \xrightarrow{d} & \mathscr{E}^1 & \xrightarrow{d} & \mathscr{E}^2 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \pi^{0,1}} & & \downarrow{\scriptstyle \pi^{0,2}} & & \\ 0 & \longrightarrow & \mathscr{O} & \longrightarrow & \mathscr{E}^{0,0} & \xrightarrow{\bar\partial} & \mathscr{E}^{0,1} & \xrightarrow{\bar\partial} & \mathscr{E}^{0,2} & \longrightarrow & \cdots . \end{array}$$

For $\eta \in \mathscr{E}^q(U)$ on some open subset $U \subseteq M$, we have a decomposition

$$\eta = \eta^{q,0} + \eta^{q-1,1} + \cdots + \eta^{1,q-1} + \eta^{0,q}.$$

The only part of η that can contribute to the $(0, q+1)$-part of $d\eta$ is $\eta^{0,q}$, so $\pi^{0,q+1} d\eta = \bar\partial \pi^{0,q}\eta$. Thus the above sheaf diagram commutes, and it follows that $\pi^{0,q}$ descends to a well-defined linear map from $H^q_{\mathrm{dR}}(M;\mathbb{C})$ to $H^{0,q}(M)$. Commutativity of (6.29) is an immediate consequence of the naturality statement of the de Rham–Weil theorem. □

It is important to note that the above proof does not apply to $\pi^{p,q}$ for $p \neq 0$, because the $(p, q+1)$-part of $d\eta$ can involve contributions from both $\eta^{p,q}$ and $\eta^{p-1,q+1}$. Later we will see that for a restricted class of manifolds (compact Kähler manifolds), there are well-defined projections $\pi_*^{p,q} : H^{p+q}(M;\mathbb{C}) \to H^{p,q}(M)$, but the proof is considerably more complicated (see Thm. 9.44).

By combining the isomorphism $H^k_{\mathrm{dR}}(M;\mathbb{R}) \cong H^k(M;\underline{\mathbb{R}})$ of Theorem 6.20 with the isomorphism $H^k(M;\underline{\mathbb{R}}) \cong H^k_{\mathrm{Sing}}(M;\mathbb{R})$ of Theorem 6.18, we obtain the classical de Rham isomorphism $H^k_{\mathrm{dR}}(M;\mathbb{R}) \cong H^k_{\mathrm{Sing}}(M;\mathbb{R})$ and its complex analogue. But we need a more precise version of this, which says explicitly that the isomorphism is given by integration of differential forms. For that purpose, we introduce a modified version of singular cohomology.

Here is the setup. Suppose M is a smooth manifold and k is a nonnegative integer. A singular k-simplex $\sigma\colon \Delta_k \to M$ is called a ***smooth singular simplex*** if it is smooth in the sense that it can be extended to a smooth map defined on a neighborhood of Δ_k in $\mathbb{R}^k$. A chain $c \in \mathrm{Sing}_k(M)$ is called a ***smooth (singular) chain*** if it can be written as a formal linear combination of smooth singular simplices. Let $\mathrm{Sing}^\infty_k(M) \subseteq \mathrm{Sing}_k(M)$ denote the subgroup consisting of smooth singular chains in M. Since the singular boundary operator maps smooth chains to smooth chains, the sequence

$$\cdots \to \mathrm{Sing}^\infty_{k+1}(M) \xrightarrow{\partial} \mathrm{Sing}^\infty_k(M) \xrightarrow{\partial} \mathrm{Sing}^\infty_{k-1}(M) \to \cdots$$

is a chain complex, whose homology groups are called ***smooth singular homology groups*** and denoted by $H^\infty_k(M)$.

The fundamental fact about smooth singular homology is the following theorem, whose proof can be found in [**LeeSM**, Thm. 18.7].

Theorem 6.22 (Smooth Singular vs. Singular Homology). *For every smooth manifold M and nonnegative integer k, the map $i_*\colon H^\infty_k(M) \to H_k(M)$ induced by inclusion $i\colon \mathrm{Sing}^\infty_k(M) \hookrightarrow \mathrm{Sing}_k(M)$ is an isomorphism.*

Now let G be an abelian group and let

$$\mathrm{Sing}^{k,\infty}(M;G) = \mathrm{Hom}\big(\mathrm{Sing}^\infty_k(M), G\big).$$

The singular coboundary maps $\delta\colon \mathrm{Sing}^{k,\infty}(M;G) \to \mathrm{Sing}^{k+1,\infty}(M;G)$ are defined just as in the case of continuous cochains, and they satisfy $\delta\circ\delta = 0$. Thus we have a cochain complex

$$\cdots \to \mathrm{Sing}^{k-1,\infty}(M;G) \xrightarrow{\delta} \mathrm{Sing}^{k,\infty}(M;G) \xrightarrow{\delta} \mathrm{Sing}^{k+1,\infty}(M;G) \to \cdots.$$

Its cohomology groups, which we denote by $H^k_{\mathrm{Sing},\infty}(M;G)$, are called the ***smooth singular cohomology groups of M with coefficients in G***,

The dual of the inclusion map $i\colon \mathrm{Sing}^\infty_k(M) \hookrightarrow \mathrm{Sing}_k(M)$ is a homomorphism $R\colon \mathrm{Sing}^k(M;G) \to \mathrm{Sing}^{k,\infty}(M;G)$, which is just restriction: if φ is a singular k-cochain on M, then $R(\varphi)$ is just the restriction of φ to smooth singular chains. It commutes with the coboundary operators and thus descends to a linear map $R_*\colon H^k_{\mathrm{Sing}}(M;G) \to H^k_{\mathrm{Sing},\infty}(M;G)$. Because inclusion of smooth chains induces isomorphisms in homology, the next corollary follows immediately from Proposition 6.17.

Corollary 6.23 (Smooth Singular vs. Singular Cohomology). *Suppose M is a smooth manifold and G is an abelian group or a real or complex vector space. The map $R_*: H^k_{\mathrm{Sing}}(M;G) \to H^k_{\mathrm{Sing},\infty}(M;G)$ is an isomorphism for each nonnegative integer k.* □

Let $\mathscr{E}^k(M)$ denote the space of smooth complex-valued k-forms on M. If $\eta \in \mathscr{E}^k(M)$ and $c = \sum_j a_j \sigma_j$ is a smooth singular k-chain, we define the ***integral of η over c*** by

$$\int_c \eta = \sum_j a_j \int_{\Delta_k} \sigma_j^* \eta.$$

Define a linear map $I: \mathscr{E}^k(M) \to \mathrm{Sing}^{k,\infty}(M;\mathbb{C})$ by

$$I(\eta)(c) = \int_c \eta$$

for every smooth k-form η and smooth chain c. A version of Stokes's theorem for smooth singular chains [**LeeSM**, Thm. 18.12] shows that

$$\int_c d\eta = \int_{\partial c} \eta. \tag{6.30}$$

This implies

$$\delta(I(\eta))(c) = I(\eta)(\partial c) = \int_{\partial c} \eta = \int_c d\eta = I(d\eta)(c),$$

which is to say that

$$\delta \circ I = I \circ d. \tag{6.31}$$

Thus I descends to a linear map $I_*: H^k_{\mathrm{dR}}(M;\mathbb{C}) \to H^k_{\mathrm{Sing},\infty}(M;\mathbb{C})$.

Here is the theorem that describes the detailed relationship between de Rham and singular cohomology. (For a different proof, see [**LeeSM**, Thm. 18.14].)

Theorem 6.24 (De Rham Theorem, Integration Version). *For every smooth manifold M and every integer $k \geq 0$, the map $I_*: H^k_{\mathrm{dR}}(M;\mathbb{C}) \to H^k_{\mathrm{Sing},\infty}(M;\mathbb{C})$ is an isomorphism. The analogous statement holds for real de Rham cohomology.*

Proof. The proofs for real and complex coefficients are essentially identical, so we just treat the complex case. Let M be a smooth manifold. For each nonnegative integer k, the assignment $U \mapsto \mathrm{Sing}^{k,\infty}(U;\mathbb{C})$ is a presheaf on M, and there is a presheaf morphism $I: \mathscr{E}^k \to \mathrm{Sing}^{k,\infty}$, with $I: \mathscr{E}^k(U) \to \mathrm{Sing}^{k,\infty}(U;\mathbb{C})$ defined by integration over smooth chains in U as above. Let $\mathrm{Sing}^{k,\infty+}$ be the sheafification of the presheaf $\mathrm{Sing}^{k,\infty}$, and let $I^+: \mathscr{E}^k \to \mathrm{Sing}^{k,\infty+}$ be the sheafification of I (where we identify the sheaf $\mathscr{E}^k$ with its sheafification). Consider the following diagram of sheaf morphisms:

$$
(6.32)\qquad
\begin{array}{ccccccccc}
0 \longrightarrow \underline{\mathbb{C}} & \longrightarrow & \mathscr{E}^0 & \xrightarrow{\;d\;} & \mathscr{E}^1 & \xrightarrow{\;d\;} & \mathscr{E}^2 & \xrightarrow{\;d\;} & \cdots \\
\Big\downarrow{=} & & \Big\downarrow{I^+} & & \Big\downarrow{I^+} & & \Big\downarrow{I^+} & & \\
0 \longrightarrow \underline{\mathbb{C}} & \longrightarrow & \operatorname{Sing}^{0,\infty+} & \xrightarrow[\delta^+]{} & \operatorname{Sing}^{1,\infty+} & \xrightarrow[\delta^+]{} & \operatorname{Sing}^{2,\infty+} & \xrightarrow[\delta^+]{} & \cdots .
\end{array}
$$

It commutes by virtue of (6.31) and the functoriality of sheafification. The top row is an acyclic resolution of $\underline{\mathbb{C}}$. The proof of Theorem 6.18 showed that the analogue of the second row, with $\operatorname{Sing}^{k,+}$ in place of $\operatorname{Sing}^{k,\infty+}$, is also an acyclic resolution; the same proof applies in this case, once we observe that the smooth singular cohomology groups of a contractible open set are trivial by virtue of Theorem 6.22 and the universal coefficient theorem.

Applying the naturality statement of the de Rham–Weil theorem to (6.32), we find that the following diagram commutes for each k:

$$
\begin{array}{ccc}
H^k(M;\underline{\mathbb{C}}) & \xrightarrow{\;\cong\;} & H^k_{\mathrm{dR}}(M;\mathbb{C}) \\
{=}\Big\downarrow & & \Big\downarrow (I^+)_* \\
H^k(M;\underline{\mathbb{C}}) & \xrightarrow[\cong]{} & H^k_{\mathrm{Sing},\infty+}(M;\mathbb{C}).
\end{array}
$$

Thus $(I^+)_*$ is an isomorphism. On the other hand, Theorem 5.9 shows there is a commutative sheaf diagram

$$
\begin{array}{ccc}
\mathscr{E}^k & \xrightarrow{\;I\;} & \operatorname{Sing}^{k,\infty} \\
{=}\Big\downarrow & & \Big\downarrow \theta \\
\mathscr{E}^k & \xrightarrow[I^+]{} & \operatorname{Sing}^{k,\infty+}
\end{array}
$$

(still identifying $\mathscr{E}^k$ with its sheafification), and passing to induced cohomology maps, we find that the isomorphism $(I^+)_*$ is equal to the composition

$$
(6.33)\qquad H^k_{\mathrm{dR}}(M;\mathbb{C}) \xrightarrow{I_*} H^k_{\mathrm{Sing},\infty}(M;\mathbb{C}) \xrightarrow{\theta_*} H^k_{\mathrm{Sing},\infty+}(M;\mathbb{C}).
$$

Thus to show that I_* is an isomorphism, we need to show that the map θ_* above is an isomorphism. The proof of Theorem 6.18 showed that the analogous map $\theta_*: H^k_{\mathrm{Sing}}(M;\mathbb{C}) \to H^k_{\mathrm{Sing},+}(M;\mathbb{C})$ is an isomorphism. The same proof applies here; the only additional observation that needs to be made is that when the subdivision operators of [**LeeTM**, Prop. 13.19] are applied to smooth chains, they produce smooth chains, because the new singular simplices are defined as compositions of the original smooth simplices with affine maps between subsets of Euclidean spaces. □

The next corollary is an immediate consequence of Corollary 6.23 and Theorem 6.24.

Corollary 6.25. *For a smooth manifold M and a nonnegative integer k, define $\mathscr{I} : H^k_{\mathrm{dR}}(M;\mathbb{C}) \to H^k_{\mathrm{Sing}}(M;\mathbb{C})$ by $\mathscr{I} = R_*^{-1} \circ I_*$, where I_* is the map of Theorem* 6.24 *and R_* is the map of Corollary* 6.23. *Then $\mathscr{I}$ is an isomorphism.* □

Corollary 6.26. *Let M be a smooth manifold. For any closed k-form η and singular k-cycle c, the Kronecker pairing between $\mathscr{I}[\eta] \in H^k_{\mathrm{Sing}}(M;\mathbb{C})$ and $[c] \in H_k(M)$ is given by*

$$\langle \mathscr{I}[\eta],\ [c]\rangle = \int_{\tilde{c}} \eta,$$

where $\tilde{c}$ is any smooth cycle cohomologous to c.

Proof. Let η and c be given. By definition, $\mathscr{I}[\eta] = R_*^{-1} \circ I_*[\eta]$. Thus there is a singular cocycle $\varphi \in \mathrm{Sing}^k(M;\mathbb{C})$ such that $R(\varphi) \in \mathrm{Sing}^{k,\infty}(M;\mathbb{C})$ is cohomologous to $I(\eta)$, and $\mathscr{I}[\eta] = [\varphi]$. By definition of the Kronecker pairing, $\langle \mathscr{I}[\eta], [c]\rangle = \varphi(c)$. Theorem 6.22 shows that there is a smooth cycle $\tilde{c}$ homologous to c, and since the cocycle φ gives the same value on homologous cycles, the Kronecker pairing is also equal to $\varphi(\tilde{c})$. Since $\tilde{c}$ is smooth, this is the same as $R(\varphi)(\tilde{c}) = I(\eta)(\tilde{c}) = \int_{\tilde{c}} \eta$. □

In Chapters 9 and 10, it will be important to know which closed forms represent integral cohomology classes in the following sense: If M is a smooth manifold, we say that a cohomology class in $H^k_{\mathrm{dR}}(M;\mathbb{C})$ (or either of the isomorphic groups $H^k(M;\underline{\mathbb{C}})$ or $H^k_{\mathrm{Sing}}(M;\mathbb{C})$) is ***integral*** if it lies in the image of the coefficient homomorphism $i_* : H^k_{\mathrm{Sing}}(M;\mathbb{Z}) \to H^k_{\mathrm{Sing}}(M;\mathbb{C})$ (or equivalently $H^k(M;\underline{\mathbb{Z}}) \to H^k(M;\underline{\mathbb{C}})$) induced by inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{C}$.

The next lemma gives a simple criterion for detecting when a de Rham cohomology class is integral.

Lemma 6.27. *Let M be a smooth manifold and let η be a closed k-form on M. Then η represents an integral cohomology class if and only if $\int_c \eta \in \mathbb{Z}$ for every smooth k-cycle c in M. If $H_k(M)$ is finitely generated, then the set of integral classes is a free abelian subgroup of $H^k_{\mathrm{dR}}(M;\mathbb{C})$, a basis for which is also a basis for $H^k_{\mathrm{dR}}(M;\mathbb{C})$ over $\mathbb{C}$ and a basis for $H^k_{\mathrm{dR}}(M;\mathbb{R})$ over $\mathbb{R}$.*

Proof. Consider the following commutative diagram of group homomorphisms:

$$\begin{array}{ccccc}
 & & H^k_{\mathrm{Sing}}(M;\mathbb{Z}) & \xrightarrow{\kappa_{\mathbb{Z}}} & \mathrm{Hom}(H_k(M),\mathbb{Z}) \\
 & & \Big\downarrow i_* & & \Big\downarrow i_\# \\
H^k_{\mathrm{dR}}(M;\mathbb{C}) & \xrightarrow[\mathscr{I}]{} & H^k_{\mathrm{Sing}}(M;\mathbb{C}) & \xrightarrow[\kappa_{\mathbb{C}}]{} & \mathrm{Hom}(H_k(M),\mathbb{C}),
\end{array}$$

where $\kappa_{\mathbb{Z}}$ and $\kappa_{\mathbb{C}}$ are the respective Kronecker homomorphisms and $i_\#$ is post-composition with $i: \mathbb{Z} \hookrightarrow \mathbb{C}$. Corollary 6.26 shows that the composition $\kappa_{\mathbb{C}} \circ \mathscr{I}[\eta]$ is the homomorphism that sends $[c] \in H_k(M)$ to $\int_{\tilde{c}} \eta$ for a smooth cycle $\tilde{c}$ homologous to c.

On the one hand, if $[\eta]$ is integral, then there is a class $\gamma \in H^k_{\mathrm{Sing}}(M;\mathbb{Z})$ such that $i_*\gamma = \mathscr{I}[\eta]$, which implies

$$\kappa_{\mathbb{C}} \circ \mathscr{I}[\eta] = \kappa_{\mathbb{C}} \circ i_*\gamma = i_\# \circ \kappa_{\mathbb{Z}}\gamma,$$

and thus the homomorphism $\kappa_{\mathbb{C}} \circ \mathscr{I}[\eta]$ takes only integer values.

Conversely, if $\kappa_{\mathbb{C}} \circ \mathscr{I}[\eta]$ takes only integer values, then $\kappa_{\mathbb{C}} \circ \mathscr{I}[\eta] = i_\#\varphi$ for $\varphi \in \operatorname{Hom}(H_k(M);\mathbb{Z})$ (given by the same formula). Since $\kappa_{\mathbb{Z}}$ is surjective by the universal coefficient theorem (Prop. 6.16), it follows that there is some $\gamma \in H^k_{\mathrm{Sing}}(M;\mathbb{Z})$ such that $\kappa_{\mathbb{Z}}\gamma = \varphi$, which implies

$$\kappa_{\mathbb{C}} \circ \mathscr{I}[\eta] = i_\# \circ \kappa_{\mathbb{Z}}\gamma = \kappa_{\mathbb{C}} \circ i_*\gamma.$$

Since $\kappa_{\mathbb{C}}$ is an isomorphism, this implies $\mathscr{I}[\eta] = i_*\gamma$, so $[\eta]$ is integral.

To prove the last statement, assume $H_k(M)$ is finitely generated, and let $T \subseteq H_k(M)$ be its ***torsion subgroup*** (the subgroup of elements x that satisfy $nx = 0$ for some positive integer n). Then $H_k(M)/T \cong \mathbb{Z}^m$ for some nonnegative integer m. Let $(\gamma_1, \dots, \gamma_m)$ be elements of $H_k(M)$ whose images in the quotient group form a basis for $H_k(M)/T$. Since every homomorphism to $\mathbb{Z}$ annihilates torsion elements, $\operatorname{Hom}(H_k(M),\mathbb{Z}) \cong \operatorname{Hom}(H_k(M)/T,\mathbb{Z})$, so it is a free abelian group with basis $(\varphi^1, \dots, \varphi^m)$, where $\varphi^i(\gamma_j) = \delta^i_j$. These same elements are also a basis for the vector space $\operatorname{Hom}(H_k(M),\mathbb{C}) \cong \operatorname{Hom}(H_k(M)/T,\mathbb{C})$ over $\mathbb{C}$, and therefore their images in the isomorphic space $H^k_{\mathrm{dR}}(M;\mathbb{C})$ are a basis over $\mathbb{Z}$ for the subgroup of integral classes and also a basis over $\mathbb{C}$ for $H^k_{\mathrm{dR}}(M;\mathbb{C})$ itself. The argument for $H^k_{\mathrm{dR}}(M;\mathbb{R})$ is the same. □

Chern Classes of Line Bundles

For our next application, suppose M is a complex manifold and $\mathscr{O}^*$ is its sheaf of nonvanishing holomorphic functions. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of M. The computations in Example 6.3 showed that a Čech 1-cocycle on $\mathcal{U}$ with coefficients in $\mathscr{O}^*$ consists of a collection of nonvanishing holomorphic functions $\tau_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}$ that satisfy

$$\tau_{\alpha\gamma} = \tau_{\alpha\beta}\tau_{\beta\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

This is exactly the data for constructing a holomorphic line bundle, so we get a map

$$\mathscr{L}: Z^1(\mathcal{U};\mathscr{O}^*) \to \operatorname{Pic}(M)$$

where $Z^1(\mathcal{U};\mathscr{O}^*)$ is the group of Čech cycles on $\mathcal{U}$ with coefficients in $\mathscr{O}^*$, and $\operatorname{Pic}(M)$ is the Picard group of M (the group of isomorphism classes of holomorphic line bundles under the tensor product).

Proposition 6.28. *Let M be a complex manifold. The map $\mathscr{L}$ defined above descends to a group isomorphism*

$$\mathscr{L}_* : H^1(M;\mathcal{O}^*) \cong \operatorname{Pic}(M).$$

Similarly, if M is a smooth manifold, $H^1(M;\mathscr{E}^*)$ is isomorphic to the group of isomorphism classes of smooth complex line bundles on M; and if M is any topological space, $H^1(M;\mathscr{C}^*)$ is isomorphic to the group of isomorphism classes of topological complex line bundles.

Proof. We first note that for a fixed open cover $\mathscr{U}$ of M, the map $\mathscr{L}$ is a group homomorphism from $Z^1(\mathscr{U};\mathcal{O}^*)$ to $\operatorname{Pic}(M)$, because the transition functions for a tensor product $L \otimes L'$ are the products of the ones for L and the ones for L'.

Next we show that for a cocycle $\tau \in Z^1(\mathscr{U};\mathcal{O}^*)$, the isomorphism class of the bundle $\mathscr{L}(\tau)$ depends only on the cohomology class of τ. If τ and τ' are cohomologous, the computations in Example 6.3 show that there is a 0-cocycle ψ such that

$$\tau_{\alpha\beta} = \tau'_{\alpha\beta}\psi_\beta\psi_\alpha^{-1} \text{ on } U_\alpha \cap U_\beta,$$

so Proposition 3.7 shows that the bundles $\mathscr{L}(\tau)$ and $\mathscr{L}(\tau')$ are isomorphic. Thus $\mathscr{L}$ descends to a homomorphism from $H^1(\mathscr{U};\mathcal{O}^*)$ to $\operatorname{Pic}(M)$.

If $\mathscr{V} = \{V_\beta : \beta \in B\}$ is a refinement of $\mathscr{U}$ and ρ is a refining map, then we obtain a trivialization of the bundle $\mathscr{L}(\tau)$ over each subset V_β by restricting the one over $U_{\rho(\beta)}$, and the transition maps between these trivializations are the restrictions of those over $\mathscr{U}$, which is to say

$$\tau_{\alpha\beta} = \tau_{\rho(\alpha)\rho(\beta)}\big|_{V_\alpha \cap V_\beta}.$$

This implies that $\mathscr{V}$ is a trivializing cover for both $\mathscr{L}(\tau)$ and $\mathscr{L}(\rho^\#\tau)$ with the same transition functions, so these two bundles are isomorphic. Therefore, $\mathscr{L}$ passes to the direct limit to define a homomorphism $\mathscr{L}_* : H^1(M;\mathcal{O}^*) \to \operatorname{Pic}(M)$.

If $L \to M$ is any holomorphic line bundle, we can choose a trivializing cover $\mathscr{U}$, and the argument above shows that the transition functions determine a 1-cocycle and therefore an element of $H^1(M;\mathcal{O}^*)$ whose image under $\mathscr{L}_*$ is the isomorphism class of L; thus $\mathscr{L}_*$ is surjective.

On the other hand, to show that $\mathscr{L}_*$ is injective, it suffices to show that its kernel consists only of the zero cohomology class. Suppose $\mathscr{L}_*(\gamma)$ is the equivalence class of the trivial bundle for some $\gamma \in H^1(M;\mathcal{O}^*)$. If τ is a cocycle representing γ over some open cover $\mathscr{U}$, then Corollary 3.9 shows there is a 0-cochain $\psi \in C^0(\mathscr{U};\mathcal{O}^*)$ such that $\tau_{\alpha\beta} = \psi_\beta\psi_\alpha^{-1}$ on $U_\alpha \cap U_\beta$, which means $\tau = \delta\psi$. Thus τ represents the trivial cohomology class, which is to say $\gamma = 0$.

The analogous results for smooth and topological line bundles are proved in exactly the same way, with only minor changes in notation. □

Suppose M is a smooth manifold. Consider the smooth exponential sheaf sequence on M (Example 5.23(e)):

$$0 \to \underline{\mathbb{Z}} \hookrightarrow \mathscr{E} \xrightarrow{\varepsilon} \mathscr{E}^* \to 0. \tag{6.34}$$

The connecting homomorphism of the associated long exact sequence is a group homomorphism $\delta_* : H^1(M;\mathscr{E}^*) \to H^2(M;\underline{\mathbb{Z}})$. By Proposition 6.28, $H^1(M;\mathscr{E}^*)$ is isomorphic to the group of isomorphism classes of smooth complex line bundles. Given such a bundle L, we define the ***(sheaf-theoretic) Chern class of L*** to be $c(L) = -\delta_*([L]) \in H^2(M;\underline{\mathbb{Z}})$, or its image in $H^2_{\text{Sing}}(M;\mathbb{Z})$ under the isomorphism given by Theorem 6.18. (The negative sign is a normalization constant that will be explained later. Some authors define the sheaf-theoretic Chern class as $c(L) = \delta^*([L])$, without the negative sign; see the remark following the proof of Theorem 7.14 below.)

Theorem 6.29 (Classification of Smooth Line Bundles). *Let M be a smooth manifold. Smooth complex line bundles over M are classified up to isomorphism by their Chern classes: for every cohomology class $\gamma \in H^2(M;\underline{\mathbb{Z}})$, there is a smooth complex line bundle L with $c(L) = \gamma$, and two smooth complex line bundles are smoothly isomorphic if and only if their Chern classes are equal.*

Proof. The long exact sequence associated with (6.34) contains the following segment:

$$H^1(M;\mathscr{E}) \to H^1(M;\mathscr{E}^*) \xrightarrow{\delta_*} H^2(M;\underline{\mathbb{Z}}) \to H^2(M;\mathscr{E}).$$

Because $\mathscr{E}$ is a fine sheaf, the leftmost and rightmost groups are zero, so $-c = \delta_* : H^1(M;\mathscr{E}^*) \to H^2(M;\underline{\mathbb{Z}})$ is an isomorphism. □

When we turn our attention to *holomorphic* line bundles, something different happens. Let M be a complex manifold, and consider the holomorphic version of the exponential sequence:

$$0 \to \underline{\mathbb{Z}} \hookrightarrow \mathscr{O} \xrightarrow{\varepsilon} \mathscr{O}^* \to 0. \tag{6.35}$$

Now the long exact sequence reads

$$H^1(M;\mathscr{O}) \to H^1(M;\mathscr{O}^*) \xrightarrow{\delta_*} H^2(M;\underline{\mathbb{Z}}) \to H^2(M;\mathscr{O}). \tag{6.36}$$

By Proposition 6.28, $H^1(M;\mathscr{O}^*)$ is isomorphic to the Picard group of M. Again, we define the Chern class of a holomorphic line bundle L by $c(L) = -\delta_*([L])$. By naturality of the long exact sequence applied to the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathbb{Z}} & \hookrightarrow & \mathscr{O} & \xrightarrow{\varepsilon} & \mathscr{O}^* & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\mathbb{Z}} & \hookrightarrow & \mathscr{E} & \xrightarrow[\varepsilon]{} & \mathscr{E}^* & \longrightarrow & 0,
\end{array}$$

the following diagram commutes:

$$\begin{array}{ccc} H^1(M;\mathcal{O}^*) & \xrightarrow{\delta_*} & H^2(M;\underline{\mathbb{Z}}) \\ \downarrow & & \downarrow = \\ H^1(M;\mathcal{E}^*) & \xrightarrow[\delta_*]{} & H^2(M;\underline{\mathbb{Z}}), \end{array}$$

and therefore $c(L)$ depends only on the smooth isomorphism class of L.

By the previous theorem, two holomorphic line bundles are *smoothly* isomorphic if and only if their Chern classes are equal. However, since $\mathcal{O}$ is not fine, it need not be the case that the groups on the ends of (6.36) are zero, so the map $c\colon \operatorname{Pic}(M) \to H^2(M;\underline{\mathbb{Z}})$ might not be injective and/or surjective. Thus if $H^2(M;\mathcal{O}) \neq 0$, there might be smooth line bundles on M that admit no holomorphic structures; and if $H^1(M;\mathcal{O}) \neq 0$, there might be smooth line bundles that admit multiple nonisomorphic holomorphic structures.

The kernel of the Chern class homomorphism is called the ***Picard variety of M*** and denoted by $\operatorname{Pic}^0(M)$. It is the group of isomorphism classes of holomorphic line bundles with zero Chern class, or equivalently holomorphic structures on the trivial smooth line bundle. In Chapter 9, we will see that for a large class of compact complex manifolds, including all projective manifolds and all compact Riemann surfaces, $\operatorname{Pic}^0(M)$ has the structure of a complex torus (see Thm. 9.66).

Line Bundles on Riemann Surfaces

For line bundles on compact Riemann surfaces, we can say more. Suppose M is a connected compact Riemann surface. For any smooth complex line bundle $L \to M$, the ***degree of L*** is defined to be the integer

$$\deg(L) = \langle c(L), [M] \rangle,$$

where $c(L) \in H^2_{\text{Sing}}(M;\mathbb{Z})$ is the Chern class of L, $[M] \in H_2(M)$ is the fundamental class of M, and $\langle \cdot, \cdot \rangle$ represents the Kronecker pairing.

Proposition 6.30. *Let M be a connected compact Riemann surface.*

(a) *The degree map descends to a surjective homomorphism from* $\operatorname{Pic}(M)$ *to* $\mathbb{Z}$.

(b) *Two holomorphic line bundles on M are smoothly isomorphic if and only they have the same degree.*

Proof. The fact that $c(L)$ depends only on the isomorphism class of L implies immediately that the degree map descends to $\operatorname{Pic}(M)$.

Because M has complex dimension 1, there are no nonzero $(0,2)$-forms, so the Dolbeault group $H^{0,2}(M)$ is zero. Thus $H^2(M;\mathcal{O}) = 0$ by the Dolbeault theorem, so (6.36) implies that the map $\delta_* :\ H^2(M;\mathcal{O}^*) \to :\ H^2(M;\underline{\mathbb{Z}})$ is surjective, and thus so is the Chern class map $c :\ \operatorname{Pic}(M) \to H^2_{\text{Sing}}(M;\mathbb{Z})$. Because M is connected, compact, and orientable, the classification theorem for compact surfaces [**LeeTM**, Thms. 6.15 & 10.22] shows that it is homeomorphic either to $\mathbb{S}^2$ or to a connected sum of one or more copies of the torus $\mathbb{S}^1 \times \mathbb{S}^1$. Let g be the ***genus*** of M, which is defined to be 0 if M is homeomorphic to $\mathbb{S}^2$ and to be g if it is homeomorphic to a connected sum of $g \geq 1$ tori. The first singular homology group of M is isomorphic to the free abelian group $\mathbb{Z}^{2g}$ [**LeeTM**, Cor. 13.15], and therefore the universal coefficient theorem (Prop. 6.16) implies that the Kronecker map $\kappa :\ H^2_{\text{Sing}}(M;\mathbb{Z}) \to \operatorname{Hom}(H_2(M);\mathbb{Z})$ is an isomorphism. On the other hand, since $H_2(M)$ is an infinite cyclic group generated by the fundamental class $[M]$ (Prop. 6.15), it follows that the evaluation map $E :\ \operatorname{Hom}(H_2(M);\mathbb{Z}) \to \mathbb{Z}$ given by $E(\varphi) = \varphi([M])$ is also an isomorphism.

The degree map can be written as the following composition:

$$\operatorname{Pic}(M) \xrightarrow{\ c\ } H^2_{\text{Sing}}(M;\mathbb{Z}) \xrightarrow{\ \kappa\ } \operatorname{Hom}(H_2(M),\mathbb{Z}) \xrightarrow{\ E\ } \mathbb{Z}.$$

Since c is a surjective homomorphism and κ and E are both isomorphisms, claim (a) follows. Then (b) follows immediately from Theorem 6.29. □

The degree homomorphism is surjective, but in most cases it is not injective. Its kernel is the Picard variety $\operatorname{Pic}^0(M)$ defined above.

Later, we will develop some effective methods for computing the degrees of line bundles on a Riemann surface.

Other Sheaf Cohomology Theories

The Čech construction of sheaf cohomology that we have described is convenient for differential geometry because it makes the interconnections among holomorphic line bundles, differential forms, and algebraic topology relatively transparent. But as we noted earlier, its effectiveness is limited to paracompact Hausdorff spaces. For some applications, this is an intolerable restriction. This is especially true in algebraic geometry, because the Zariski topology is not Hausdorff.

There are other definitions of sheaf cohomology that are more general. Since you might encounter those definitions in other books, for completeness we briefly describe the most important ones here and show that they yield cohomology groups that are isomorphic to the Čech groups when applied to paracompact Hausdorff spaces. The material in the remainder of this chapter will not be used anywhere else in the book, so you are free to ignore it if you wish.

Most other approaches to sheaf cohomology groups are based on constructing a certain acyclic resolution and *defining* the sheaf cohomology groups to be the cohomology groups of the associated global section sequence. Of course, “acyclic” does not make sense until the sheaf cohomology groups have been defined, so the definitions proceed by defining a canonical resolution by a certain class of sheaves that will turn out to be acyclic in the given theory.

There are two common approaches to defining sheaf cohomology in this way. (A less common third approach is described in Problem 6-7.) The first, due to Roger Godement [**God73**], uses a resolution by *flasque sheaves*: A sheaf $\mathscr{F}$ on a topological space M is said to be ***flasque*** if for every open subset $U \subseteq M$, the restriction map $r^M_U : \mathscr{F}(M) \to \mathscr{F}(U)$ is surjective. It follows that every restriction map $r^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ is surjective because $r^U_V \circ r^M_U = r^M_V$. (Some authors use the term ***flabby***, which is the English translation of the French word *flasque*.) Godement showed that every sheaf of abelian groups admits a canonical resolution by flasque sheaves (the ***Godement resolution***, described in Problem 6-6), and defined the sheaf cohomology groups to be the cohomology groups of the associated global section sequence of that resolution. The advantage of this definition is that short exact sheaf sequences yield long exact cohomology sequences on any topological space, not just a paracompact Hausdorff one.

The second common approach, due to Alexander Grothendieck [**Gro57**], uses a resolution by *injective sheaves:* a sheaf $\mathscr{S}$ on M is ***injective*** if whenever $\mathscr{B}$ is a sheaf on M and $\mathscr{A}$ is a subsheaf of $\mathscr{B}$, every sheaf morphism $F : \mathscr{A} \to \mathscr{S}$ extends to $\mathscr{B}$; that is, there is a sheaf morphism $\widetilde{F} : \mathscr{B} \to \mathscr{S}$ making the following diagram commute:

$$\begin{array}{ccc} \mathscr{B} & & \\ \uparrow & \searrow^{\widetilde{F}} & \\ \mathscr{A} & \xrightarrow[F]{} & \mathscr{S}. \end{array}$$

Grothendieck showed that every sheaf of abelian groups admits a canonical resolution by injective sheaves, and defined sheaf cohomology groups as the cohomology groups of the associated global section sequence. This is the most general definition of all, because it can be expressed in purely category-theoretic terms, and thus makes sense for sheaves over objects even more general than topological spaces.

Both of these constructions lead to definitions of sheaf cohomology groups that satisfy all the essential properties that we have proved for the Čech groups: they are functorial; zero-degree cohomology is naturally isomorphic to the group of global sections; and short exact sheaf sequences yield long exact cohomology sequences. The details can be found, for example, in [**Bre97**, Chap. II]. In this section, we will prove that both flasque sheaves and injective sheaves on a paracompact Hausdorff space are acyclic in Čech cohomology, so it follows from the de Rham–Weil theorem that the cohomology groups defined by either the Godement construction or the Grothendieck construction are isomorphic to the ones we have defined.

We begin by describing the relationship between these two types of sheaves.

Theorem 6.31. *On any topological space, every injective sheaf of abelian groups is flasque.*

Proof. Let $\mathscr{S}$ be an injective sheaf of abelian groups on a topological space M. Suppose $U \subseteq M$ is an open subset and $s \in \mathscr{S}(U)$. Let $\underline{\mathbb{Z}}$ be the constant sheaf on M with coefficients in $\mathbb{Z}$, and define a subsheaf $\underline{\mathbb{Z}}_U \subseteq \underline{\mathbb{Z}}$ by

$$\underline{\mathbb{Z}}_U(V) = \{k \in \underline{\mathbb{Z}}(V) : k(x) = 0 \text{ for } x \in V \smallsetminus U\}.$$

(Recall that elements of $\underline{\mathbb{Z}}(V)$ are locally constant functions from V to $\mathbb{Z}$.) Because $\mathscr{S}$ is a sheaf of $\underline{\mathbb{Z}}$-modules (see Example 5.4(c)), we can use the section s to define a sheaf morphism $I : \underline{\mathbb{Z}}_U \to \mathscr{S}$ as follows: given an open subset $V \subseteq M$ and a section $k \in \underline{\mathbb{Z}}_U(V)$, let $I(k) \in \mathscr{S}(V)$ be the section satisfying

$$\begin{aligned} I(k)|_{V \cap U} &= (k|_{V \cap U})(s|_{V \cap U}), \\ I(k)|_{V \cap k^{-1}(0)} &= 0. \end{aligned}$$

The fact that k is locally constant implies that $k^{-1}(0)$ is open, so the gluing and locality properties of $\mathscr{S}$ ensure that $I(k)$ is uniquely defined. Because $\mathscr{S}$ is injective, I extends to a morphism $\widetilde{I} : \underline{\mathbb{Z}} \to \mathscr{S}$. Define a section $\tilde{s} \in \mathscr{S}(M)$ by $\tilde{s} = \widetilde{I}(1)$, where $1 \in \underline{\mathbb{Z}}(M)$ is the constant function with value 1. It satisfies

$$\tilde{s}|_U = \widetilde{I}(1)|_U = \widetilde{I}(1|_U) = I(1|_U) = s,$$

so $r^M_U(\tilde{s}) = s$, showing that r^M_U is surjective. □

Problem 6-5 describes an example that shows the converse of this theorem is not true.

Here are two essential properties of flasque sheaves.

Lemma 6.32 (Properties of Flasque Sheaves). *Suppose $\mathscr{R}$, $\mathscr{S}$, and $\mathscr{T}$ are sheaves of abelian groups on a topological space M, and the following sheaf sequence is exact:*

$$0 \to \mathscr{R} \xrightarrow{F} \mathscr{S} \xrightarrow{G} \mathscr{T} \to 0. \tag{6.37}$$

(a) *If $\mathscr{R}$ is flasque, then for every open subset $U \subseteq M$, the following sequence of abelian groups is exact:*

$$0 \to \mathscr{R}(U) \xrightarrow{F_U} \mathscr{S}(U) \xrightarrow{G_U} \mathscr{T}(U) \to 0. \tag{6.38}$$

(b) *If $\mathscr{R}$ and $\mathscr{S}$ are flasque, then so is $\mathscr{T}$.*

Proof. Suppose first that $\mathscr{R}$ is flasque. We know from Proposition 5.24 that the sequence (6.38) is exact at $\mathscr{R}(U)$ and $\mathscr{S}(U)$, so we need only prove that the homomorphism $G_U : \mathscr{S}(U) \to \mathscr{T}(U)$ is surjective.

Let $t \in \mathscr{T}(U)$ be arbitrary. The fact that the sheaf sequence (6.37) is exact means that for each $p \in U$, there is a neighborhood V of p contained in U and a section $s \in \mathscr{S}(V)$ such that $G_V(s) = t|_V$. Let P be the set of all pairs of the form (V, s) where V is an open subset of U and $s \in \mathscr{S}(V)$ satisfies $G_V(s) = t|_V$. Give P a partial order by saying that $(V, s) \le (V', s')$ if $V \subseteq V'$ and $s'|_V = s$. If $Q = \{(V_\alpha, s_\alpha) : \alpha \in A\}$ is any totally ordered subset of P, then the pair (V_∞, s_∞) is an upper bound for Q, where $V_\infty = \bigcup_\alpha V_\alpha$ and $s_\infty \in \mathscr{S}(V_\infty)$ is defined as follows: the fact that Q is totally ordered guarantees that for any $\alpha, \beta \in A$, either $V_\alpha \subseteq V_\beta$ or vice versa, and in either case $s_\alpha|_{V_\alpha \cap V_\beta} = s_\beta|_{V_\alpha \cap V_\beta}$, so by the gluing property there exists $s_\infty \in \mathscr{S}(V_\infty)$ whose restriction to each V_α is equal to s_α. It follows that $G_{V_\infty}(s_\infty) = t|_{V_\infty}$, so $(V_\infty, s_\infty) \in P$. By Zorn's lemma, there is a maximal element $(V, s) \in P$.

We will show that $V = U$, which means that $G_U(s) = t$, proving that G_U is surjective. Assume for contradiction that $V \ne U$, and let p be a point of $U \smallsetminus V$. Then there exist a neighborhood W of p in U and a section $\sigma \in \mathscr{S}(W)$ such that $G_W(\sigma) = t|_W$. Since $G_{V\cap W}(\sigma|_{V\cap W}) = t|_{V\cap W} = G_{V\cap W}(s|_{V\cap W})$, the fact that $0 \to \mathscr{R}(V \cap W) \to \mathscr{S}(V \cap W) \to \mathscr{T}(V \cap W)$ is exact implies there exists $r \in \mathscr{R}(V \cap W)$ such that

$$s|_{V\cap W} - \sigma|_{V\cap W} = F_{V\cap W}(r). \tag{6.39}$$

Because $\mathscr{R}$ is flasque, there is a section $\tilde{r} \in \mathscr{R}(W)$ that restricts to r on $V \cap W$. Let $\widetilde{\sigma} = \sigma + F_W(\tilde{r}) \in \mathscr{S}(W)$. It follows from (6.39) that the restrictions of s and $\widetilde{\sigma}$ agree on $V \cap W$, so there is a section $\tilde{s} \in \mathscr{S}(V \cup W)$ that satisfies

$$\tilde{s}|_V = s, \qquad \tilde{s}|_W = \widetilde{\sigma}.$$

Then $G_{V\cup W}(\tilde{s})$ agrees with $G_V(s)$ on V and with $G_W(\widetilde{\sigma}) = G_W(\sigma)$ on W, so by the locality property $G_{V\cup W}(\tilde{s}) = t|_{V\cup W}$, showing that $(V \cup W, \tilde{s}) \in P$. Since it is strictly larger than (V, s), this contradicts the maximality of (V, s), thus proving (a).

To prove (b), assume in addition that $\mathscr{S}$ is flasque, and suppose $U \subseteq M$ is open and $t \in \mathscr{T}(U)$. By the result of part (a), there is a section $s \in \mathscr{S}(U)$ such that $G_U(s) = t$. Since $\mathscr{S}$ is flasque, there is a section $\tilde{s} \in \mathscr{S}(M)$ such that $\tilde{s}|_U = s$, and then $\tilde{t} = G_M(\tilde{s}) \in \mathscr{T}(M)$ satisfies

$$\tilde{t}|_U = G_M(\tilde{s})|_U = G_U(\tilde{s}|_U) = G_U(s) = t,$$

showing that $\mathscr{T}$ is flasque. □

The principal source of flasque sheaves is the following construction. For any sheaf $\mathscr{S}$, we define the ***sheaf of rough sections of $\mathscr{S}$***, denoted by $\widehat{\mathscr{S}}$, by letting $\widehat{\mathscr{S}}(U)$

be the set of *all* functions $\sigma : U \to \mathrm{Et}(\mathscr{S})$ satisfying $\pi \circ \sigma = \mathrm{Id}_U$; it is a sheaf by the result of Exercise 5.1(c). If $\mathscr{S}$ is a sheaf of abelian groups, then so is $\widehat{\mathscr{S}}$ with addition defined pointwise, because each stalk of $\mathrm{Et}(\mathscr{S})$ inherits an abelian group structure from $\mathscr{S}$. (In the literature, $\widehat{\mathscr{S}}$ is often called the "sheaf of discontinuous sections of $\mathscr{S}$"; but of course not all of the sections in $\widehat{\mathscr{S}}(U)$ are discontinuous.)

Lemma 6.33 (Properties of the Sheaf of Rough Sections). *Suppose $\mathscr{S}$ is a sheaf of abelian groups on a topological space M, and $\widehat{\mathscr{S}}$ is its sheaf of rough sections.*

(a) *$\widehat{\mathscr{S}}$ is flasque.*

(b) *There is a canonical injective sheaf morphism $\iota_{\mathscr{S}} : \mathscr{S} \hookrightarrow \widehat{\mathscr{S}}$.*

(c) *Given a sheaf morphism $F : \mathscr{S} \to \mathscr{T}$, there is a unique sheaf morphism $\widehat{F} : \widehat{\mathscr{S}} \to \widehat{\mathscr{T}}$ making the following diagram commute:*

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{F} & \mathscr{T} \\ {\scriptstyle \iota_{\mathscr{S}}}\downarrow & & \downarrow{\scriptstyle \iota_{\mathscr{T}}} \\ \widehat{\mathscr{S}} & \xrightarrow{\widehat{F}} & \widehat{\mathscr{T}}. \end{array} \tag{6.40}$$

(d) *If M is a paracompact Hausdorff space, then $\widehat{\mathscr{S}}$ is fine.*

Proof. Every section in $\widehat{\mathscr{S}}(U)$ extends to a section in $\widehat{\mathscr{S}}(M)$ by defining it to be zero outside of U, so $\widehat{\mathscr{S}}$ is flasque.

The morphism $\iota_{\mathscr{S}} : \mathscr{S} \to \widehat{\mathscr{S}}$ is defined by sending $s \in \mathscr{S}(U)$ to the (continuous) section $s^+ \in \widehat{\mathscr{S}}(U)$ defined by (5.6). Since two elements of $\mathscr{S}(U)$ that have the same germ at every point are equal by the locality property, $\iota_{\mathscr{S}}$ is injective.

If $F : \mathscr{S} \to \mathscr{T}$ is a sheaf morphism, the morphism $\widehat{F} : \widehat{\mathscr{S}} \to \widehat{\mathscr{T}}$ is defined by

$$\widehat{F}_U(\sigma)(p) = F_p(\sigma(p))$$

for $\sigma \in \widehat{\mathscr{S}}(U)$, where F_p is the stalk homomorphism determined by F. Since $\widehat{F}_U$ is defined pointwise, it commutes with restrictions and thus defines a sheaf morphism; and $\widehat{F}$ respects the group structure on stalks because each F_p does. For any section $s \in \mathscr{S}(U)$,

$$\begin{aligned} \widehat{F}_U \circ \iota_{\mathscr{S}}(s)(p) &= F_p(s^+(p)) = F_p([s]_p) = [F_U(s)]_p \\ &= (F_U(s))^+(p) = \iota_{\mathscr{T}} \circ F_U(s)(p), \end{aligned}$$

showing that (6.40) commutes. For every section $\sigma \in \widehat{\mathscr{S}}(U)$, the value $\sigma(p)$ at each $p \in U$ is the germ at p of some section $s \in \mathscr{S}(V)$ on some neighborhood V of p, and

$$\widehat{F}_U(\sigma)(p) = F_p([s]_p) = [F_V(s)]_p.$$

This shows that $\widehat{F}$ is completely determined by F, thus proving uniqueness.

Now suppose M is a paracompact Hausdorff space. To see that $\widehat{\mathscr{S}}$ is fine, suppose $\{U_\beta\}_{\beta\in B}$ is a locally finite open cover of M, and let $\{\psi_\beta\}_{\beta\in B}$ be a subordinate topological partition of unity. Choose a well-ordering of the index set B, and for each $\beta \in B$ define a morphism $\eta_\beta : \widehat{\mathscr{S}} \to \widehat{\mathscr{S}}$ on the stalk level by

$$(\eta_\beta \sigma)(x) = \begin{cases} \sigma(x) & \text{if } \beta \text{ is the smallest index for which } \psi_\beta(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The support of η_β is contained in that of ψ_β and therefore in U_β, and it is straightforward to check that the collection $\{\eta_\beta\}_{\beta \in B}$ is a sheaf partition of unity. □

For convenience, we will indulge in a slight abuse of notation and consider $\mathscr{S}$ to be a subsheaf of $\widehat{\mathscr{S}}$ by identifying it with the image of the morphism $\iota_{\mathscr{S}}$, which is exactly the sheafification $\mathscr{S}^+$.

Theorem 6.34. *Let M be a paracompact Hausdorff space. Every flasque sheaf of abelian groups on M is acyclic in Čech cohomology.*

Proof. We will prove by induction on q that for every $q \geq 1$ and every flasque sheaf $\mathscr{S}$ on M, we have $H^q(M;\mathscr{S}) = 0$.

Suppose $\mathscr{S}$ is a flasque sheaf on M, and let $\widehat{\mathscr{S}}$ be its sheaf of rough sections. We have an exact sheaf sequence

$$0 \to \mathscr{S} \hookrightarrow \widehat{\mathscr{S}} \to \widehat{\mathscr{S}}/\mathscr{S} \to 0. \tag{6.41}$$

The corresponding long exact sequence in Čech cohomology contains the segment

$$H^0(M;\widehat{\mathscr{S}}) \to H^0(M;\widehat{\mathscr{S}}/\mathscr{S}) \to H^1(M;\mathscr{S}) \to H^1(M;\widehat{\mathscr{S}}). \tag{6.42}$$

Since $\widehat{\mathscr{S}}$ is fine, the rightmost group in (6.42) is trivial. By Theorem 6.7, the first map is equivalent to the global section map $\widehat{\mathscr{S}}(M) \to (\widehat{\mathscr{S}}/\mathscr{S})(M)$, which is surjective by Lemma 6.32; thus $H^1(M;\mathscr{S}) = 0$ for every flasque sheaf $\mathscr{S}$.

Now let $q \geq 1$, and assume we have proved that $H^q(M;\mathscr{S}) = 0$ for every flasque sheaf $\mathscr{S}$. Suppose $\mathscr{S}$ is flasque, and consider the following portion of the long exact sequence associated with (6.41):

$$H^q(M;\widehat{\mathscr{S}}) \to H^q(M;\widehat{\mathscr{S}}/\mathscr{S}) \to H^{q+1}(M;\mathscr{S}) \to H^{q+1}(M;\widehat{\mathscr{S}}).$$

The first and last groups above are zero because $\widehat{\mathscr{S}}$ is fine. It follows from Lemma 6.32 that $\widehat{\mathscr{S}}/\mathscr{S}$ is flasque, so the inductive hypothesis implies $H^{q+1}(M;\mathscr{S}) \cong H^q(M;\widehat{\mathscr{S}}/\mathscr{S}) = 0$, thus completing the induction. □

Corollary 6.35. *On a paracompact Hausdorff space, the sheaf cohomology groups defined by either the Godement construction or the Grothendieck construction are isomorphic to those defined by the Čech construction.*

Proof. Either construction starts with a resolution by sheaves that are acyclic in Čech cohomology by Theorems 6.31 and 6.34, so the result follows from the de Rham–Weil theorem. □

Problems

6-1. Suppose M is a positive-dimensional complex manifold. Show that the sheaf $\mathscr{O}$ of holomorphic functions on M and the constant sheaf $\underline{\mathbb{C}}$ on M are not fine. [Hint: Let U, V be proper open subsets of M whose union is M and whose intersection is nonempty, and show that there is no sheaf partition of unity subordinate to the cover $\mathcal{U} = \{U, V\}$.]

6-2. Let M be a paracompact Hausdorff space and $\mathscr{R}$ a fine sheaf of commutative rings on M. Show that every sheaf of $\mathscr{R}$-modules on M is fine.

6-3. Let M be a smooth manifold, and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an indexed open cover of M such that each nonempty finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is contractible. (Such a cover can be constructed by choosing a Riemannian metric on M and choosing a geodesically convex neighborhood of each point [**LeeRM**, Thm. 6.17], and noting that intersections of geodesically convex sets are geodesically convex and therefore contractible.) By following through the proof of the de Rham–Weil theorem, show that the de Rham–Weil isomorphisms $\mathscr{R}_1 : H^1_{\mathrm{dR}}(M; \mathbb{C}) \to H^1(M; \underline{\mathbb{C}})$ and $\mathscr{R}_2 : H^2_{\mathrm{dR}}(M; \mathbb{C}) \to H^2(M; \underline{\mathbb{C}})$ can be described as follows.

(a) Let η be a closed 1-form on M. For each α, there is a smooth function u_α on U_α such that $\eta\big|_{U_\alpha} = du_\alpha$. Then

$$a_{\alpha\beta} = u_\beta\big|_{U_\alpha\cap U_\beta} - u_\alpha\big|_{U_\alpha\cap U_\beta}$$

defines a 1-cocycle a on $\mathcal{U}$ with coefficients in $\underline{\mathbb{C}}$, and $\mathscr{R}_1[\eta] = [[a]]$.

(b) Let η be a closed 2-form on M. For each α, there is a smooth 1-form φ_α on U_α such that $\eta|_{U_\alpha} = d\varphi_\alpha$; and for each α and β such that $U_\alpha \cap U_\beta \neq \varnothing$, there is a smooth function $u_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that $\varphi_\beta\big|_{U_\alpha\cap U_\beta} - \varphi_\alpha\big|_{U_\alpha\cap U_\beta} = du_{\alpha\beta}$. Then

$$a_{\alpha\beta\gamma} = \left(u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}\right)\big|_{U_\alpha\cap U_\beta\cap U_\gamma}$$

defines a 2-cocycle on $\mathcal{U}$ with coefficients in $\underline{\mathbb{C}}$, and $\mathscr{R}_2[\eta] = [[a]]$.

6-4. Let M be a complex manifold. Let $\mathscr{P}$ denote the sheaf of pluriharmonic functions on M (see Problem 4-7), and $\mathscr{E}_{\mathbb{R}}$ the sheaf of smooth real-valued functions. For each $q \geq 1$, let $\mathscr{F}^q$ denote the sheaf of smooth real $(q+1)$-forms whose $(q+1, 0)$ and $(0, q+1)$-parts are zero; in other words, $\mathscr{F}^q$ is the sheaf of smooth real-valued forms taking values in $\Lambda^{q,1}M \oplus \cdots \oplus \Lambda^{1,q}M$. Show that the following sheaf sequence is a fine resolution of $\mathscr{P}$:

$$0 \to \mathscr{P} \hookrightarrow \mathscr{E}_{\mathbb{R}} \xrightarrow{i\partial\bar\partial} \mathscr{F}^1 \xrightarrow{d} \mathscr{F}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{F}^q \xrightarrow{d} \cdots .$$

Conclude that for $q \geq 2$,

$$H^q(M;\mathscr{P}) \cong \frac{\operatorname{Ker}\left(d: \mathscr{F}^q(M) \to \mathscr{F}^{q+1}(M)\right)}{\operatorname{Im}\left(d: \mathscr{F}^{q-1}(M) \to \mathscr{F}^q(M)\right)}.$$

State the analogous result for $q = 1$. [Hint: For the proof of exactness at $\mathscr{F}^q$, if β is a local section of $\mathscr{F}^q$ and $\beta = d\alpha$, write $\alpha = \alpha^{(q,0)} + \widetilde{\alpha} + \alpha^{(0,q)}$ with $\widetilde{\alpha}$ a section of $\mathscr{F}^{q-1}$, and show that locally $d\alpha^{(q,0)} = d\bar{\partial}\sigma$ for some $(q-1,0)$ form σ.]

6-5. Let M be any nonempty topological space, and let $\mathbb{Z}_p$ and $\mathbb{R}_p$ be the skyscraper sheaves at a point $p \in M$ with values in $\mathbb{Z}$ and $\mathbb{R}$, respectively, so $\mathbb{Z}_p$ is a subsheaf of $\mathbb{R}_p$. Show that $\mathbb{Z}_p$ is flasque but not injective, by showing that the identity morphism $\mathbb{Z}_p \to \mathbb{Z}_p$ does not extend to a morphism $\mathbb{R}_p \to \mathbb{Z}_p$.

6-6. This problem describes how to construct the Godement resolution of a sheaf. Let $\mathscr{S}$ be a sheaf of abelian groups on a topological space M. Define $\mathscr{S}^0 = \widehat{\mathscr{S}}$ (the sheaf of rough sections of $\mathscr{S}$). Next, let $\mathscr{Q}^1 = \mathscr{S}^0/\mathscr{S}$, where we identify $\mathscr{S}$ with a subsheaf of $\widehat{\mathscr{S}}$ via the canonical injection $\iota_{\mathscr{S}}$ given by Lemma 6.33; and let $\mathscr{S}^1 = \widehat{\mathscr{Q}^1}$. Let $d_0 : \mathscr{S}^0 \to \mathscr{S}^1$ be the composition

$$\mathscr{S}^0 \xrightarrow{\Pi} \mathscr{Q}^1 \xrightarrow{\iota_{\mathscr{Q}^1}} \widehat{\mathscr{Q}^1},$$

where the first map is given by Proposition 5.22 and the second by Lemma 6.33. Then by induction define $\mathscr{Q}^{j+1} = \widehat{\mathscr{Q}^j}/\mathscr{Q}^j$ and $\mathscr{S}^{j+1} = \widehat{\mathscr{Q}^{j+1}}$, with the morphism $d_j : \mathscr{S}^j \to \mathscr{S}^{j+1}$ defined as above. The ***Godement resolution of $\mathscr{S}$*** is the following sequence of sheaf morphisms

$$0 \to \mathscr{S} \xrightarrow{\iota_{\mathscr{S}}} \mathscr{S}^0 \xrightarrow{d_0} \mathscr{S}^1 \xrightarrow{d_1} \mathscr{S}^2 \to \cdots.$$

Prove that this is a flasque resolution.

6-7. Another approach that is sometimes used to define sheaf cohomology groups (e.g., in [**Wel08**, Chap. 2]) is based on resolutions by *soft sheaves*, defined as follows. If $\mathscr{S}$ is a sheaf of abelian groups on a space M and $K \subseteq M$ is a closed subset, let $\mathscr{S}(K)$ denote the group of continuous sections of the étalé space $\operatorname{Et}(\mathscr{S})$ over K. (This can be naturally identified with the inverse image sheaf $i^{-1}(\mathscr{S})$, where $i: K \to M$ is inclusion. See Problem 5-13). We define a restriction operator $r^M_K : \mathscr{S}(M) \to \mathscr{S}(K)$ by letting $r^M_K(s)$ be the restriction to K of the continuous function $s^+ : M \to \operatorname{Et}(\mathscr{S})$ defined by (5.6). The sheaf $\mathscr{S}$ is said to be ***soft*** if for every closed subset $K \subseteq M$, the restriction map r^M_K is surjective.

(a) Let M be a paracompact Hausdorff space. Prove that both conclusions of Lemma 6.32 hold if "flasque" is replaced by "soft."

(b) For every sheaf $\mathscr{S}$, prove that the sheaf of rough sections $\widehat{\mathscr{S}}$ is soft.

(c) Prove that every soft sheaf on a paracompact Hausdorff space is acyclic in Čech cohomology.

(d) Let $\mathcal{S}$ be a sheaf of abelian groups on a paracompact Hausdorff space, and suppose we are given a resolution of $\mathcal{S}$ by soft sheaves. Prove that the cohomology groups of the corresponding global section sequence are isomorphic to the Čech cohomology groups of $\mathcal{S}$.

6-8. Prove that every fine sheaf on a paracompact Hausdorff space is soft.

6-9. Let $\mathcal{S}$ be a sheaf of abelian groups on a paracompact Hausdorff space M. Show that the following are equivalent:

(a) $\mathcal{S}$ is fine.

(b) The sheaf $\mathcal{H}om(\mathcal{S}, \mathcal{S})$ is soft (see Problems 5-7 and 6-7).

(c) For each pair of disjoint closed subsets $K, L \subseteq M$, there is a sheaf morphism $F : \mathcal{S} \to \mathcal{S}$ that restricts to the identity on a neighborhood of K and to the zero morphism on a neighborhood of L.

[Hint: For (b) $\Rightarrow$ (a), let $\{U_\alpha\}_{\alpha \in A}$ be a locally finite open cover of M, and choose another open cover $\{V_\alpha\}_{\alpha \in A}$ satisfying $V_\alpha \subseteq \overline{V}_\alpha \subseteq U_\alpha$. Let Ψ be the set of collections $\{\psi_\beta\}_{\beta \in B}$ where $B \subseteq A$, each ψ_β is a sheaf morphism $\mathcal{S} \to \mathcal{S}$ satisfying $\operatorname{supp} \psi_\beta \subseteq U_\beta$, and $\sum_{\beta \in B} \psi_\beta$ restricts to the identity on $\bigcup_{\beta \in B} \overline{V}_\beta$. Give Ψ a partial order by saying $\{\psi_\beta\}_{\beta \in B} \leq \{\psi'_\beta\}_{\beta \in B'}$ if $B \subseteq B'$ and $\psi_\beta = \psi'_\beta$ for $\beta \in B$, and use Zorn's lemma.]

6-10. This problem requires some knowledge of simplicial cohomology. (See [**Mun84**, Chap. 5] or [**Hat02**, Chap. 3].) Suppose K is a simplicial complex and $|K|$ is its underlying topological space. For each vertex v of K, the ***star of v***, denoted by $\operatorname{St} v$, is the union of the interiors of all simplices that have v as a vertex. If $v_0, \dots, v_p$ are the vertices of a simplex σ of K, then $\operatorname{St} v_0 \cap \cdots \cap \operatorname{St} v_p$ is a neighborhood of σ in $|K|$. Let G be an abelian group and let $\mathcal{U}$ be the open cover of $|K|$ consisting of the stars of all the vertices.

(a) Show that $H^p(\mathcal{U}; \underline{G})$ is isomorphic to the simplicial cohomology group $H^p(K; G)$.

(b) If $\mathcal{V}$ is a refinement of $\mathcal{U}$, show that there is a subdivision of K whose covering by open stars provides a further refinement of $\mathcal{V}$, and then use the isomorphism between singular and simplicial cohomology to provide another proof that $H^p_{\mathrm{Sing}}(M; G) \cong H^p(M; \underline{G})$ for a triangulable topological space M.

10.1090/gsm/244/07

Chapter 7

Connections

Among the main tools in Riemannian geometry are *connections*, or covariant differentiation operators, on vector bundles. Connections play an equally important role in complex geometry, and this chapter is dedicated to exploring their fundamental properties.

We begin with the definition of a connection on a smooth complex vector bundle, and show how to construct connections and how to do computations with them. Next we describe the curvature of a connection, which can be thought of as an obstruction to the existence of parallel local frames. We then show how a connection can be used to construct a cohomology class associated with every smooth complex vector bundle, called the *first real Chern class*; for line bundles, it is closely related to the sheaf-theoretic Chern class defined in Chapter 3.

The last part of the chapter focuses on holomorphic vector bundles. If we endow such a bundle with a Hermitian fiber metric, there are many connections on the bundle that are compatible with the metric. But for *holomorphic* Hermitian vector bundles, there is an additional condition, called *compatibility with the holomorphic structure*, that allows us to single out a unique connection, called the *Chern connection*. Here we introduce the Chern connection and study some of its properties.

Connections on Complex Vector Bundles

Let E be a smooth complex vector bundle of rank m on a smooth manifold M. A ***connection*** on E is a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, written $(X, \sigma) \mapsto \nabla_X \sigma$, that is linear over $C^\infty(M)$ in X, linear over $\mathbb{C}$ in σ, and satisfies the product rule $\nabla_X(f\sigma) = f\nabla_X\sigma + (Xf)\sigma$ for every smooth complex-valued function f. The expression $\nabla_X\sigma$ is called the ***covariant derivative of σ in the direction X***. The value of $\nabla_X\sigma$ at a point $p \in M$ depends only on the value of X at p and the value of σ in an arbitrarily small neighborhood of p (see [**LeeRM**, Lemma 4.5], where this is proved for real vector bundles; the proof for complex bundles is just the same). Consequently any connection ∇ on E determines a connection, still denoted by ∇, on the restriction of E to any open subset of M [**LeeRM**, Prop. 4.3]. We will often make use of this fact without further comment.

In addition to determining covariant derivatives with respect to specific vector fields, a connection also determines a ***total covariant derivative*** of each smooth section σ: this is the section $\nabla\sigma$ of the bundle $T^*M \otimes E$, which is canonically isomorphic to $\operatorname{Hom}(TM, E)$, defined by $(\nabla\sigma)(X) = \nabla_X\sigma$.

We extend our connections by complex linearity to accept complex vector fields as well as real ones: if ∇ is a connection on E and $Z = X + iY$ is a smooth complex vector field on M, we set

$$\nabla_Z\sigma = \nabla_X\sigma + i\nabla_Y\sigma.$$

This operation is easily seen to be linear over $C^\infty(M;\mathbb{C})$ in Z, and it allows us to regard the total covariant derivative of σ (still denoted by $\nabla\sigma$) as a section of $T^*_{\mathbb{C}}M \otimes E \cong \operatorname{Hom}(T_{\mathbb{C}}M, E)$.

Because connections are determined locally, we can carry out most computations related to them in terms of local frames. Thus suppose $(s_j) = (s_1, \dots, s_m)$ is a smooth local frame for E over a subset $U \subseteq M$. For a given vector field X on U, we can express each of the covariant derivatives $\nabla_X(s_j)$ in terms of the same frame (using the summation convention) as

$$\nabla_X s_j = \theta_j^k(X)s_k \tag{7.1}$$

for some smooth coefficients $\theta_j^k(X)$. Formula (7.1) determines the covariant derivative of an arbitrary section $\sigma = \sigma^j s_j$ by

$$\nabla_X(\sigma^j s_j) = (X\sigma^j)s_j + \sigma^j\theta_j^k(X)s_k. \tag{7.2}$$

Because ∇ is linear over $C^\infty(M)$ in X, the coefficients θ_j^k define a matrix of complex 1-forms, called the ***connection forms*** with respect to this frame. They are smooth because their action on any smooth vector field X can be written as $\theta_j^k(X) = \varepsilon^k(\nabla_X s_j)$, where (ε^k) is the frame for E^* dual to (s_j). In terms of these forms, we can write the total covariant derivative as

$$\nabla s_j = \theta_j^k \otimes s_k, \tag{7.3}$$

$$\nabla(\sigma^j s_j) = d\sigma^j \otimes s_j + \sigma^j\theta_j^k \otimes s_k. \tag{7.4}$$

Conversely, given an arbitrary matrix of smooth complex 1-forms θ_j^k on the domain U of a smooth local frame for E, formula (7.2) determines a connection on E over U.

If we have another local frame $(\tilde{s}_k)$, then where they overlap we can write

$$\tilde{s}_k = \tau_k^j s_j,$$

for a $\mathrm{GL}(m, \mathbb{C})$-valued transition function $\tau = (\tau_k^j)$. To see how the connection forms change, we compute

$$\nabla \tilde{s}_k = d\tau_k^j \otimes s_j + \tau_k^j \theta_j^l \otimes s_l = d\tau_k^j \otimes (\tau^{-1})_j^p \tilde{s}_p + \tau_k^j \theta_j^l \otimes (\tau^{-1})_l^p \tilde{s}_p.$$

Comparing this with $\nabla \tilde{s}_k = \tilde{\theta}_k^p \otimes \tilde{s}_p$, we find that

$$\tilde{\theta}_k^p = (\tau^{-1})_j^p d\tau_k^j + (\tau^{-1})_l^p \theta_j^l \tau_k^j,$$

or in matrix notation,

$$\tilde{\theta} = \tau^{-1} d\tau + \tau^{-1} \theta \tau. \tag{7.5}$$

(Note that the $d\tau$ and θ factors above are matrices of complex 1-forms, while the other factors on the right-hand side are matrices of complex-valued functions. It is important to observe the order of factors because matrix multiplication does not commute in general.)

Now suppose E is endowed with a Hermitian fiber metric. A connection ∇ on E is ***compatible with the metric***, or a ***metric connection***, if the following identity holds for all $X \in \Gamma(TM)$ and $\sigma, \tau \in \Gamma(E)$:

$$X\langle \sigma, \tau \rangle = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle. \tag{7.6}$$

When we apply this to a complex vector field Z, the equation for compatibility with the metric reads

$$Z\langle \sigma, \tau \rangle = \langle \nabla_Z \sigma, \tau \rangle + \langle \sigma, \nabla_{\overline{Z}} \tau \rangle, \tag{7.7}$$

because of the conjugate linearity of the Hermitian inner product in its second argument.

The next proposition gives an important property of the connection forms for a metric connection.

Proposition 7.1. *Suppose $E \to M$ is a smooth complex vector bundle with a Hermitian fiber metric and ∇ is a metric connection on E. The matrix of connection 1-forms with respect to any local orthonormal frame is skew-Hermitian:*

$$\theta_j^k = -\overline{\theta_k^j}.$$

Proof. Let (s_j) be a local orthonormal frame for E, and let θ_j^k be the corresponding connection 1-forms. For every local complex vector field Z, compatibility with the metric implies

$$\begin{aligned}0 = Z(\delta_{jk}) = Z\langle s_j, s_k\rangle &= \langle \nabla_Z s_j,\ s_k\rangle + \langle s_j,\ \nabla_{\overline{Z}} s_k\rangle\\ &= \langle \theta_j^l(Z)s_l,\ s_k\rangle + \langle s_j,\ \theta_k^l(\overline{Z})s_l\rangle = \theta_j^l(Z)\delta_{lk} + \overline{\theta_k^l(\overline{Z})}\delta_{jl}\\ &= \theta_j^k(Z) + \overline{\theta_k^j}(Z).\end{aligned}$$

□

There are always plenty of connections compatible with a given metric: For example, we can choose an open cover of M by domains U_α on each of which there exists a local orthonormal frame for E, together with a smooth partition of unity $\{\psi_\alpha\}$ subordinate to this cover; then define a connection ∇^α on $E|_{U_\alpha}$ by setting all of the connection 1-forms to be zero, and define a global connection ∇ on E by blending these together with the partition of unity:

$$\nabla_X \sigma = \sum_\alpha \psi_\alpha \nabla_X^\alpha \sigma. \tag{7.8}$$

► **Exercise 7.2.** Prove that (7.8) defines a metric connection on E.

The following lemma is a useful technical result about metric connections that can simplify some computations. We will use it in Chapter 10.

Lemma 7.3. *Suppose $E \to M$ is a smooth Hermitian vector bundle and ∇ is a metric connection on E. In a neighborhood of each $x_0 \in M$, there is a smooth orthonormal frame (s_j) for E that satisfies $\nabla s_j = 0$ at x_0.*

Proof. Start with any orthonormal frame (s_j), and let θ_j^k be the corresponding connection forms. Proposition 7.1 shows that the matrix (θ_j^k) is skew-Hermitian. Choose any smooth coordinates (x^a) centered at x_0 on an open set $U \subseteq M$, and write

$$\theta_j^k\big|_{x_0} = A_{ja}^k dx^a\big|_{x_0}$$

for some constants A_{ja}^k. Let $\mathrm{M}(r, \mathbb{C})$ denote the space of all complex $r \times r$ matrices (where r is the rank of E), and let $F : U \to \mathrm{M}(r, \mathbb{C})$ be the matrix-valued function

$$F_j^k(x) = A_{ja}^k x^a.$$

Then the fact that (θ_j^k) is skew-Hermitian implies that $F(x)$ is skew-Hermitian for all x. Define another matrix-valued function $B(x) = e^{-F(x)}$; it satisfies

$$B_j^k(x_0) = \delta_j^k; \qquad \partial_a B_j^k(x_0) = -\partial_a F_j^k(x_0) = -A_{ja}^k.$$

Since the space of skew-Hermitian matrices is the Lie algebra of the unitary group, it follows that $B(x)$ is unitary for each x. Thus the local frame $(\tilde{s}_k)$ defined by

$$\tilde{s}_k(x) = B_k^j(x) s_j(x)$$

is orthonormal. At x_0, $\tilde{s}_k = s_k$, and

$$\nabla_{\partial_a} \tilde{s}_k = \partial_a B^j_k(x_0) s_j + \delta^j_k A^l_{ja} s_l = 0.$$

Thus $(\tilde{s}_k)$ satisfies the conclusion of the lemma. □

Covariant Derivatives Along Curves

A familiar construction in Riemannian geometry is to use a connection on the tangent bundle to define a covariant derivative operator acting on vector fields along curves (see [**LeeRM**, Thm. 4.24]). The same construction can be carried out for arbitrary vector bundles; here we briefly describe how that works.

Suppose $E \to M$ is a smooth complex vector bundle and $\gamma : I \to M$ is a smooth curve (here $I \subseteq \mathbb{R}$ is some interval). A ***section of E along γ*** is a continuous map $\sigma : I \to E$ satisfying $\sigma(t) \in E_{\gamma(t)}$ for each $t \in I$. A section along γ is said to be ***extendible*** if there is a section $\widetilde{\sigma}$ of E on an open set containing the image of γ such that $\sigma(t) = \widetilde{\sigma}(\gamma(t))$ for each $t \in I$.

Proposition 7.4 (Covariant Derivative Along a Curve). *Suppose $E \to M$ is a smooth complex vector bundle and ∇ is a connection on E. For each smooth curve $\gamma : I \to M$, there is a unique operator D_t that takes smooth sections of E along γ to smooth sections along γ, satisfying*

(i) LINEARITY OVER $\mathbb{R}$: $D_t(a\sigma + b\tau) = aD_t\sigma + bD_t\tau$ *for* $a, b \in \mathbb{R}$.

(ii) PRODUCT RULE: $D_t(f\sigma) = f'\sigma + fD_t\sigma$ *for* $f \in C^\infty(I)$.

(iii) *If σ is an extendible smooth section along γ, then for every smooth extension $\widetilde{\sigma}$ of σ, we have $D_t\sigma(t) = \nabla_{\gamma'(t)}\widetilde{\sigma}$.*

► **Exercise 7.5.** Show how to adapt the proof of [**LeeRM**, Thm. 4.24] to prove this proposition.

For a smooth vector bundle $E \to M$ with a connection ∇, a smooth section σ of E along a smooth curve $\gamma : I \to M$ is said to be ***parallel along γ*** if $D_t\sigma \equiv 0$.

Proposition 7.6 (Existence and Uniqueness of Parallel Transport). *Let $E \to M$ be a smooth vector bundle endowed with a connection ∇, and let $\gamma : I \to M$ be a smooth curve. Given $t_0 \in I$ and an element $\sigma_0 \in E_{\gamma(t_0)}$, there is a unique parallel section σ along γ such that $\sigma(t_0) = \sigma_0$, called the **parallel transport of σ_0 along γ**.*

Proof. First assume the image of γ is contained in the domain of a smooth local frame $(s_1, \dots, s_m)$ for E, and let θ^k_j be the connection 1-forms for this frame. We can write $\sigma(t) = f^j(t)s_j(\gamma(t))$ (using the summation convention) for some smooth functions $f^1, \dots, f^m : I \to \mathbb{C}$. Since the sections s_j are extendible, Proposition

7.4 implies

$$\begin{aligned} D_t\sigma(t) &= \dot f^j(t)s_j(\gamma(t)) + f^j(t)\nabla_{\gamma'(t)}s_j \\ &= \dot f^j(t)s_j(\gamma(t)) + f^j(t)\theta_j^k(\gamma'(t))s_k(\gamma(t)) \\ &= \left(\dot f^k(t) + f^j(t)\theta_j^k(\gamma'(t))\right)s_k(\gamma(t)) \end{aligned}$$

(where dots represent derivatives with respect to t). Thus σ is parallel if and only if its component functions satisfy the following system of linear ODEs on I:

$$\dot f^k(t) = -f^j(t)\theta_j^k(\gamma'(t)).$$

It follows from [**LeeRM**, Thm. 4.31] that this system has a unique solution on all of I satisfying the initial conditions $f^j(t_0) = \sigma_0^j$, where $\sigma_0 = \sigma_0^j s_j(\gamma(t_0))$.

For the general case, we can let β be the supremum of all $b > t_0$ such that a unique parallel transport σ exists on $[0, b]$. If $\beta < \sup I$, we can choose a local frame on an open set containing $\gamma(\beta - \delta, \beta + \delta)$ for some small $\delta > 0$, and find a parallel section $\widetilde\sigma$ on that interval with initial value $\widetilde\sigma(\beta - \delta/2) = \sigma(\beta - \delta/2)$; by uniqueness, $\widetilde\sigma$ agrees with σ on their common domain, so $\widetilde\sigma$ provides a parallel extension of σ past β, which is a contradiction. The same argument works for $t < t_0$. □

A local section $\sigma : U \to E$ defined on an open subset $U \subseteq M$ is said to be ***parallel*** if it is parallel along every smooth curve in U. We end this section with two properties of parallel transport whose proofs are left as exercises.

Proposition 7.7. *If $E \to M$ is a smooth vector bundle endowed with a connection ∇, a local section $\sigma : U \to E$ is parallel if and only if $\nabla\sigma \equiv 0$.*

Proposition 7.8. *Suppose $E \to M$ is a smooth Hermitian vector bundle and ∇ is a metric connection on E.*

(a) *If σ and τ are parallel sections of E along a smooth curve γ, then $\langle\sigma, \tau\rangle$ is constant along γ.*

(b) *If σ is a parallel section of E (that is, $\nabla E \equiv 0$) on a connected open subset $U \subseteq M$, then $|\sigma|$ is constant.*

► **Exercise 7.9.** Prove the two preceding propositions.

Curvature

A fundamental local invariant of a Riemannian metric is the curvature of its Levi-Civita connection. For connections on complex vector bundles, there is an analogous local invariant, defined in essentially the same way.

Suppose $E \to M$ is a smooth complex vector bundle and ∇ is a connection on E. We define a map $\Theta : \Gamma(T_{\mathbb{C}}M) \times \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) \to \Gamma(E)$, called the ***curvature of*** ∇, by

$$\Theta(X, Y)\sigma = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma.$$

The same argument as in the Riemannian case [**LeeRM**, Prop. 7.3] shows that Θ is linear over $C^\infty(M;\mathbb{C})$ in all three arguments X, Y, σ, and thus defines a smooth section of the bundle $T^*_{\mathbb{C}}M \otimes T^*_{\mathbb{C}}M \otimes E^* \otimes E$. Moreover, Θ is clearly antisymmetric in X and Y, and $E^* \otimes E \cong E \otimes E^*$ is canonically isomorphic to the bundle $\operatorname{End}(E)$; so Θ can be viewed as an element of $\Gamma\big(\Lambda^2_{\mathbb{C}}M \otimes \operatorname{End}(E)\big) = \mathscr{E}^2(M;\operatorname{End}(E))$. A connection ∇ is said to be ***flat*** if its curvature is identically zero.

To study the curvature more deeply, let us choose a smooth local frame (s_j) for E, and let θ_j^k be the corresponding matrix of connection forms. To compute the curvature, it suffices to compute its action on each basis section s_j for arbitrary X, Y:

$$\begin{aligned}
\Theta(X,Y)s_j &= \nabla_X\big(\theta_j^k(Y)s_k\big) - \nabla_Y\big(\theta_j^k(X)s_k\big) - \theta_j^k\big([X,Y]\big)s_k \\
&= X\big(\theta_j^k(Y)\big)s_k + \theta_j^k(Y)\theta_k^l(X)s_l - Y\big(\theta_j^k(X)\big)s_k - \theta_j^k(X)\theta_k^l(Y)s_l \\
&\quad - \theta_j^k\big([X,Y]\big)s_k \\
&= \big(d\theta_j^l(X,Y) + \big(\theta_k^l \wedge \theta_j^k\big)(X,Y)\big)s_l,
\end{aligned}$$

where in the last line we have used the invariant formula for the exterior derivative of a 1-form [**LeeSM**, Prop. 14.29]. Thus Θ is represented locally by the matrix $\big(\Theta_j^l\big)$ of 2-forms, called the ***curvature forms of*** ∇ with respect to this frame, given by

$$\Theta_j^l = d\theta_j^l + \theta_k^l \wedge \theta_j^k. \tag{7.9}$$

If we interpret θ_j^l and Θ_j^l as the local expressions for $\operatorname{End}(E)$-valued forms, we can use the wedge product of endomorphism-valued forms defined by (4.18) to write (7.9) in the form

$$\Theta = d\theta + \theta \wedge \theta.$$

There are a couple of caveats that need to be observed about this formula. First, since θ is a matrix-valued 1-form, the wedge product $\theta \wedge \theta$ need not be zero, as it would be if θ were a scalar-valued 1-form. Second, although Θ is a globally defined $\operatorname{End}(E)$-valued 2-form as described above, we can only interpret θ as an endomorphism-valued form in the domain of a particular frame. If it were the local matrix representation for a global $\operatorname{End}(E)$-valued 1-form, then its transformation law under a change of frames would be $\tilde\theta = \tau^{-1}d\tau$ instead of the more complicated formula (7.5). Even so, the definition of wedge products involving θ works the same as for a globally defined form.

The next proposition gives a qualitative interpretation of the curvature of a connection: it is the obstruction to the existence of parallel local frames. (A local frame $(s_1, \dots, s_k)$ for E is said to be ***parallel*** if each of the sections s_j is parallel.)

Proposition 7.10. *Let $E \to M$ be a smooth complex vector bundle endowed with a connection ∇. Then ∇ is flat if and only if every point of M has a neighborhood on which there exists a parallel local frame for E.*

Proof. One direction is easy: if $(s_1, \dots, s_k)$ is a parallel local frame, then the connection forms θ_j^k are all identically zero, so the curvature forms Θ_j^k are zero as well.

Conversely, suppose $\Theta \equiv 0$. Given $p \in M$, we begin by showing that every element $\sigma_0 \in E_p$ has a parallel extension to a neighborhood of p. The proof is an adaptation of [**LeeRM**, Lemma 7.8]. Choose smooth coordinates $(x^1, \dots, x^N)$ on some neighborhood U of p, such that p has coordinates $(0, \dots, 0)$ and the image of the coordinate map is a cube in $\mathbb{R}^N$. We define σ on U as follows: first parallel transport σ_0 along the x^1-axis; then from each point on the x^1-axis, parallel transport σ along the x^2-curve through that point; and continue by induction to obtain a section σ defined on all of U. It is smooth because solutions to ODEs depend smoothly on their initial conditions.

For each $i = 1, \dots, N$ let $M_i \subseteq U$ be the slice defined by $x^{i+1} = \cdots = x^N = 0$. We will prove by induction on j that

$$\nabla_{\partial_1}\sigma = \cdots = \nabla_{\partial_j}\sigma = 0 \text{ on } M_j. \tag{7.10}$$

For $j = 1$, this is true by construction, and for $j = N$, it is the statement that σ is parallel on U. Assuming (7.10) is true for some $j \geq 1$, we also have $\nabla_{\partial_{j+1}}\sigma \equiv 0$ on M_{j+1} by construction. For $1 \leq i \leq j$, since $\nabla_{\partial_i}\sigma = 0$ on M_j, if we can show that $\nabla_{\partial_{j+1}}(\nabla_{\partial_i}\sigma) = 0$ on M_{j+1}, it will follow from uniqueness of parallel transport that $\nabla_{\partial_i}\sigma = 0$ on all of M_{j+1}. By definition of the curvature,

$$\nabla_{\partial_{j+1}}(\nabla_{\partial_i}\sigma) = \nabla_{\partial_i}(\nabla_{\partial_{j+1}}\sigma) + \nabla_{[\partial_{j+1},\partial_i]}\sigma + \Theta(\partial_1, \partial_2)\sigma.$$

Each of the terms on the right-hand side is zero on M_{j+1}: the first because $\partial_{j+1}\sigma \equiv 0$ there; the second because $[\partial_{j+1}, \partial_i] = 0$; and the third because ∇ is flat. This completes the induction and shows that every $\sigma_0 \in E_p$ has a parallel extension to a neighborhood of p.

Now choose a basis $(b_1, \dots, b_k)$ for E_p, and for each $i = 1, \dots, k$, let s_i be a parallel extension of b_i to a neighborhood of p. By continuity, the sections $(s_1, \dots, s_k)$ will continue to be linearly independent in some neighborhood of p, so they constitute a parallel local frame. □

Here is another way to look at curvature. Given a smooth complex vector bundle $E \to M$ and a connection ∇ on E, we can define a sequence of maps on bundle-valued differential forms

$$\mathscr{E}^0(M; E) \xrightarrow{D} \mathscr{E}^1(M; E) \xrightarrow{D} \mathscr{E}^2(M; E) \to \cdots,$$

called ***exterior covariant differentiation***, as follows.

Proposition 7.11 (The Exterior Covariant Derivative). *Suppose M is a complex manifold, $E \to M$ is a smooth complex vector bundle, and ∇ is a connection on E. Then there are operators $D\colon \mathscr{E}^q(M; E) \to \mathscr{E}^{q+1}(M; E)$ for each $q \geq 0$ satisfying* *the following properties:*

(i) *For* $\sigma \in \mathscr{E}^0(M; E) = \Gamma(E)$*,* $D\sigma = \nabla\sigma$*.*

(ii) *For* $\alpha \in \mathscr{E}^q(M)$ *and* $\beta \in \mathscr{E}^{q'}(M; E)$*,*

$$D(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^q \alpha \wedge D\beta.$$

(iii) $D^2\alpha = \Theta \wedge \alpha$*, where* $\Theta \in \mathscr{E}^2(M; \operatorname{End}(E))$ *is the curvature form of* ∇*.*

Proof. The proof is closely parallel to that of Proposition 4.16 for the Cauchy–Riemann operator $\bar{\partial}_E$. For an E-valued 0-form (smooth section of E), we simply define $D\sigma = \nabla\sigma$. To extend this to higher-degree forms, choose a smooth local frame (s_j) for E. For a section $\alpha \in \mathscr{E}^q(M; E)$ expressed locally as $\alpha = \alpha^j \otimes s_j$ where the α^j's are scalar q-forms (using the summation convention), we wish to let $D\alpha$ be the element of $\mathscr{E}^{q+1}(M; E)$ whose local expression is

$$D\alpha = d\alpha^j \otimes s_j + (-1)^q \alpha^j \wedge Ds_j. \tag{7.11}$$

If $\tilde{s}_k = \tau^j_k s_j$ is another local frame, we can write $\alpha = \widetilde{\alpha}^k \otimes \tilde{s}_k$ with $\tau^j_k \widetilde{\alpha}^k = \alpha^j$, and use the product rule for ∇ to compute

$$\begin{aligned}
d\alpha^j \otimes s_j &+ (-1)^q \alpha^j \wedge Ds_j \\
&= d\big(\tau^j_k \widetilde{\alpha}^k\big) \otimes \big((\tau^{-1})^l_j \tilde{s}_l\big) + (-1)^q \big(\tau^j_k \widetilde{\alpha}^k\big) \wedge \nabla\big((\tau^{-1})^l_j \tilde{s}_l\big) \\
&= d\tau^j_k \wedge \widetilde{\alpha}^k \otimes (\tau^{-1})^l_j \tilde{s}_l + \tau^j_k d\widetilde{\alpha}^k \otimes (\tau^{-1})^l_j \tilde{s}_l \\
&\qquad + (-1)^q \tau^j_k \widetilde{\alpha}^k \wedge d(\tau^{-1})^l_j \otimes \tilde{s}_l + (-1)^q \tau^j_k \widetilde{\alpha}^k \wedge (\tau^{-1})^l_j \nabla \tilde{s}_l \\
&= d\widetilde{\alpha}^k \otimes \tilde{s}_k + (-1)^q \widetilde{\alpha}^k \wedge D\tilde{s}_k,
\end{aligned}$$

where the two terms involving derivatives of τ^j_k cancel because $d\big((\tau^{-1})^l_j \tau^j_k\big) = d(\delta^l_k) = 0$. This proves that D is well defined, and properties (i) and (ii) follow immediately from the definition.

To prove (iii), we note first that for $\alpha \in \mathscr{E}^q(M)$ and $\sigma \in \mathscr{E}^0(M; E) = \Gamma(E)$, property (ii) implies

$$\begin{aligned}
D^2(\alpha \wedge \sigma) &= D\big(d\alpha \wedge \sigma + (-1)^q \alpha \wedge D\sigma\big) \\
&= (-1)^{q+1} d\alpha \wedge D\sigma + (-1)^q d\alpha \wedge D\sigma + \alpha \wedge D^2\sigma \\
&= \alpha \wedge D^2\sigma.
\end{aligned}$$

This implies two important facts: First, it shows that the action of D^2 on $\mathscr{E}^q(M; E)$ is determined by that on $\mathscr{E}^0(M; E)$. And second, by taking α to be a 0-form $f \in C^\infty(M)$, it shows that D^2 is linear over $C^\infty(M)$ and is thus a smooth bundle homomorphism, so it suffices to check (iii) on elements of a local frame for E.

Given a smooth local frame (s_j) with connection 1-forms θ_j^k, we have

$$D^2 s_j = D\big(\theta_j^k \otimes s_k\big) = d\theta_j^k \otimes s_k - \theta_j^k \wedge \theta_k^l \otimes s_l = \big(d\theta_j^l + \theta_k^l \wedge \theta_j^k\big) \otimes s_l = \Theta_j^l \otimes s_l,$$

which is (iii) in this case. □

The First Real Chern Class

For any connection ∇ on a smooth complex vector bundle $E \to M$, the curvature form Θ associated with ∇ is a global endomorphism-valued 2-form. Because the trace of an endomorphism is independent of the choice of basis, we can form a global *scalar* 2-form, called the ***first Chern form of*** ∇, by

$$c_1(\nabla) = \frac{i}{2\pi} \operatorname{tr} \Theta.$$

The reason for the coefficient will emerge shortly. (The adjective "first" reflects the fact that there are also higher Chern forms that can be defined using higher-degree polynomials in the curvature forms; but we will not need those. These forms, and the cohomology classes they determine, were introduced by Shiing-Shen Chern in 1946 [**Che46**].)

Theorem 7.12. *For any connection on a smooth complex vector bundle, the first Chern form is closed, and its de Rham cohomology class is independent of the choice of connection.*

Proof. Let $E \to M$ be a smooth complex vector bundle and ∇ a connection on E. In terms of a smooth local frame, we have

$$c_1(\nabla) = \frac{i}{2\pi}\Theta_j^j = \frac{i}{2\pi}\big(d\theta_j^j + \theta_l^j \wedge \theta_j^l\big).$$

Because wedge products of (scalar) 1-forms anticommute, we can write the second term above as

$$\theta_l^j \wedge \theta_j^l = -\theta_j^l \wedge \theta_l^j = -\theta_l^j \wedge \theta_j^l,$$

where the second equality follows from interchanging the dummy indices j and l. Thus the wedge product term is identically zero, so in the domain of the local frame we have

$$c_1(\nabla) = \frac{i}{2\pi} d\theta_j^j, \tag{7.12}$$

which is locally exact and thus closed. (It is typically not globally exact, though, because the 1-forms θ_j^k are not globally defined.)

To see that its cohomology class is independent of the choice of connection, suppose $\widetilde{\nabla}$ is another connection on E. Define the ***difference tensor*** between $\widetilde{\nabla}$ and ∇ as the map $\mathscr{D} : \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) \to \Gamma(E)$ given by

$$\mathscr{D}(X)\sigma = \widetilde{\nabla}_X \sigma - \nabla_X \sigma.$$

A straightforward computation shows that $\mathscr{D}$ is linear over $C^\infty(M)$ in both arguments, so $\mathscr{D}$ is actually a global endomorphism-valued 1-form, that is, a section of $\mathscr{E}^1(\operatorname{End}(E))$. It follows that its trace is a global scalar 1-form, and we have

$$c_1(\widetilde{\nabla}) - c_1(\nabla) = \frac{i}{2\pi}\left(\operatorname{tr}\widetilde{\Theta} - \operatorname{tr}\Theta\right) = \frac{i}{2\pi}\left(\operatorname{tr} d\tilde{\theta} - \operatorname{tr} d\theta\right) = \frac{i}{2\pi}(\operatorname{tr} d\mathscr{D})$$
$$= \frac{i}{2\pi} d(\operatorname{tr}\mathscr{D}).$$

Thus $c_1(\widetilde{\nabla})$ and $c_1(\nabla)$ differ by an exact form, so they define the same de Rham cohomology class. □

We need one more property of this cohomology class, which explains why the factor of i is included in the definition of the Chern form.

Proposition 7.13. *Suppose $E \to M$ is a smooth complex vector bundle with a Hermitian fiber metric. If ∇ is a metric connection on E, then $c_1(\nabla)$ is a real 2-form.*

Proof. Let ∇ be a metric connection on E. In a neighborhood of each point, we may choose an orthonormal local frame (s_j), and let θ_j^k be the corresponding connection 1-forms. Proposition 7.1 shows that

$$\theta_j^k + \overline{\theta_k^j} = 0.$$

Taking the trace of this equation, we find that the scalar 1-form θ_j^j is purely imaginary, and thus so is $\Theta_j^j = d\theta_j^j$. Thus $c_1(\nabla) = \frac{i}{2\pi}\Theta_j^j$ is real. □

Let $E \to M$ be a smooth complex vector bundle. We can always choose a Hermitian fiber metric on ∇ and a connection ∇ compatible with it, so that $c_1(\nabla)$ is represented by a real 2-form. Thus the cohomology class determined by $c_1(\nabla)$ lies in $H^2_{\mathrm{dR}}(M;\mathbb{R})$ (considered as the subspace of $H^2_{\mathrm{dR}}(M;\mathbb{C})$ consisting of cohomology classes that are invariant under conjugation). We define the ***first real Chern class of E***, denoted by $c_1^{\mathbb{R}}(E) \in H^2_{\mathrm{dR}}(M;\mathbb{R})$, to be the cohomology class of $c_1(\nabla)$, where ∇ is any connection on E. Under the de Rham–Weil isomorphisms, we can also consider it as an element of the sheaf cohomology group $H^2(M;\underline{\mathbb{R}})$ or the singular cohomology group $H^2_{\mathrm{Sing}}(M;\mathbb{R})$.

Line Bundles

Now we examine how this looks in the case of line bundles. Let $L \to M$ be a smooth complex line bundle, and let ∇ be a connection on L. The endomorphism bundle $\operatorname{End}(L)$ is a trivial line bundle because the identity endomorphism is a canonical global frame, so we can consider the connection form Θ as a global scalar-valued 2-form. If we choose a trivializing cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ with a local frame s_α (that is, a nonvanishing local section) on each set U_α, the connection is determined in U_α by a 1×1 matrix of 1-forms, that is, an ordinary scalar 1-form. Let us denote the 1-form associated with the local frame s_α by θ_α, so $\nabla s_\alpha = \theta_\alpha \otimes s_\alpha$. When U_α and U_β overlap, equation (3.4) shows that the local frames s_α and s_β are related by $s_\beta = \tau_{\alpha\beta} s_\alpha$, so the transition formula (7.5) for the connection forms becomes

$$\theta_\beta = \tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta} + \theta_\alpha \tag{7.13}$$

(because 1×1 matrices commute).

The curvature form is the globally defined 2-form Θ that satisfies $\Theta|_{U_\alpha} = d\theta_\alpha$ for each α; and the Chern form is $c_1(\nabla) = \frac{i}{2\pi}\Theta$, a globally defined closed 2-form, which is real if ∇ is a metric connection.

The next theorem explains the relationship between the first real Chern class of a line bundle and the sheaf-theoretic Chern class defined in Chapter 5. It is the reason the factor of $i/(2\pi)$ is included in the definition of $c_1(\nabla)$, and the negative sign in the definition of $c(L)$. (See the remark following the proof for a comment about the sign.)

Theorem 7.14. *Let $L \to M$ be a smooth complex line bundle. Under the composition*

$$H^2(M; \underline{\mathbb{Z}}) \to H^2(M; \underline{\mathbb{R}}) \to H^2_{\mathrm{dR}}(M; \mathbb{R}),$$

in which the first map is the homomorphism induced by the sheaf inclusion $\iota : \underline{\mathbb{Z}} \hookrightarrow \underline{\mathbb{R}}$ and the second is the inverse of the de Rham–Weil isomorphism, the sheaf-theoretic Chern class $c(L)$ maps to the first real Chern class $c_1^{\mathbb{R}}(L)$.

Proof. This is a matter of unwinding the definitions of two maps: first, the sheaf-theoretic Chern class map $c : H^1(M; \mathscr{E}^*) \to H^2(M; \underline{\mathbb{Z}})$, and second, the isomorphism $\mathscr{R}_2 : H^2_{\mathrm{dR}}(M; \mathbb{R}) \to H^2(M; \underline{\mathbb{R}})$ given by the de Rham–Weil theorem.

We begin by choosing an open cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ of M by convex geodesic balls with respect to some Riemannian metric, so that each U_α and all finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are contractible, as in Problem 6-3. By choosing the geodesic balls small enough, we can also ensure that over each U_α there is a smooth local frame s_α (that is, a nonvanishing local section) for L.

Recall that the sheaf-theoretic Chern class is defined by $c(L) = -\delta_*([L])$, where $\delta_* : H^1(M; \mathscr{E}^*) \to H^2(M; \underline{\mathbb{Z}})$ is the connecting homomorphism in the long exact sequence arising from the short exact sheaf sequence

$$0 \to \underline{\mathbb{Z}} \overset{\iota}{\hookrightarrow} \mathscr{E} \overset{\varepsilon}{\longrightarrow} \mathscr{E}^* \to 0.$$

The homomorphism δ_* is characterized by (6.6). To compute it explicitly, we begin by letting $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(1, \mathbb{C})$ be the transition functions for the given local frames; taken together, they define a cocycle $\tau \in C^1(\mathscr{U}; \mathscr{E}^*)$ that represents $[L] \in H^1(M; \mathscr{E}^*)$. Because $U_\alpha \cap U_\beta$ is contractible, the nonvanishing complex-valued function $\tau_{\alpha\beta}$ has a complex logarithm there by the result of Problem 5-11, so we can choose a smooth function $b_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C}$ such that $\tau_{\alpha\beta} = \varepsilon(b_{\alpha\beta}) = e^{2\pi i b_{\alpha\beta}}$. Let $b \in C^1(\mathscr{U}; \mathscr{E})$ denote the cochain defined by these functions. Our construction ensures that $\varepsilon_\# b = \tau$, where $\varepsilon_\#\colon C^1(\mathscr{U};\mathscr{E}) \to C^1(\mathscr{U};\mathscr{E}^*)$ is the homomorphism defined by (6.2).

Now consider $\delta b \in C^2(\mathscr{U};\mathscr{E})$:

$$(\delta b)_{\alpha\beta\gamma} = (b_{\beta\gamma} - b_{\alpha\gamma} + b_{\alpha\beta})\big|_{U_\alpha \cap U_\beta \cap U_\gamma}.$$

The fact that $\tau_{\alpha\beta}\tau_{\beta\gamma} = \tau_{\alpha\gamma}$ implies that $(\delta b)_{\alpha\beta\gamma}$ is integer-valued, and because $U_\alpha \cap U_\beta \cap U_\gamma$ is connected, this continuous integer-valued function is constant. Thus the assignment

$$k_{\alpha\beta\gamma} = (b_{\beta\gamma} - b_{\alpha\gamma} + b_{\alpha\beta})\big|_{U_\alpha \cap U_\beta \cap U_\gamma} \tag{7.14}$$

defines a 2-cocyle $k \in C^2(\mathscr{U};\underline{\mathbb{Z}})$ that satisfies $\iota_\# k = \delta b$. We see from (6.6) that $\delta_*[\tau]$ is represented by k, so the sheaf-theoretic Chern class $c(L)$ is represented by $-k$.

On the other hand, Problem 6-3 shows how the isomorphism $\mathscr{R}_2\colon H^2_{\mathrm{dR}}(M;\mathbb{R}) \to H^2(M;\underline{\mathbb{R}})$ is constructed. Starting with a global closed 2-form η, on each U_α we need to find a 1-form φ_α such that $d\varphi_\alpha = \eta|_{U_\alpha}$, and on each nonempty intersection $U_\alpha \cap U_\beta$ a smooth function $u_{\alpha\beta}$ such that $\varphi_\beta|_{U_\alpha\cap U_\beta} - \varphi_\alpha|_{U_\alpha\cap U_\beta} = du_{\alpha\beta}$; and then $\mathscr{R}_2([\eta]) = [[a]]$ where $a_{\alpha\beta\gamma}$ is the restriction to $U_\alpha \cap U_\beta \cap U_\gamma$ of the constant function $u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}$.

Let ∇ be a connection on L that is compatible with some Hermitian fiber metric, and for each α let θ_α be the connection 1-form on U_α; the proof of Proposition 7.13 shows that θ_α is purely imaginary. Let Θ be the connection form of ∇, and let $\eta = \frac{i}{2\pi}\Theta$ be its first Chern form. On each set U_α, we have $\Theta|_{U_\alpha} = d\theta_\alpha$, so to apply the de Rham isomorphism to η we can take φ_α to be the real 1-form $\frac{i}{2\pi}\theta_\alpha$ on U_α. On each nonempty overlap $U_\alpha \cap U_\beta$, (7.13) shows that

$$\theta_\beta - \theta_\alpha = \tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta}.$$

Using the same functions $b_{\alpha\beta}$ as above, we have $\tau_{\alpha\beta} = e^{2\pi i b_{\alpha\beta}}$, so

$$\varphi_\beta - \varphi_\alpha = \frac{i}{2\pi}(\theta_\beta - \theta_\alpha) = -db_{\alpha\beta},$$

and we can take $u_{\alpha\beta} = -b_{\alpha\beta}$. This yields $\mathscr{R}_2[c_1(\nabla)] = [[a]]$, where a is the $\underline{\mathbb{C}}$-valued 2-cocycle given by

$$a_{\alpha\beta\gamma} = -(b_{\beta\gamma} - b_{\alpha\gamma} + b_{\alpha\beta})\big|_{U_\alpha \cap U_\beta \cap U_\gamma}. \tag{7.15}$$

Comparing (7.14) and (7.15), we see that $a = -\iota_\# k$, so $\iota_* c(L) = -[[\iota_\# k]] = [[a]] = \mathscr{R}_2([c_1(\nabla)])$. □

Remark. As we noted in Chapter 6, some authors write the definition of the sheaf-theoretic Chern class as $c(L) = \delta^*([L])$, without the negative sign. In some cases, this may just be a matter of choosing a different convention for the transition functions, with (3.4) replaced by $s_\alpha = \tau_{\alpha\beta} s_\beta$. But in other cases, it seems to be based on a computational error. For example, in [**GH94**, p. 141], the discrepancy results from an incorrect transformation equation between the connection forms θ_β and θ_α: their transition functions $g_{\alpha\beta}$ play the same role as our $\tau_{\alpha\beta}$, but their transformation formula reverses the roles of θ_α and θ_β compared to our equation (7.13), resulting in an incorrect sign for the image of $c_1(\nabla)$ in $H^2(M; \underline{\mathbb{Z}})$. Be sure to check all such computations carefully before using them.

Corollary 7.15. *Suppose M is a connected compact Riemann surface, $L \to M$ is a smooth complex line bundle, and Ω is a closed 2-form representing $c_1^{\mathbb{R}}(L)$. The degree of L is given by the formula*

$$\deg(L) = \int_M \Omega.$$

Proof. Let $\varphi \in \operatorname{Sing}^2(M; \mathbb{Z})$ be a singular cocycle representing the sheaf-theoretic Chern class $c(L) \in H^2_{\mathrm{Sing}}(M; \mathbb{Z})$, and let μ be a singular cycle representing the fundamental class $[M] \in H_2(M)$. Then by definition $\deg(L) = \varphi(\mu)$. Because M is a smooth oriented compact manifold, we can choose μ to be a smooth chain obtained from a smooth triangulation of M.

Now the image of $c(L)$ in $H^2_{\mathrm{Sing}}(M; \mathbb{C})$ under the coefficient homomorphism $\mathbb{Z} \hookrightarrow \mathbb{C}$ is represented by the same cocycle φ, but now thought of as a homomorphism from $\operatorname{Sing}_2(M)$ into $\mathbb{C}$. Similarly, its image in $H^2_{\mathrm{Sing},\infty}(M; \mathbb{C})$ is again the same cocycle, but with its action restricted to smooth chains. On the other hand, Theorem 7.14 shows that this cocycle is also represented by the cocycle $\widetilde{\varphi} \in \operatorname{Sing}^{2,\infty}(M; \mathbb{C})$ given by $\widetilde{\varphi}(c) = \int_c \Omega$. Since integrating over μ is the same as integrating over M [**LeeSM**, Prop. 16.8], the result follows. □

It is important to realize that the coefficient homomorphism $H^2(M; \underline{\mathbb{Z}}) \to H^2(M; \underline{\mathbb{C}})$ (or equivalently $H^2_{\mathrm{Sing}}(M; \mathbb{Z}) \to H^2_{\mathrm{Sing}}(M; \mathbb{C})$) need not be injective, so the sheaf-theoretic Chern class may contain more information than the first real Chern class. For example, if M is any smooth manifold for which $H^2_{\mathrm{Sing}}(M; \mathbb{Z})$ contains a nontrivial torsion element γ (meaning $k\gamma = 0$ for some positive integer k), then the image of γ in $H^2_{\mathrm{dR}}(M; \mathbb{R})$ is zero. (Simple examples of such manifolds are the real projective spaces $\mathbb{RP}^n$ for $n \geq 2$.) By Theorem 6.29, there is a nontrivial smooth complex line bundle over M whose sheaf-theoretic Chern class is equal to γ; but its first real Chern class is zero.

The Chern Connection

Now we turn our attention to holomorphic vector bundles on complex manifolds. We can always choose a Hermitian fiber metric on such a bundle, and as we mentioned earlier, there is a connection on the bundle that is compatible with the metric. As you can see from the use of a partition of unity in the construction of such connections, such a connection is far from unique. But when the bundle is holomorphic, we can require an additional condition that will guarantee uniqueness.

It is useful to consider the case of Riemannian metrics for comparison. On a Riemannian manifold, there are many connections on the tangent bundle that are compatible with the Riemannian metric; but there is a unique connection, the Levi-Civita connection, that satisfies the additional condition of being ***torsion-free***, meaning that $\nabla_X Y - \nabla_Y X = [X, Y]$ for all smooth vector fields X and Y. But the torsion-free condition makes sense only for connections on the tangent bundle, so we need another condition to determine a unique connection on a holomorphic bundle.

Suppose M is a complex manifold and $E \to M$ is a holomorphic vector bundle. Using the decomposition $T^*_{\mathbb{C}}M = \Lambda^1_{\mathbb{C}}M = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$, we can decompose a connection on E as $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$, where $\nabla^{(1,0)}\sigma \in \Gamma(\Lambda^{1,0}M \otimes E)$ and $\nabla^{(0,1)}\sigma \in \Gamma(\Lambda^{0,1}M \otimes E)$.

Here is the additional condition we wish to impose in the holomorphic case. A connection on a holomorphic vector bundle $E \to M$ is said to be ***compatible with the holomorphic structure*** if $\nabla^{(0,1)}$ is equal to the operator $\bar\partial_E$ defined by Proposition 4.16.

Proposition 7.16 (Compatibility with the Holomorphic Structure). *Let $E \to M$ be a holomorphic vector bundle and ∇ a connection on E. The following are equivalent:*

(a) ∇ *is compatible with the holomorphic structure* (*i.e.*, $\nabla^{(0,1)} = \bar\partial_E$).

(b) *Whenever σ is a holomorphic local section of E and $\overline{Z}$ is a smooth local section of $T''M$, we have $\nabla_{\overline{Z}}\sigma = 0$.*

(c) *For each holomorphic local frame (s_j), we have $\nabla s_j = \theta_j^k \otimes s_k$ where the* 1*-forms θ_j^k are all of type* $(1, 0)$.

Proof. This is a local issue, so we may work in an open set over which there is a holomorphic local frame (s_j). Let θ_j^k be the connection 1-forms with respect to this frame. Taking the projection of both sides of (7.4) onto $\Lambda^{0,1}M \otimes E$, we have

$$\nabla^{(0,1)}(\sigma^j s_j) = \bar\partial\sigma^j \otimes s_j + \sigma^j (\theta_j^k)^{(0,1)} \otimes s_k.$$

On the other hand, (4.20) shows that

$$\bar\partial_E(\sigma^j s_j) = \bar\partial\sigma^j \otimes s_j.$$

Comparing these two equations, we see that $\nabla^{(0,1)} = \bar{\partial}_E$ if and only if $(\theta_j^k)^{(0,1)} = 0$ for all j and k, which shows that (a) $\Leftrightarrow$ (c).

To prove that (c) $\Rightarrow$ (b), suppose (c) holds. Let σ be a holomorphic local section of E and $\overline{Z}$ be a smooth local section of $T''M$. We can write $\sigma = \sigma^j s_j$, where now the component functions σ^j are holomorphic, and compute

$$\nabla_{\overline{Z}}\sigma = \overline{Z}(\sigma^j)s_j + \theta_j^k(\overline{Z})s_k = 0 + 0.$$

Conversely if (b) holds, then for any local section $\overline{Z}$ of $T''M$,

$$0 = \nabla_{\overline{Z}} s_j = \theta_j^k(\overline{Z})s_k.$$

This shows that each form θ_j^k vanishes on $T''M$, which is equivalent to being of type $(1,0)$, so (b) $\Rightarrow$ (c). □

Here is the fundamental fact about connections on Hermitian holomorphic bundles.

Theorem 7.17 (Chern Connection Theorem). *On every Hermitian holomorphic vector bundle there is a unique connection, called the* ***Chern connection****, that is compatible with the metric and the holomorphic structure.*

Proof. We begin by proving uniqueness. Suppose ∇ is such a connection on $E \to M$, and let (s_j) be a holomorphic local frame for E over an open subset $U \subseteq M$. Writing $\nabla s_j = \theta_j^k \otimes s_k$, we note first that Proposition 7.16 implies that the forms θ_j^k are all of type $(1,0)$. If we let $h_{jk} = \langle s_j, s_k \rangle$, then compatibility with the metric implies the following equation for every local section Z of $T'M$:

$$\begin{aligned} Z(h_{jk}) = Z\langle s_j, s_k\rangle &= \langle \nabla_Z s_j,\ s_k\rangle + \langle s_j, \nabla_{\overline{Z}} s_k\rangle \\ &= \langle \theta_j^l(Z)s_l,\ s_k\rangle + 0 \\ &= \theta_j^l(Z)h_{lk}. \end{aligned} \tag{7.16}$$

Since the matrix (h_{jk}) is positive definite, it has an inverse matrix, denoted by (h^{jk}). Multiplying both sides of (7.16) by h^{km} and simplifying, we obtain

$$h^{km}Z(h_{jk}) = \theta_j^m(Z).$$

Since this is true for every section Z of $T'M$, it implies

$$\theta_j^m = h^{km}\partial h_{jk}. \tag{7.17}$$

This shows that ∇ is uniquely determined if it exists.

To prove existence, we use (7.17) to define ∇ in terms of each holomorphic local frame. These forms are of type $(1,0)$ by definition, so the resulting connection is compatible with the holomorphic structure; and reverse-engineering the computation above (together with the analogous computation for $\overline{Z}$) shows that it is also compatible with the metric. Then uniqueness guarantees that the definitions associated with different local frames agree where they overlap, so ∇ is globally defined. □

The relative simplicity of (7.17) should be compared with the much more complicated formula for the connection coefficients of the Levi–Civita connection on a Riemannian manifold [**LeeRM**, eq. (5.12)].

Proposition 7.18 (Curvature of the Chern Connection is Type (1, 1)). *Suppose M is a complex manifold and $E \to M$ is a Hermitian holomorphic vector bundle. The curvature form Θ associated with the Chern connection on E lies in $\mathscr{E}^{1,1}(M; \mathrm{End}(E))$.*

Proof. In terms of a holomorphic local frame (s_j), we can write

$$\Theta_j^m = \bar\partial\theta_j^m + \partial\theta_j^m + \theta_k^m \wedge \theta_j^k. \tag{7.18}$$

Because the forms θ_j^m are all of type $(1,0)$, the first term on the right-hand side of (7.18) is of type $(1,1)$ and the last two are of type $(2,0)$; thus it suffices to show that the last two terms sum to zero. This is just a computation using (7.17):

$$\begin{aligned}\partial\theta_j^m + \theta_k^m \wedge \theta_j^k &= \partial\big(h^{km}\partial h_{jk}\big) + \big(h^{pm}\partial h_{kp}\big) \wedge \big(h^{qk}\partial h_{jq}\big)\\ &= \partial h^{km} \wedge \partial h_{jk} + h^{pm}h^{qk}\partial h_{kp} \wedge \partial h_{jq}.\end{aligned}$$

Now differentiating $h^{kp}h_{pl} = \delta_l^k$ and multiplying by h^{lm} yields

$$\partial h^{km} = -h^{lm}h^{kp}\partial h_{pl},$$

and substituting this above and renaming the dummy indices proves the result. □

The Chern Connection on a Line Bundle

In the case of a holomorphic line bundle, the formulas for the Chern connection and its curvature simplify even further. Let $L \to M$ be a Hermitian holomorphic line bundle and let ∇ be its Chern connection. Given a local holomorphic frame s for L (that is, a nonvanishing holomorphic local section) over an open set $U \subseteq M$, let θ be the corresponding connection 1-form. The fiber metric is completely determined in U by the strictly positive smooth function $h = |s|^2 = \langle s, s\rangle$. In this situation, (7.17) reduces to

$$\theta = h^{-1}\partial h = \partial(\log h), \tag{7.19}$$

and its curvature is the globally defined 2-form Θ whose expression in terms of each holomorphic local frame s is

$$\Theta = d\theta = \bar\partial\partial(\log h) = \bar\partial\partial\big(\log|s|^2\big). \tag{7.20}$$

Thus the Chern form for this connection has the local expression

$$c_1(\nabla) = \frac{i}{2\pi}\bar\partial\partial(\log h) = \frac{i}{2\pi}\bar\partial\partial(\log|s|^2).$$

Next we will show how these formulas allow us to easily compute the connection and curvature forms for various related line bundles—tensor product bundles, dual bundles, and pullback bundles. We begin with tensor products. For a tensor product of line bundles $L \otimes L'$, having chosen Hermitian fiber metrics for L and L', we obtain a fiber metric for $L \otimes L'$, called the ***tensor product metric***, by setting $|\sigma \otimes \sigma'|^2 = |\sigma|^2 |\sigma'|^2$.

Proposition 7.19 (Curvature of a Tensor Product of Line Bundles). *Let $L, L' \to M$ be Hermitian holomorphic line bundles, and let Θ_L and $\Theta_{L'}$ be curvature forms of their Chern connections. With respect to the tensor product metric, the curvature of the Chern connection on $L \otimes L'$ is given by*

$$\Theta_{L \otimes L'} = \Theta_L + \Theta_{L'}.$$

Proof. With respect to holomorphic local frames s for L and s' for L', if we set $h = |s|^2$ and $h' = |s'|^2$, then the fiber metric on $L \otimes L'$ is represented locally by the function $hh' = |s \otimes s'|^2$. Thus the curvature form of the Chern connection on $L \otimes L'$ is given by

$$\Theta_{L \otimes L'} = \bar{\partial}\partial(\log hh') = \bar{\partial}\partial(\log h + \log h') = \Theta_L + \Theta_{L'}. \qquad \square$$

Next we consider the dual bundle.

Proposition 7.20 (Curvature of the Dual Bundle). *Let $L \to M$ be a Hermitian holomorphic line bundle and let Θ_L be the curvature form of its Chern connection. When L^* is endowed with the dual metric (see Problem 7-3), the curvature of its Chern connection is given by*

$$\Theta_{L^*} = -\Theta_L.$$

Proof. This follows easily from the results of Problems 7-4 and 7-5. $\square$

Finally, we consider pullbacks of line bundles. Suppose M and M' are complex manifolds and $L \to M$ is a holomorphic line bundle. Given a holomorphic map $f : M' \to M$, recall that the ***pullback bundle*** $f^*L \to M'$ is defined as

$$f^*L = \{(x, v) \in M' \times L : v \in L_{f(x)}\}.$$

Thus the fiber of f^*L over $x \in M'$ is $\{x\} \times L_{f(x)}$, which we can canonically identify with $L_{f(x)}$. Using this identification, we can define a fiber metric on f^*L, called the ***pullback metric***, by

$$\langle \tilde{v}, \widetilde{w} \rangle_{f^*L} = \langle v, w \rangle_L \quad \text{for } \tilde{v} = (x, v), \widetilde{w} = (x, w) \in (f^*L)_x.$$

For any smooth local frame s for L, we get a smooth local frame f^*s for f^*L defined by $f^*s(x) = (x, s(f(x)))$, and the pullback metric satisfies $|f^*s(x)|_{f^*L} = |s(f(x))|_L$, so it is smooth by composition.

Proposition 7.21 (Curvature of a Pullback Bundle). *Let M and M' be complex manifolds, $L \to M$ a holomorphic line bundle, and $f : M' \to M$ a holomorphic map. With respect to any Hermitian fiber metric on L and the pullback metric on f^*L, the curvatures of the Chern connections on L and f^*L satisfy*

$$\Theta_{f^*L} = f^*\Theta_L.$$

Proof. Given $p \in M'$, let s be a holomorphic local frame for L in a neighborhood of $f(p)$, and let $s' = f^*s$, which is a holomorphic local frame for f^*L. If we write $h = |s|^2$ and $h' = |s'|^2$, then $h' = h \circ f$, and therefore the curvature form of the Chern connection on f^*L is given locally by

$$\Theta_{f^*L} = \bar\partial\partial(\log h') = \bar\partial\partial(\log f^*h) = f^*\bar\partial\partial(\log h) = f^*\Theta_L.$$

Since this is true in a neighborhood of each point, it follows that $\Theta_{f^*L} = f^*\Theta_L$ globally. □

The next theorem gives an extremely useful way to compute the degrees of line bundles on Riemann surfaces. Recall that if $L \to M$ is a holomorphic line bundle on a compact Riemann surface and σ is a meromorphic section of L, the ***divisor of σ***, denoted by (σ), is the formal sum of zeros and poles of σ with coefficients indicating their orders (positive for zeros, negative for poles). If $D = \sum_{\alpha=1}^k m_\alpha p_\alpha$ is a divisor, we define the ***degree of D*** to be the sum of its coefficients, that is,

$$\deg D = \sum_{\alpha=1}^k m_\alpha.$$

Theorem 7.22 (Degree of a Line Bundle Associated with a Divisor). *Let M be a connected compact Riemann surface. For any divisor D on M, the degree of the line bundle L_D is equal to the degree of the divisor D.*

Proof. Let $D = \sum_{\alpha=1}^k m_\alpha p_\alpha$ be a divisor on M. Proposition 3.41 shows that L_D has a meromorphic section σ whose divisor is D. Choose any Hermitian metric on L, and let ∇ be the corresponding Chern connection and $\Theta \in \mathscr{E}^{1,1}(M)$ its curvature form.

For each point p_α, choose a holomorphic coordinate z_α on an open disk U_α centered at p_α. For $\varepsilon > 0$, let $U_\alpha(\varepsilon) \subseteq M$ be the set $\{q \in U_\alpha : |z_\alpha(q)| < \varepsilon\}$. For any ε small enough that $\overline{U}_\alpha(\varepsilon) \subseteq U_\alpha$ for each α and the sets $\overline{U}_1(\varepsilon), \dots, \overline{U}_k(\varepsilon)$ are disjoint, let $M_\varepsilon = M \smallsetminus (U_1(\varepsilon) \cup \dots \cup U_k(\varepsilon))$.

On $M \smallsetminus \{p_1, \dots, p_k\}$, the section σ is holomorphic and nonvanishing, so it can be used as a holomorphic frame for L with which to compute Θ on that set. Formula (7.20) shows that on that set we have

$$\Theta = \bar\partial\partial\big(\log|\sigma|^2\big) = d\partial\big(\log|\sigma|^2\big).$$

Thus by Stokes's theorem,

$$\int_{M_\varepsilon} \Theta = \int_{\partial M_\varepsilon} \partial\big(\log|\sigma|^2\big) = -\sum_{\alpha=1}^{k} \int_{\partial \overline{U}_\alpha(\varepsilon)} \partial\big(\log|\sigma|^2\big),$$

where the minus sign in the last expression comes from the fact that the Stokes orientation on $\partial\overline{U}_\alpha(\varepsilon)$, considered as the boundary of $\overline{U}_\alpha(\varepsilon)$, is opposite to its orientation considered as part of ∂M_ε. It follows from Corollary 7.15 that

$$\deg(L) = \frac{i}{2\pi}\int_M \Theta = -\lim_{\varepsilon\to 0}\sum_{\alpha=1}^{k} \frac{i}{2\pi}\int_{\partial \overline{U}_\alpha(\varepsilon)} \partial\big(\log|\sigma|^2\big). \tag{7.21}$$

Now consider one of the disks $\overline{U}_\alpha(\varepsilon)$, and write the corresponding holomorphic coordinate as $z = z_\alpha$ for convenience. Choose a nonvanishing holomorphic section s of L on U_α, and write $\sigma(z) = z^{m_\alpha} f(z)s(z)$, where m_α is the coefficient of p_α in D and f is a nonvanishing holomorphic function on $\overline{U}_\alpha(\varepsilon)$. Let $h(z) = |s(z)|^2$, a nonvanishing smooth function. In these coordinates, since $|\sigma|^2$ does not vanish on $\partial\overline{U}_\alpha(\varepsilon)$, we have

$$\begin{aligned}\partial\big(\log|\sigma|^2\big) &= \frac{\partial\big(z^{m_\alpha}\bar{z}^{m_\alpha}|f(z)|^2 h(z))\big)}{z^{m_\alpha}\bar{z}^{m_\alpha}|f(z)|^2 h(z)}\\ &= m_\alpha\frac{dz}{z} + u(z)\,dz,\end{aligned}$$

where u is a smooth function of z on $\overline{U}_\alpha(\varepsilon)$.

We can parametrize $\partial\overline{U}_\alpha(\varepsilon)$ in z-coordinates by $z = \varepsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$, so $dz = i\varepsilon e^{i\theta}\,d\theta$. This yields

$$\frac{i}{2\pi}\int_{\partial\overline{U}_\alpha(\varepsilon)} \partial\big(\log|\sigma|^2\big) = \frac{i}{2\pi}\int_0^{2\pi} m_\alpha i\,d\theta + \frac{i}{2\pi}\int_0^{2\pi} u\big(\varepsilon e^{i\theta}\big) i\varepsilon e^{i\theta}\,d\theta.$$

The first term on the right is equal to $-m_\alpha$, and the second approaches zero as $\varepsilon \to 0$. Applying this to each term in the sum (7.21) proves the result. □

Corollary 7.23. *Let M be a connected compact Riemann surface. If $L \to M$ is a holomorphic line bundle of negative degree, then L admits no nontrivial holomorphic sections.*

Proof. Theorem 7.22 implies the contrapositive: if L admits a nontrivial holomorphic section σ, then it is isomorphic to the bundle L_D for $D = (\sigma)$; and since D is an effective divisor (i.e., it has only nonnegative coefficients), its degree is either zero or a sum of positive integers. □

Corollary 7.24. *Let M be a connected compact Riemann surface and f be a meromorphic function on M. Then* $\deg(f) = 0$.

Proof. We can consider f as a meromorphic section of the trivial holomorphic line bundle, so the line bundle L_D associated with the divisor $D = (f)$ is trivial. Since the trivial line bundle has degree 0, so does (f). □

Proposition 7.25 (Characterizations of $\mathbb{CP}^1$). *Let M be a connected compact Riemann surface. The following are equivalent.*

(a) *There exists a meromorphic function on M with only a single pole of order* 1.

(b) *There are distinct points $p, q \in M$ such that the point bundles $L_{\{p\}}$ and $L_{\{q\}}$ are isomorphic.*

(c) *There is a holomorphic line bundle $L \to M$ of degree* 1 *whose space of global holomorphic sections has dimension* 2.

(d) *M is biholomorphic to $\mathbb{CP}^1$.*

Proof. We will prove (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (a). First, assuming (a), let f be a meromorphic function with a simple pole at $p \in M$. Corollary 7.24 implies that f must have exactly one simple zero, say at q. Let L denote the point bundle $L_{\{p\}}$. It has a holomorphic section σ_p with a simple zero at p by Theorem 3.39. Then $f\sigma_p$ will have a removable singularity at p, so it is a holomorphic section of L with a simple zero at q. Thus Theorem 3.39 shows that $L_{\{p\}} \cong L_{\{q\}}$.

Next, assuming (b), let us denote the bundle $L_{\{p\}} \cong L_{\{q\}}$ by L. By Theorem 3.39, there are holomorphic sections $s_p, s_q \in \mathcal{O}(M; L)$ that vanish simply at p and q, respectively, and nowhere else. Because they vanish at different points, they cannot be linearly dependent. If $\sigma \in \mathcal{O}(M; L)$ is an arbitrary nontrivial holomorphic section, because L has degree 1 the divisor of σ must consist of a single point x. If x is equal to p, then Theorem 3.41 shows that σ is a constant multiple of s_p. Otherwise, $s_p(x) \neq 0$, so there is a complex number a such that $s_q(x) = as_p(x)$. The section $\sigma' = as_p - s_q$ also vanishes at x, and is not identically zero because s_p and s_q are linearly independent. Because $\deg L = 1$, Theorem 7.22 shows that σ' cannot vanish anywhere else in M, so it follows from Theorem 3.41 that σ is a constant multiple of σ'. In either case, we have shown that s_p and s_q span $\mathcal{O}(M; L)$. This proves (c).

Now assume $L \to M$ is a holomorphic line bundle of degree 1 such that $\dim \mathcal{O}(M; L) = 2$. We will show that the associated map $F : M \to \mathbb{CP}^1$ is a biholomorphism. By Lemma 3.42, F will be a holomorphic embedding provided $\mathcal{O}(M; L)$ separates points and directions.

Let (s_1, s_2) be a basis for $\mathcal{O}(M; L)$. To see that they separate points, let $p, q \in M$ be arbitrary distinct points. If $s_1(p) = 0$, then $s_1(q) \neq 0$ because each section has only one simple zero. Otherwise, there is a complex number a such that $s_2(p) = as_1(p)$, and the section $\sigma = as_1 - s_2$ vanishes at p. As before, it can have no other zeros, so $\sigma(q) \neq 0$.

To see that they separate directions, again let $p \in M$ be arbitrary and let σ be the section defined above. Theorem 7.22 shows that σ cannot have a zero of order greater than 1 at p, so in terms of a local holomorphic frame s for L we have $\sigma = fs$ where $f(p) = 0$ and $df_p \neq 0$. Thus given any nonzero $v \in T'_p M$, we have $vf(p) = df_p(v) \neq 0$.

Therefore F is a holomorphic embedding. Since M and $\mathbb{CP}^1$ are connected compact manifolds of the same dimension, F is both an open and closed map and therefore surjective, so it is a biholomorphism.

Finally, to prove (d) $\Rightarrow$ (a), just note that the function $f : \mathbb{CP}^1 \smallsetminus \{[0,1]\} \to \mathbb{C}$ defined by $f([w^0, w^1]) = w^1/w^0$ has a simple pole at $[0,1]$ and is holomorphic elsewhere. □

Here are some examples that illustrate how to use these formulas.

Example 7.26 (Curvatures of Projective Line Bundles). Let $T \to \mathbb{CP}^n$ be the tautological bundle. Since each fiber is canonically identified with a line in $\mathbb{C}^{n+1}$, it inherits a Hermitian metric $\langle \cdot, \cdot \rangle$ from the standard metric on $\mathbb{C}^{n+1}$. On each open subset $U_\alpha = \{[w] \in \mathbb{CP}^n : w^\alpha \neq 0\}$, we have a nonvanishing section s_α of T given by (3.6), and its norm with respect to this metric satisfies

$$|s_\alpha([w])|^2 = \frac{|w|^2}{|w^\alpha|^2}.$$

On U_α, the Chern connection for this metric is given by the connection form

$$\theta_\alpha = \partial \log |s_\alpha|^2 = \partial \log \frac{|w|^2}{|w^\alpha|^2},$$

and its curvature form is

$$\Theta_T\big|_{U_\alpha} = \bar\partial\partial \log |s_\alpha|^2 = \bar\partial\partial \log \frac{|w|^2}{|w^\alpha|^2}.$$

For the hyperplane bundle $H = T^*$ with the dual fiber metric, Proposition 7.20 shows that

$$\Theta_H\big|_{U_\alpha} = -\Theta_T\big|_{U_\alpha} = \bar\partial\partial \log \frac{|w^\alpha|^2}{|w|^2},$$

and more generally for a tensor power H^m with $m \in \mathbb{Z}$,

$$\Theta_{H^m}\big|_{U_\alpha} = m\bar\partial\partial \log \frac{|w^\alpha|^2}{|w|^2}.$$

Let us compute the curvature form Θ_H explicitly in terms of affine coordinates $(z^1, \dots, z^n) \leftrightarrow [z^1, \dots, 1, \dots, z^n]$ on U_α:

$$\begin{aligned} \Theta_H\big|_{U_\alpha} &= \bar\partial\partial \log \frac{1}{1+|z|^2} = \partial\bar\partial \log(1+|z|^2) \\ &= \frac{\sum_j dz^j \wedge d\bar z^j}{1+|z|^2} - \frac{\sum_{j,k} \bar z^j z^k \, dz^j \wedge d\bar z^k}{(1+|z|^2)^2}. \end{aligned} \tag{7.22}$$

It is notable that the formula is exactly the same in all the affine charts. //

Example 7.27 (Degrees of Line Bundles on $\mathbb{CP}^1$). The holomorphic sections of the hyperplane bundle $H \to \mathbb{CP}^1$ correspond to complex-linear functionals on $\mathbb{C}^2$, and each such section vanishes simply on a projective hyperplane, which in this case is a single point. Thus it follows from Theorem 7.22 that $\deg(H) = 1$. Since the degree map is a homomorphism from the Picard group to $\mathbb{Z}$, for any tensor power H^d we have $\deg(H^d) = d$.

We can also verify this using the Chern form. On $\mathbb{CP}^1$, formula (7.22) reduces to

$$\Theta_H\big|_{U_\alpha} = \frac{(1+|z|^2)dz \wedge d\bar z - |z|^2 \, dz \wedge d\bar z}{(1+|z|^2)^2} = \frac{dz \wedge d\bar z}{(1+|z|^2)^2}.$$

By Corollary 7.15, the degree of H is equal to the integral $\frac{i}{2\pi}\int_{\mathbb{CP}^1} \Theta_H$. This can be computed explicitly by integrating over the subset U_0 (which is biholomorhpic to $\mathbb{C}$), since $\mathbb{CP}^1 \smallsetminus U_0$ has measure zero. Making the change of variables $z = re^{i\theta}$, $dz \wedge d\bar z = -2ir \, dr \wedge d\theta$, we obtain

$$\begin{aligned} \deg(H) &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{dz \wedge d\bar z}{(1+|z|^2)^2} \\ &= \frac{i}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{-2ir}{(1+r^2)^2} \, dr \, d\theta. \end{aligned}$$

Then a further change of variables $s = 1 + r^2$ yields

$$\deg(H) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{1}{s^2} \, ds \, d\theta = 1.$$ //

The first real Chern class gives us another useful notion of positivity for line bundles. A $(1,1)$-form ω on a complex manifold M is said to be ***positive*** if it is real and $\omega(v, Jv) > 0$ for every nonzero real tangent vector v; a ***negative*** $(\mathbf{1},\mathbf{1})$***-form*** is defined analogously. (Note that this is a special definition for forms on complex manifolds; it does not make sense to ask that a differential form be literally positive in the sense of taking only positive values, because changing the sign of one of its arguments will always convert a positive value to a negative one. See Problem 7-9 for a generalization to (p,p)-forms.)

The next lemma gives some alternative characterizations of positive forms.

Lemma 7.28. *Suppose ω is a real $(1,1)$-form on a complex manifold M. The following are equivalent:*

(a) *ω is positive.*

(b) *$-i\omega(v,\bar{v}) > 0$ for every nonzero $v \in T'M$.*

(c) *ω restricts to a positively oriented volume form on every 1-dimensional complex submanifold of M.*

Proof. Problem 7-8. □

A complex line bundle L on a complex manifold is said to be ***positive*** if its first real Chern class is represented by a positive $(1,1)$-form. It is ***negative*** if $c_1^{\mathbb{R}}(L)$ has a negative representative.

The next proposition shows that for Riemann surfaces, this new notion of positivity coincides with positivity of the degree.

Proposition 7.29. *Let M be a connected, compact Riemann surface and $L \to M$ a holomorphic line bundle. Then L is positive if and only if* $\deg L > 0$.

Proof. First suppose L is positive. Then there is a positive 2-form ω representing $c_1(L)$, and Corollary 7.15 shows that that $\deg L = \int_M \omega$. Since Lemma 7.28 shows that ω is a positively oriented volume form for M, it follows that $\deg L = \int_M \omega > 0$.

Conversely, suppose $\deg L > 0$, and let dV_g be the Riemannian volume form for some Riemannian metric on M. Since there are no nontrivial $(2,0)$-forms or $(0,2)$-forms on a complex 1-manifold, dV_g is a form of type $(1,1)$. Because $H^2(M;\mathbb{R}) \cong \operatorname{Hom}(H_2(M),\mathbb{R})$ is 1-dimensional, it is spanned by the cohomology class of dV_g. Thus the class $c_1^{\mathbb{R}}(L)$ is represented by $a\,dV_g$ for some real constant a. The hypothesis implies $0 < \deg L = \int_M a\,dV_g = a\operatorname{Vol}(M)$. Thus $a > 0$, which means $a\,dV_g$ is a positive $(1,1)$-form representing $c_1^{\mathbb{R}}(L)$. □

We end the chapter with an important result about algebraic curves in $\mathbb{CP}^2$.

Proposition 7.30 (Degree of the Hyperplane Bundle on a Complex Curve). *Suppose $M \subseteq \mathbb{CP}^2$ is a nonsingular algebraic variety defined by a homogeneous polynomial $P\colon \mathbb{C}^3 \to \mathbb{C}$ of degree d. Let $L \to M$ be the restriction of the hyperplane bundle $H \to \mathbb{CP}^2$. Then* $\deg L = d$.

Proof. We will show that $\deg(L) = d$ by showing that L has a holomorphic section with exactly d simple zeros.

First we dispose of the easy case. If $d = 1$, then M is a projective line. After a projective transformation, we may assume M is the standard embedding of $\mathbb{CP}^1$ in $\mathbb{CP}^2$, and the restriction of H to M is isomorphic to the hyperplane bundle of $\mathbb{CP}^1$ (as can be verified by examining transition functions). Thus $\deg(L) = 1$ by the result of Example 7.27.

For the case $d > 1$, we start by finding a projective line $\Pi \subseteq \mathbb{CP}^2$ that intersects M transversely. Let $(\mathbb{CP}^2)^*$ denote the dual projective space (Example 2.32), and consider the map $\Phi\colon M \to (\mathbb{CP}^2)^*$ that sends a point $p \in M$ to the projective tangent space to M at p (see Prop. 2.33). This map is smooth (as can be verified by writing it in any affine coordinates), so its image has measure zero by an easy application of Sard's theorem [**LeeSM**, Cor. 6.11]. Thus there is some projective line $\Pi \in (\mathbb{CP}^2)^*$ not in the image of Φ, which is to say that $T'_p\Pi \neq T'_pM$ at each point $p \in \Pi \cap M$. Since both holomorphic tangent spaces are 1-dimensional, their intersection is trivial, and then by conjugation it follows that $T''_p\Pi \cap T''_pM = \{0\}$ for each such p, and thus $T_p\Pi \cap T_pM = \{0\}$. By applying a projective transformation, we can arrange that Π is the projective line $\left\{[w^0, w^1, w^2] : w^2 = 0\right\}$.

Since the polynomial P has degree $d > 1$, there is some point $b = [b^0, b^1, 0]$ that lies on Π but not on M. The projective transformation $A[w^0, w^1, w^2] = [b^1w^0 + b^0w^1, b^1w^1 - b^0w^0, w^2]$ takes $[0, 1, 0]$ to b and leaves Π invariant, so after replacing M by $A^{-1}(M)$, we can arrange that $[0, 1, 0] \notin M$.

By formula (3.15), the linear function $w^2\colon \mathbb{C}^3 \to \mathbb{C}$ defines a holomorphic section φ_{w^2} of H that vanishes exactly on Π. Let σ be the section of $L = H|_M$ obtained by restricting φ_{w^2}. Its zeros are exactly the points where M intersects Π. Every such point can be written in the form $[w^0, w^1, 0]$, where $P(w^0, w^1, 0) = 0$. Since $[0, 1, 0]$ is not one of those points, we can write each such point uniquely in the form $[1, \zeta, 0]$, for ζ a solution to the polynomial equation $q(\zeta) = 0$, where $q(\zeta) = P(1, \zeta, 0)$. Thus all such points lie in the affine coordinate domain $U_0 = \{[w^0, w^1, w^2] : w^0 \neq 0\}$.

We need to check several things: that q is actually a polynomial of degree d; that each zero of q has multiplicity 1; and correspondingly that each zero of σ has multiplicity 1. Once we have verified these facts, it will follow from the fundamental theorem of algebra that q has exactly d zeros, all of multiplicity 1; and therefore σ has d zeros of multiplicity 1. Then Theorem 7.22 implies that $\deg(L) = d$.

To see that q has degree d, note first that it has degree at most d because it is obtained from P by setting some variables equal to constants. Our normalizations have ensured that $P(0, 1, 0) \neq 0$. Since $P(0, 1, 0)$ is equal to the coefficient of $(w^1)^d$ in the expression for P, it follows that this coefficient is nonzero, so $q(\zeta)$ has a nonzero ζ^d term.

To see that the zeros of q all have multiplicity 1, assume for contradiction that $a \in \mathbb{C}$ is a zero of multiplicity greater than 1. This implies $0 = q'(a) = (\partial P/\partial w^1)(1, a, 0)$. In affine coordinates $(z^1, z^2) \leftrightarrow [1, z^1, z^2]$ on U_0, $M \cap U_0$ is defined by the equation $P(1, z^1, z^2) = 0$, and $\Pi \cap U_0$ is defined by $z^2 = 0$. Since $(\partial P/\partial w^1)(1, a, 0) = 0$, it follows that both $T'_{(a,0)}M$ and $T'_{(a,0)}\Pi$ are spanned in these coordinates by $\partial/\partial z^1$, which contradicts the fact that M and Π intersect transversely.

Finally, to see that the zeros of σ are simple, note that the restriction of H to U_0 has a global nonvanishing section $s = \varphi_{w^0}|_{U_0}$, and in terms of that section $\sigma|_{M\cap U_0}$ can be written $\sigma(z^1, z^2) = z^2 s$. At a point $(a,0) \in M \cap \Pi \cap U_0$, the argument above shows that $(\partial P/\partial w^1)(1,a,0) = q'(a) \neq 0$, so by the holomorphic implicit function theorem we can solve the equation $P(1, z^1, z^2) = 0$ for z^1 in some neighborhood W of $(a,0)$, and write $M \cap W = \{(z^1, z^2) : z^1 = f(z^2)\}$ for some holomorphic function f that satisfies $P(1, f(z^2), z^2) = 0$ and $f(0) = a$. This gives a local parametrization of M in the form $(z^1, z^2) = \chi(\zeta) = (f(\zeta), \zeta)$, with $\chi(0) = (a, 0)$. Pulling back σ by this parametrization, we find $\chi^*\sigma(\zeta) = \zeta\chi^* s$, which has a zero of multiplicity 1 at $\zeta = 0$. This completes the proof that $\deg(L) = d$. □

Problems

7-1. Prove the converse of Proposition 7.1: if ∇ is a connection on a Hermitian vector bundle whose matrix of connection 1-forms is skew-Hermitian with respect to every local orthonormal frame, then ∇ is a metric connection.

7-2. Suppose $E \to M$ is a holomorphic vector bundle with a Hermitian fiber metric h, and $E' \subseteq E$ is a holomorphic subbundle with the fiber metric h' given by restricting h. For any section σ of E, let $\sigma^\top$ and $\sigma^\perp$ be the orthogonal projections of σ onto E' and $(E')^\perp$, respectively. Denoting the Chern connections on E and E' by ∇ and ∇', respectively, show that $\nabla'_X\sigma = (\nabla_X\sigma)^\top$ for every smooth vector field X on M and smooth section σ of E'.

7-3. Let $E \to M$ be a smooth complex vector bundle endowed with a Hermitian fiber metric h.

(a) Show that h determines a smooth conjugate-linear bundle isomorphism $\widehat{h}\colon E \to E^*$ by

$$\widehat{h}(\sigma)(\tau) = \langle \tau, \sigma\rangle_h.$$

(Note the reversal of order on the right-hand side.)

(b) Show that the formula

$$\langle \varphi, \psi\rangle_{h^*} = \langle \widehat{h}^{-1}(\psi), \widehat{h}^{-1}(\varphi)\rangle_h.$$

defines a Hermitian fiber metric h^* on the dual bundle E^*, called the ***dual metric***.

7-4. Let $E \to M$ be a smooth complex vector bundle, and let ∇ be a connection on E. Define a map $\nabla^*\colon \Gamma(T_{\mathbb{C}}M) \times \Gamma(E^*) \to \Gamma(E^*)$ by

$$(\nabla^*_X\varphi)(\sigma) = X(\varphi(\sigma)) - \varphi(\nabla_X\sigma)$$

for $X \in \Gamma(T_{\mathbb{C}}M)$, $\varphi \in \Gamma(E^*)$, and $\sigma \in \Gamma(E)$.

(a) Show that ∇^* is a connection on E^*, called the ***dual connection***.

(b) Now suppose E is endowed with a Hermitian fiber metric h, and let h^* be the dual metric on E^* (Problem 7-3). Show that if ∇ is a metric connection, then

$$\nabla^*_X\big(\widehat{h}(\sigma)\big) = \widehat{h}\big(\nabla_{\overline{X}}\sigma\big),$$

and conclude that ∇^* is also a metric connection.

(c) Show that if E is a holomorphic Hermitian vector bundle and ∇ is its Chern connection, then ∇^* is the Chern connection of E^*.

7-5. Let $E \to M$ be a smooth complex vector bundle, let ∇ be a connection on E, and let ∇^* be the dual connection on E^* (Problem 7-5). Suppose (s_j) is a holomorphic local frame for E, and θ^j_k and Θ^j_k are its connection and curvature forms, respectively. Let (ε_j) be the dual frame for E^* defined by $\varepsilon_j(s_k) = \delta_{jk}$, and let θ^{*k}_j and Θ^{*k}_j be the connection and curvature forms of ∇^*, satisfying

$$\nabla^*_X\varepsilon_j = \theta^{*k}_j(X)\varepsilon_k, \qquad \Theta^*(X,Y)\varepsilon_j = \Theta^{*k}_j(X,Y)\varepsilon_k.$$

(We write the dual frame with lower indices instead of upper ones so as to ensure that the indices of the connection and curvature forms for ∇^* function the same as those for ∇.) Show that

$$\theta^{*k}_j = -\theta^j_k, \qquad \Theta^{*k}_j = -\Theta^j_k.$$

7-6. Let $E, E' \to M$ be smooth complex vector bundles endowed with connections ∇ and ∇'.

(a) Show that there is a unique connection $\nabla^{\otimes}$ on $E \otimes E'$, called the ***tensor product connection***, that satisfies $\nabla^{\otimes}_X(\sigma\otimes\sigma') = \nabla_X\sigma\otimes\sigma' + \sigma\otimes\nabla'_X\sigma'$ for all $\sigma \in \Gamma(E)$ and $\sigma' \in \Gamma(E')$.

(b) Show that the curvature of $\nabla^{\otimes}$ satisfies

$$\Theta^{\otimes}(X,Y)(\sigma\otimes\tau) = \big(\Theta(X,Y)\sigma\big)\otimes\tau + \sigma\otimes\big(\Theta'(X,Y)\tau\big).$$

(c) Show that if E and E' are holomorphic Hermitian bundles and ∇ and ∇' are their Chern connections, then $\nabla^{\otimes}$ is the Chern connection on $E \otimes E'$.

7-7. Let ∇ be a connection on a smooth vector bundle $E \to M$ and Θ its curvature. Prove the ***differential Bianchi identity*** $D\Theta = 0$, where $D\colon \mathscr{E}^2(\operatorname{End} E) \to \mathscr{E}^3(\operatorname{End} E)$ is the exterior covariant derivative operator associated with the tensor product connection on $\operatorname{End} E \cong E \otimes E^*$.

7-8. Prove Lemma 7.28 (characterizations of positive $(1,1)$-forms).

7-9. Let M be a complex n-manifold. Define a ***positive*** (p,p)***-form*** on M to be a real (p,p)-form that restricts to a positively oriented volume form on every p-dimensional complex submanifold of M.

(a) Show that a real (p,p)-form ω is positive if and only if

$$(-i)^p \omega(v_1, \bar{v}_1, \ldots, v_p, \bar{v}_p) > 0$$

whenever $(v_1, \ldots, v_p)$ is a linearly independent p-tuple of elements of $T'_x M$ for $x \in M$.

(b) Show that if φ is a nonzero $(n,0)$-form, then $i^{n^2} \varphi \wedge \overline{\varphi}$ is positive.

7-10. Let M be a complex n-manifold, and let $\pi : E \to M$ be a smooth complex vector bundle. Given a connection ∇ on E, let D be the associated exterior covariant derivative operator, and for each $q > 0$ define an operator $D'' : \mathscr{E}^{0,q}(M;E) \to \mathscr{E}^{0,q+1}(M;E)$ by $D'' = \pi^{0,q+1} \circ D$. Show that if there is a connection on E that satisfies $D'' \circ D'' = 0$ on $\mathscr{E}^{0,0}(M;E)$, then E has a unique structure as a holomorphic vector bundle such that the holomorphic sections of E are exactly those in the kernel of D''. [Hint: Show that for any smooth local frame $(s_1, \ldots, s_m)$ for E over $U \subseteq M$, the $(0,1)$-forms θ^j_k on U defined by $D''(s_k) = \theta^j_k \otimes s_j$ satisfy $\bar{\partial}\theta^j_k + \theta^j_l \wedge \theta^l_k = 0$. Let (z^j) be local holomorphic coordinates for U and let $(z,b) = (z^1, \ldots, z^n, b^1, \ldots, b^m)$ be the (complex-valued) coordinates on the open set $\pi^{-1}(U) \subseteq E$ defined by the local frame (s_k), via the correspondence $(z,b) \leftrightarrow b^k s_k(z)$. Show that there is a unique integrable almost complex structure on the total space of E such that $\Lambda^{1,0}(E)$ is locally spanned by $\left\{\pi^* dz^j,\ db^l + b^k \pi^* \theta^l_k \ :\ j = 1, \ldots, n,\ l = 1, \ldots, m\right\}$, and apply the Newlander–Nirenberg theorem.]

10.1090/gsm/244/08

Chapter 8

Hermitian and Kähler Manifolds

In this chapter, we explore the interplay between holomorphic structures and metrics on the tangent bundle. We begin by discussing basic properties of Hermitian metrics on the tangent bundle and their relationships with Riemannian metrics. Then for the rest of the chapter we focus on the special case of Kähler metrics, which are Hermitian metrics that satisfy an additional condition ensuring a closer relationship between the Riemannian structure and the holomorphic structure. Complex manifolds that admit Kähler metrics are far and away the most important class of complex manifolds.

Hermitian Metrics on the Tangent Bundle

Let M be a complex manifold. Because the almost complex structure J turns the tangent bundle into a complex vector bundle $T_J M$, we can look for Hermitian fiber metrics on $T_J M$. Because J plays the role of multiplication by i on $T_J M$, a Hermitian fiber metric in this case is a map $h : \Gamma(T_J M) \times \Gamma(T_J M) \to C^\infty(M; \mathbb{C})$ that satisfies all of the following:

- h is bilinear over $C^\infty(M; \mathbb{R})$;
- $h(JX, Y) = ih(X, Y)$;
- $h(X, JY) = -ih(X, Y)$;
- $h(Y, X) = \overline{h(X, Y)}$;
- $h(X, X) > 0$ at points where $X \neq 0$.

A Hermitian fiber metric is not a Riemannian metric, because it cannot take on only real values; but the next lemma shows that its real part is.

Lemma 8.1. *Suppose M is a complex manifold and h is a Hermitian fiber metric on $T_J M$. Then $g = \operatorname{Re} h$ is a Riemannian metric on M.*

Proof. It follows directly from the definition that g is smooth, positive definite, and bilinear over $C^\infty(M;\mathbb{R})$. It remains only to show that it is symmetric. We compute

$$g(X,Y) = \tfrac{1}{2}\big(h(X,Y) + \overline{h(X,Y)}\big) = \tfrac{1}{2}\big(h(X,Y) + h(Y,X)\big),$$

which is unchanged when X and Y are swapped. □

It is natural to ask what can be said about the imaginary part. The next lemma answers that question.

Lemma 8.2. *Suppose M is a complex manifold and h is a Hermitian fiber metric on $T_J M$. Then $\omega = -\operatorname{Im} h$ is a 2-form of type $(1,1)$.*

Proof. As before, it is immediate that ω is smooth and bilinear over $C^\infty(M;\mathbb{R})$, so we need only show that it is antisymmetric and of type $(1,1)$. Antisymmetry is another simple computation:

$$\omega(X,Y) = -\frac{1}{2i}\big(h(X,Y) - \overline{h(X,Y)}\big) = -\frac{1}{2i}\big(h(X,Y) - h(Y,X)\big),$$

and this last expression changes sign when X and Y are swapped.

To see that ω is of type $(1,1)$, note that the properties of h guarantee that $h(JX,JY) = (i)(-i)h(X,Y) = h(X,Y)$ for all real vector fields X and Y. It follows that

$$\omega(JX,JY) = -\operatorname{Im} h(JX,JY) = -\operatorname{Im} h(X,Y) = \omega(X,Y),$$

so ω is of type $(1,1)$ by the result of Problem 4-1. □

The reasons for the choice of a negative sign in the definition of ω will become clear below (see Example 8.6 and Proposition 8.9).

Another natural question to ask is this: Given a Riemannian metric g on a complex manifold M, can we find a 2-form ω such that $h = g - i\omega$ is a Hermitian fiber metric on $T_J M$? The first thing to note is that if there is such a form ω, it is uniquely determined by g.

Lemma 8.3. *Suppose M is a complex manifold and h is a Hermitian fiber metric on $T_J M$. Let $g = \operatorname{Re} h$ and $\omega = -\operatorname{Im} h$. Then*

$$\omega(X,Y) = g(JX,Y) \quad \text{for all } X,Y \in \Gamma(TM). \tag{8.1}$$

Proof. Once again, a computation:

$$\begin{aligned} g(JX,Y) &= \frac{1}{2}\big(h(JX,Y) + \overline{h(JX,Y)}\big) \\ &= \frac{1}{2}\big(ih(X,Y) + \overline{ih(X,Y)}\big) \\ &= \frac{i}{2}\big(h(X,Y) - \overline{h(X,Y)}\big) \\ &= \omega(X,Y). \end{aligned}$$

□

So now the question becomes: Given a Riemannian metric g, if we define a 2-form ω by (8.1), when is $h = g - i\omega$ a Hermitian fiber metric on $T_J M$? The next proposition answers that question in several ways.

Proposition 8.4. *Let M be a complex manifold M and $J : TM \to TM$ its associated almost complex structure. Given a Riemannian metric g on M, define ω, h, and $g_{\mathbb{C}}$ by*

- $\omega(X,Y) = g(JX,Y)$,
- $h = g - i\omega$,
- *$g_{\mathbb{C}}$ is the extension of g by complex bilinearity to act on pairs of complex vector fields.*

The following are equivalent.

(a) *h is a Hermitian fiber metric on $T_J M$.*

(b) *J is an orthogonal map with respect to g.*

(c) *J is a skew-symmetric map with respect to g.*

(d) *$g_{\mathbb{C}}(X,Y) = 0$ if X and Y are both sections of $T'M$.*

(e) *ω is skew-symmetric.*

Proof. We will prove (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (a).

(a) ⇒ (b): Suppose h is a Hermitian fiber metric. Because h is sesquilinear, it satisfies $h(JX,JY) = (i)(-i)h(X,Y) = h(X,Y)$. Thus

$$\begin{aligned} g(JX,JY) &= \tfrac{1}{2}\big(h(JX,JY) + \overline{h(JX,JY)}\big) \\ &= \tfrac{1}{2}\big(h(X,Y) + \overline{h(X,Y)}\big) = g(X,Y). \end{aligned}$$

(b) ⇒ (c): If J is orthogonal, then $g(JX,Y) = g(J^2X,JY) = -g(X,JY)$.

(c) ⇒ (d): Assuming J is skew-symmetric, its complexification $J : T_{\mathbb{C}}M \to T_{\mathbb{C}}M$ is also skew-symmetric with respect to $g_{\mathbb{C}}$. Suppose $Z, W \in \Gamma(T'M)$. Then

$$\begin{aligned} ig_{\mathbb{C}}(Z,W) &= g_{\mathbb{C}}(iZ,W) = g_{\mathbb{C}}(JZ,W) \\ &= -g_{\mathbb{C}}(Z,JW) = -g_{\mathbb{C}}(Z,iW) = -ig_{\mathbb{C}}(Z,W), \end{aligned}$$

from which it follows that $g_{\mathbb{C}}(Z,W) = 0$.

(d) ⇒ (e); Suppose $g_{\mathbb{C}}$ vanishes when applied to pairs of sections of $T'M$. Because $g_{\mathbb{C}}$ is the complexification of a real tensor field, by conjugation it also vanishes when applied to pairs of sections of $T''M$. Given $X, Y \in \Gamma(TM)$, we can use the decomposition $T_{\mathbb{C}}M = T'M \oplus T''M$ to write $X = Z + \overline{Z}$ and $Y = W + \overline{W}$, with $Z, W \in \Gamma(T'M)$. Then

$$\begin{aligned}\omega(X,Y) = g(JX,Y) &= g_{\mathbb{C}}\big(J(Z+\overline{Z}), W+\overline{W}\big)\\ &= g_{\mathbb{C}}\big(iZ - i\overline{Z}, W+\overline{W}\big)\\ &= i\big(g_{\mathbb{C}}(Z,W) + g_{\mathbb{C}}(Z,\overline{W}) - g_{\mathbb{C}}(\overline{Z},W) - g_{\mathbb{C}}(\overline{Z},\overline{W})\big)\\ &= i\big(g_{\mathbb{C}}(Z,\overline{W}) - g_{\mathbb{C}}(\overline{Z},W)\big).\end{aligned}$$

Interchanging X and Y has the effect of interchanging Z and W, and this last expression manifestly changes sign under that interchange.

(e) ⇒ (a): Now suppose ω is skew-symmetric. Then $h = g - i\omega$ is certainly smooth and bilinear over $C^\infty(M,\mathbb{R})$, and it is positive definite because the skew symmetry of ω implies

$$h(X,X) = g(X,X) - i\omega(X,X) = g(X,X) > 0, \quad \text{where } X \neq 0.$$

To see that h is conjugate symmetric, compute

$$\begin{aligned}h(X,Y) - \overline{h(Y,X)} &= g(X,Y) - i\omega(X,Y) - \overline{\big(g(Y,X) - i\omega(Y,X)\big)}\\ &= g(X,Y) - i\omega(X,Y) - g(Y,X) - i\omega(Y,X)\\ &= 0.\end{aligned}$$

And for complex linearity in the first variable, we have

$$\begin{aligned}h(JX,Y) = g(JX,Y) - i\omega(JX,Y) &= \omega(X,Y) - ig(J^2X,Y)\\ &= \omega(X,Y) + ig(X,Y) = ih(X,Y).\end{aligned}$$

Finally, conjugate linearity in the second variable follows from the two previous computations:

$$h(X,JY) = \overline{h(JY,X)} = \overline{ih(Y,X)} = -ih(X,Y). \qquad \square$$

A ***Hermitian metric*** on a complex manifold M is a Riemannian metric for which J is orthogonal, and a complex manifold endowed with a Hermitian structure is called a ***Hermitian manifold***. Given a Hermitian manifold (M,g), the 2-form $\omega = g(J\cdot,\cdot)$ is called the ***fundamental 2-form*** of the Hermitian metric. (Although this terminology is quite common, it is important to remember that the metric g itself is not a Hermitian fiber metric on T_JM; that role is played by $h = g - i\omega$.)

Lemma 8.5. *Every complex manifold has a Hermitian metric.*

Proof. Given a complex manifold M, let g_0 be an arbitrary Riemannian metric on M, and define another Riemannian metric g by

$$g(X,Y) = g_0(X,Y) + g_0(JX,JY).$$

It follows easily that $g(JX,JY) = g(X,Y)$. □

Example 8.6 (The Standard Metric on $\mathbb{C}^n$). On $\mathbb{C}^n$ with standard holomorphic coordinates $z^j = x^j + iy^j$, we have the Euclidean metric

$$g_{\mathrm{E}} = \sum_{j=1}^{n} (dx^j)^2 + (dy^j)^2.$$

Because

$$J\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j} \quad \text{and} \quad J\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j},$$

J takes an orthonormal frame to an orthonormal frame, so it is orthogonal. Thus g_{E} is Hermitian. Its fundamental 2-form ω_{E} satisfies

$$\begin{aligned}
\omega_{\mathrm{E}}\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) &= g_{\mathrm{E}}\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}\Big) = 0,\\
\omega_{\mathrm{E}}\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\Big) &= g_{\mathrm{E}}\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\Big) = \delta_{jk},\\
\omega_{\mathrm{E}}\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}\Big) &= -g_{\mathrm{E}}\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) = -\delta_{jk},\\
\omega_{\mathrm{E}}\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\Big) &= -g_{\mathrm{E}}\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\Big) = 0.
\end{aligned}$$

It follows that ω_{E} has the coordinate expression

$$\omega_{\mathrm{E}} = \sum_{j=1}^{n} dx^j \wedge dy^j. \tag{8.2}$$

This 2-form is called the ***standard symplectic form on*** $\mathbb{R}^{2n}$ (see [**LeeSM**, Example 22.9(a)]). This formula is one reason for the choice of the sign in our definition of the fundamental 2-form. //

In the discussion so far, we have assumed that we started with a Riemannian metric and looked for conditions under which it determines a Hermitian fiber metric on $T_J M$. We can also start with a 2-form ω and ask the analogous question: Is there a Hermitian metric g for which ω is the fundamental 2-form? The next lemma shows that such a g, if it exists, is determined by ω.

Lemma 8.7. *Suppose g is a Hermitian metric on a complex manifold M and ω is its fundamental 2-form. Then*

$$g(X,Y) = \omega(X,JY) \quad \textit{for all } X,Y \in \Gamma(TM).$$

► **Exercise 8.8.** Prove this lemma.

Thus another appropriate question is: Given a 2-form ω, if we define $g(X,Y) = \omega(X,JY)$, when is g a Hermitian metric? Lemma 8.2 shows that one necessary condition is that ω must be of type $(1,1)$. Another obvious necessary condition is that ω must be a positive form, because g must be positive definite. The next proposition shows that these two conditions are also sufficient.

Proposition 8.9. *Suppose M is a complex manifold and ω is a smooth 2-form on M. Define a 2-tensor g by $g(X,Y) = \omega(X,JY)$. Then g is a Hermitian metric with ω as its fundamental 2-form if and only if ω is a positive $(1,1)$-form.*

Proof. If g is a Hermitian metric with ω as its fundamental 2-form, then ω is a positive $(1,1)$-form as noted above.

Conversely, suppose ω is a smooth positive $(1,1)$-form. Problem 4-1 shows that $\omega(JX,JY) = \omega(X,Y)$ for all vector fields X,Y. Let $g(X,Y) = \omega(X,JY)$, and note that g is smooth and bilinear over $C^\infty(M)$, and positivity of ω ensures that g is positive definite. To ensure that g is a Riemannian metric, we need only check that it is symmetric. Using the observation above, we find

$$g(X,Y) = \omega(X,JY) = \omega(JX,J^2Y) = -\omega(JX,Y) = \omega(Y,JX) = g(Y,X).$$

To finish the proof, we use the fact that $g(JX,Y) = \omega(JX,JY) = \omega(X,Y)$, so ω is the fundamental 2-form associated with g; and since ω is skew-symmetric, Proposition 8.4 shows that g is Hermitian. □

Since positivity of a $(1,1)$-form is a natural condition because of its relationship with orientations of 1-dimensional complex submanifolds (Lemma 7.28), the preceding lemma gives another justification for the choice of the minus sign in the definition of the fundamental 2-form.

There is one more natural question we might ask: By virtue of the natural isomorphism $T_JM \cong T'M$ (Prop. 1.58), the bundle T_JM inherits the structure of a holomorphic vector bundle. Thus a Hermitian fiber metric $h = g - i\omega$ determines a Chern connection on TM. One might wonder whether the Chern connection of h coincides with the Levi-Civita connection of g. The answer is "not always," and we will explore the conditions under which they do coincide in the next section.

Kähler Metrics

A Hermitian metric on a complex manifold is called a ***Kähler metric*** if its fundamental 2-form ω is closed. A complex manifold equipped with a Kähler metric is called a ***Kähler manifold***, This seemingly innocuous condition turns out to have deep and wide-ranging consequences.

Because every Hermitian metric is completely determined by its fundamental 2-form, it is frequently easiest to define a Kähler metric that way. A ***Kähler form*** on a complex manifold is a smooth, closed, positive $(1,1)$-form. By virtue of Proposition

8.9, every Kähler form ω determines a Kähler metric via $g = \omega(\cdot, J\cdot)$. A Kähler form is, in particular, a ***symplectic form***, which is a smooth closed real 2-form that is ***nondegenerate*** in the sense that the map from TM to T^*M given by $v \mapsto v \lrcorner\, \omega$ is injective (see [**LeeSM**, Chap. 22]).

Because a Kähler form is closed and real, it determines a real cohomology class $[\omega] \in H^2_{\text{dR}}(M; \mathbb{R})$, called the ***Kähler class*** of the given metric.

In order to perform computations with Kähler metrics efficiently, we introduce some additional index conventions. Suppose M is an n-dimensional complex manifold. Given any local holomorphic coordinates $(z^1, \dots, z^n)$ for M, we will generally use the local frame $\left(\partial/\partial z^1, \dots, \partial/\partial z^n, \partial/\partial \bar{z}^1, \dots, \partial/\partial \bar{z}^n\right)$ for the complexified tangent bundle, numbered from 1 to $2n$. For each index $j \in \{1, \dots, n\}$, we interpret $\bar{j}$ to mean $j + n$, and $z^{\bar{j}}$ to be synonymous with $\bar{z}^j$. We use the abbreviations

$$\partial_j = \frac{\partial}{\partial z^j}, \qquad \partial_{\bar{j}} = \frac{\partial}{\partial \bar{z}^j},$$

with implied summations over such indices going from 1 to $n = \dim_{\mathbb{C}} M$. Thus, for example, the expression $V^j \partial_j + V^{\bar{j}} \partial_{\bar{j}}$ is shorthand for

$$\sum_{j=1}^n V^j \frac{\partial}{\partial z^j} + \sum_{j=1}^n V^{\bar{j}} \frac{\partial}{\partial \bar{z}^j}.$$

Now suppose g is a Hermitian metric. (For the time being, we are not assuming it is Kähler.) As in the previous section, we extend g to act on pairs of complex vector fields by complex bilinearity; but for simplicity, henceforth we will not differentiate notationally between g and its complex-bilinear extension, using the same symbol g for both.

In local holomorphic coordinates, we can write the complexification of g as

$$g = g_{jk}\, dz^j \otimes dz^k + g_{j\bar{k}}\, dz^j \otimes d\bar{z}^k + g_{\bar{j}k}\, d\bar{z}^j \otimes dz^k + g_{\bar{j}\bar{k}}\, d\bar{z}^j \otimes d\bar{z}^k,$$

where $g_{jk} = g(\partial_j, \partial_k)$, $g_{j\bar{k}} = g(\partial_j, \partial_{\bar{k}})$, etc. Because g is the complexification of a real symmetric tensor, we have

$$\overline{g_{jk}} = g_{\bar{j}\bar{k}} = g_{\bar{k}\bar{j}} \quad \text{and} \quad \overline{g_{j\bar{k}}} = g_{\bar{j}k} = g_{k\bar{j}}.$$

Moreover, because g vanishes when applied to pairs of vectors in $T'M$, we have $g_{jk} = 0$ and then also $g_{\bar{j}\bar{k}} = 0$ by conjugation. Thus we can write

$$\begin{aligned} g &= g_{j\bar{k}}\, dz^j \otimes d\bar{z}^k + g_{\bar{j}k}\, d\bar{z}^j \otimes dz^k \\ &= 2g_{j\bar{k}}\left(\frac{dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j}{2}\right) \qquad (8.3) \\ &= 2g_{j\bar{k}}\, dz^j\, d\bar{z}^k, \end{aligned}$$

where the juxtaposition $dz^j\, d\bar{z}^k$ represents the symmetric product. The Riemannian metric g is then the restriction of this tensor to pairs of real vector fields. For example, if X and Y are real vector fields, written locally as $X = X^j\partial_j + X^{\bar{j}}\partial_{\bar{j}}$ and $Y = Y^j\partial_j + Y^{\bar{j}}\partial_{\bar{j}}$, then

$$g(X,Y) = g_{j\bar{k}}\big(X^jY^{\bar{k}} + Y^jX^{\bar{k}}\big),$$

and $g(X,X) = 2g_{j\bar{k}}X^jX^{\bar{k}}$.

For many computations, we are going to need a Hermitian fiber metric on the whole complexified tangent bundle. For complex vector fields X, Y, we define

$$\langle X,Y\rangle = g(X,\overline{Y}), \tag{8.4}$$

and denote the associated norm by $|X| = \langle X,X\rangle^{1/2}$. It is easy to check that this is Hermitian, and it coincides with g when applied to real vector fields. Moreover, because $g(Z,W) = 0$ when Z and W are both sections of $T'M$ or $T''M$, this fiber metric makes $T'M$ orthogonal to $T''M$.

Of course, we have already defined a Hermitian fiber metric on T_JM by $h = g - i\omega$. One might hope that this new inner product matches h under the isomorphism $\xi\colon T_JM \to T'M$ given by $\xi(v) = v - iJv$. Unfortunately not: for real vector fields X and Y,

$$\begin{aligned}\langle\xi(X),\xi(Y)\rangle &= \langle X - iJX, Y - iJY\rangle = g(X - iJX, Y + iJY)\\ &= g(X,Y) - ig(JX,Y) + ig(X,JY) + g(JX,JY)\\ &= 2g(X,Y) - 2i\omega(X,Y) = 2h(X,Y).\end{aligned}$$

We could have avoided this discrepancy by adding a factor of $1/\sqrt{2}$ to the definition of ξ; but since we will be computing norms using exclusively the inner product $\langle\cdot,\cdot\rangle$, it is not worth the additional complication this would have brought to our formulas.

In holomorphic coordinates, for sections Z, W of $T'M$, we have

$$\langle Z,W\rangle = g\big(Z^j\partial_j,\ W^{\bar{k}}\partial_{\bar{k}}\big) = g_{j\bar{k}}Z^jW^{\bar{k}},$$

where we have written $W^{\bar{k}} = \overline{W^k}$. On the other hand, if X and Y are real, decomposed as $X = Z + \overline{Z}$ and $Y = W + \overline{W}$ with $Z, W \in \Gamma(T'M)$, then

$$\langle X,Y\rangle = g\big(Z + \overline{Z}, \overline{W} + W\big) = g(Z,\overline{W}) + g(W,\overline{Z}) = g_{j\bar{k}}(Z^jW^{\bar{k}} + W^jZ^{\bar{k}}\big).$$

In particular, for $X = Z + \overline{Z}$, this yields

$$|X|^2 = |Z + \overline{Z}|^2 = |Z|^2 + |\overline{Z}|^2 = 2|Z|^2 = 2g_{j\bar{k}}Z^jZ^{\bar{k}}. \tag{8.5}$$

We use the complexified metric g (not the Hermitian fiber metric $\langle\cdot,\cdot\rangle$) to define the ***musical isomorphisms*** $\flat : T_{\mathbb{C}}M \to T^*_{\mathbb{C}}M$ and $\sharp : T^*_{\mathbb{C}}M \to T_{\mathbb{C}}M$, called the ***flat*** and ***sharp*** operators: for a complex vector field X, the 1-form $X^\flat$ is defined by

$$X^\flat(Y) = g(X,Y) \quad \text{for all complex vector fields } Y;$$

and $\sharp$ is the inverse of $\flat$. Thus both $\sharp$ and $\flat$ are smooth (but not holomorphic) complex-linear bundle isomorphisms. For example, for $Z \in \Gamma(T'M)$ and Y any complex vector field, we have

$$Z^\flat(Y) = g(Z,Y) = g\big(Z^j\partial_j,\ Y^k\partial_k + Y^{\bar k}\partial_{\bar k}\big) = g_{j\bar k}Z^jY^{\bar k}.$$

Thus we can write the coordinate expression of $Z^\flat$ as $Z_{\bar k}d\bar z^k$, where $Z_{\bar k} = g_{j\bar k}Z^j$. Note that the flat operator maps $T'M$ to $\Lambda^{0,1}M$ and $T''M$ to $\Lambda^{1,0}M$. More generally, we use the matrix $g_{j\bar k}$ and its inverse $g^{j\bar k}$ to raise and lower indices on complex tensors of any type.

Next let us look at the coordinate expression for the fundamental 2-form ω. Since ω is a $(1,1)$-form, its only nonzero terms in holomorphic coordinates are those involving $dz^j \wedge d\bar z^k$ or $d\bar z^j \wedge dz^k$, and by antisymmetry we can combine those together after suitably renaming the indices. Thus we can write $\omega = \omega_{j\bar k}dz^j \wedge d\bar z^k$ for some coefficient functions $\omega_{j\bar k}$. To determine the coefficients, we compute

$$\omega_{j\bar k} = \omega(\partial_j, \partial_{\bar k}) = g(J\partial_j, \partial_{\bar k}) = ig(\partial_j, \partial_{\bar k}) = ig_{j\bar k}.$$

Thus

$$\omega = ig_{j\bar k}\, dz^j \wedge d\bar z^k. \tag{8.6}$$

(The reason there is a factor of 2 in formula (8.3) for g but not in this formula for ω is because of our convention regarding wedge products—we use the convention labeled the ***determinant convention*** in [**LeeSM**, p. 358], which for 1-forms ω and η yields $\omega\wedge\eta = \omega\otimes\eta - \eta\otimes\omega$, while the symmetric product is $\omega\eta = \frac{1}{2}(\omega\otimes\eta + \eta\otimes\omega)$. Using the other common convention for the wedge product, the ***Alt convention***, would result in a factor of 2 in the formula for ω as well.)

In particular, for the standard metric on $\mathbb{C}^n$, the fundamental 2-form is given by $\omega = \sum_j dx^j \wedge dy^j$. Converting this to holomorphic coordinates using $dx^j = (dz^j + d\bar z^j)/2$ and $dy^j = (dz^j - d\bar z^j)/(2i)$, we obtain

$$\omega = \frac{i}{2}\sum_j dz^j \wedge d\bar z^j, \qquad g = \sum_j dz^j\, d\bar z^j, \tag{8.7}$$

so the coefficients of the standard metric are $g_{j\bar k} = \frac{1}{2}\delta_{jk}$.

The next theorem illustrates many of the reasons Kähler metrics are special, and at the same time gives many alternative ways to characterize Kähler metrics among the Hermitian ones. Before stating it, we note that because $T'M$ is a holomorphic vector bundle, the isomorphism $\xi : T_JM \to T'M$ allows us to endow T_JM with a holomorphic bundle structure as well.

Theorem 8.10 (Characterizations of Kähler Metrics). *Let M be a complex manifold and g be a Hermitian metric on M. We use the following notations:*

- *ω is the fundamental 2-form of g.*
- *$h = g - i\omega$.*
- *∇ is the Levi-Civita connection of g.*
- *$\nabla^{(h)}$ is the Chern connection of h on $T_J M$.*
- *$\xi : T_J M \to T'M$ is the isomorphism $\xi(v) = v - iJv$.*

The following statements are equivalent:

(a) *g is Kähler (i.e., $d\omega = 0$).*

(b) *For every $p \in M$, there exists a real-valued function u on a neighborhood V of p such that $\omega|_V = i\partial\bar{\partial}u$.*

(c) *In every holomorphic coordinate chart, $\partial_j g_{l\bar{k}} = \partial_l g_{j\bar{k}}$ for all j, k, l.*

(d) *For every $a \in M$, there exists a holomorphic coordinate chart centered at a such that*

$$g_{j\bar{k}}(a) = \tfrac{1}{2}\delta_{jk} \text{ and } d(g_{j\bar{k}})|_a = 0. \tag{8.8}$$

(e) *In every holomorphic coordinate chart, all Christoffel symbols of g are identically zero except those of types Γ^l_{jk} and $\Gamma^{\bar{l}}_{\bar{j}\bar{k}}$.*

(f) *∇ restricts to the Chern connection on $T'M$ with respect to the Hermitian fiber metric $\langle \cdot, \cdot \rangle$.*

(g) *For every real or complex vector field X, ∇_X maps $\Gamma(T'M)$ to itself.*

(h) $\nabla J \equiv 0$.

(i) $\nabla \omega \equiv 0$.

(j) $\nabla \xi \equiv 0$.

(k) *∇ is compatible with the holomorphic structure on $T_J M$.*

(l) $\nabla^{(h)} = \nabla$.

(m) *$\nabla^{(h)}$ is torsion-free.*

Proof. Here is the plan:

$$\begin{array}{ccccccc}
(a) & \Longrightarrow & (b) & \Longrightarrow & (c) & & \\
\Uparrow & \nwarrow & & \swarrow & \Downarrow & & \\
(i) & & (d) & & (e) & & \\
\Uparrow & & & & \Downarrow & & \\
(h) & \Longleftarrow & (g) & \Longleftarrow & (f) & & \\
\Updownarrow & \nwarrow & & & & & \\
(j) & \Longrightarrow & (k) & \Longleftrightarrow & (l) & \Longleftrightarrow & (m).
\end{array}$$

(a) ⇒ (b): This is the local $\partial\bar\partial$-lemma (Corollary 4.15).

(b) ⇒ (c): Suppose $\omega = i\partial\bar\partial u$ on some open subset $V \subseteq M$. In holomorphic coordinates, $i\partial\bar\partial u = i\partial_j\partial_{\bar k}u\, dz^j \wedge d\bar z^k$. Comparing this to (8.6), we find that $g_{j\bar k} = \partial_j\partial_{\bar k}u$, and therefore

$$\partial_l g_{j\bar k} = \partial_l\partial_j\partial_{\bar k}u = \partial_j\partial_l\partial_{\bar k}u = \partial_j g_{l\bar k},$$

because the coordinate vector fields ∂_j and ∂_l commute.

(c) ⇒ (d): Given $a \in M$, let (z^j) be any holomorphic coordinates centered at a. Using these coordinates, we may as well consider g as a Hermitian metric on an open subset of $\mathbb{C}^n$ with $a = 0$. Using the Gram-Schmidt algorithm, we can find a preliminary complex-linear change of coordinates that achieves $g_{j\bar k}(0) = \frac{1}{2}\delta_{jk}$. To eliminate the first derivatives of g at 0, we will make a quadratic change of coordinates by setting $z = f(w)$, where

$$f^j(w) = w^j + A^j_{ml}w^m w^l,$$

for some constants A^j_{ml} to be determined, but assumed to be symmetric in m and l. Note that this implies

$$\frac{\partial f^j}{\partial w^m}(0) = \delta^j_m \quad \text{and} \quad \frac{\partial^2 f^j}{\partial w^m w^l}(0) = 2A^j_{ml}.$$

To compute the effect this has on g, we express g in w-coordinates as $g = \tilde g_{l\bar m} dw^l\, d\overline{w}^m$, where

$$\tilde g_{l\bar m} = \left(g_{j\bar k}\circ f\right)\frac{\partial f^j}{\partial w^l}\frac{\partial \overline{f}^k}{\partial \overline{w}^m}.$$

(Here $\overline{w}^m$ and $\overline{f}^k$ represents the conjugates of w^m and f^k, respectively.) Because the holomorphic Jacobian of f at the origin is the identity matrix, this change of coordinates maintains the condition $\tilde g_{l\bar m}(0) = \frac{1}{2}\delta_{lm}$.

By the chain rule (Prop. 1.47),

$$\frac{\partial \tilde g_{l\bar m}}{\partial w^q} = \left(\frac{\partial g_{j\bar k}}{\partial z^p}\circ f\right)\frac{\partial f^p}{\partial w^q}\frac{\partial f^j}{\partial w^l}\frac{\partial \overline{f}^k}{\partial \overline{w}^m} + \left(g_{j\bar k}\circ f\right)\frac{\partial^2 f^j}{\partial w^q \partial w^l}\frac{\partial \overline{f}^k}{\partial \overline{w}^m}.$$

(The other terms that would ordinarily result from the chain rule are zero because f is holomorphic.) When we evaluate this at 0, we find

$$\begin{aligned}\frac{\partial \tilde g_{l\bar m}}{\partial w^q}(0) &= \frac{\partial g_{j\bar k}}{\partial z^p}(0)\,\delta^p_q\,\delta^j_l\,\delta^{\bar k}_{\bar m} + g_{j\bar k}(0)2A^j_{ql}\delta^{\bar k}_{\bar m} \\ &= \frac{\partial g_{l\bar m}}{\partial z^q}(0) + 2g_{j\bar m}(0)A^j_{ql}.\end{aligned} \tag{8.9}$$

Let us choose

$$A^j_{ql} = -\frac{1}{2}g^{j\bar k}(0)\frac{\partial g_{l\bar k}}{\partial z^q}(0),$$

where $g^{j\bar k} = g^{\bar k j}$ is the matrix satisfying $g_{m\bar k}g^{\bar k j} = \delta^j_m$. Note that hypothesis (c) guarantees that A^j_{ql} is symmetric in q and l. It then follows easily from (8.9) (which implicitly used the symmetry of A^j_{ql}) that each first derivative $\partial\tilde{g}_{l\bar{m}}/\partial w^q$ vanishes at 0, and by conjugation so does $\partial\tilde{g}_{l\bar{m}}/\partial\overline{w}^q = \overline{\partial\tilde{g}_{m\bar{l}}/\partial w^q}$.

(d) $\Rightarrow$ (a): Since $d\omega$ is defined independently of coordinates, for each $p \in M$ we can choose a holomorphic coordinate chart in which $d(g_{j\bar{k}}) = 0$ at p, and then

$$d\omega_p = d\left(i g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k\right)\Big|_p = i\, dg_{j\bar{k}} \wedge dz^j \wedge d\bar{z}^k\Big|_p = 0.$$

(c) $\Rightarrow$ (e): Assuming (c), choose holomorphic coordinates (z^j), and write the Christoffel symbols of g as

$$\Gamma^a_{bc} = g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right),$$

where each of the indices a, b, c, d ranges through $\{1, \dots, n, \overline{1}, \dots, \bar{n}\}$. First note that the matrices g and g^{-1} have the block form

$$g = \begin{pmatrix} 0 & g_{\bar{j}k} \\ g_{j\bar{k}} & 0 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 0 & g^{k\bar{l}} \\ g^{\bar{k}l} & 0 \end{pmatrix},$$

where $(g^{\bar{k}l})$ is the inverse of the matrix $(g_{j\bar{k}})$ as above. We compute

$$\Gamma^{\bar{m}}_{jk} = \frac{1}{2} g^{l\bar{m}}\left(\partial_j g_{kl} + \partial_l g_{kj} - \partial_k g_{jl}\right) = 0,$$
$$\Gamma^{\bar{m}}_{j\bar{k}} = \frac{1}{2} g^{l\bar{m}}\left(\partial_j g_{\bar{k}l} + \partial_{\bar{k}} g_{jl} - \partial_l g_{j\bar{k}}\right) = 0,$$

where we have used the facts that $g_{jk} = 0$ since g annihilates pairs sections of $T'M$, and $\partial_l g_{j\bar{k}} = \partial_j g_{l\bar{k}} = \partial_j g_{\bar{k}l}$ since we are assuming hypothesis (c). It then follows by conjugation and symmetry that all other Christoffel symbols involving both barred and unbarred indices are also zero.

(e) $\Rightarrow$ (f): First we need to check that ∇ maps $\Gamma(T'M)$ to itself. This is just a computation in local holomorphic coordinates:

$$\nabla_{\partial_k}\partial_j = \Gamma^l_{kj}\partial_l + \cancel{\Gamma^{\bar{l}}_{kj}\partial_{\bar{l}}},$$
$$\nabla_{\partial_{\bar{k}}}\partial_j = \cancel{\Gamma^l_{\bar{k}j}\partial_l} + \cancel{\Gamma^{\bar{l}}_{\bar{k}j}\partial_{\bar{l}}}.$$

It is compatible with the Hermitian fiber metric $\langle\cdot,\cdot\rangle$ on $T'M$ because it is compatible with g. To check that it is compatible with the holomorphic structure on $T'M$, suppose $W = W^j\partial_j$ is a holomorphic section of $T'M$. Then for each k, we have

$$\nabla_{\partial_{\bar{k}}} W = \left(\partial_{\bar{k}} W^j\right)\partial_j + W^j \Gamma^l_{\bar{k}j}\partial_l = 0 + 0.$$

(f) $\Rightarrow$ (g): This is immediate.

(g) $\Rightarrow$ (h): Suppose ∇ maps $\Gamma(T'M)$ to itself. The covariant derivative of the endomorphism field J is defined by $(\nabla_X J)Y = \nabla_X(JY) - J(\nabla_X Y)$. If Y is a section of $T'M$, then so is $\nabla_X Y$, and the formula for $\nabla_X J$ (or rather its extension by complex linearity to complex vector fields) reduces to $\nabla_X(JY) - J(\nabla_X Y) = \nabla_X(iY) - i(\nabla_X Y) = 0$. Then by conjugation, the same formula holds whenever Y is a section of $T''M$. Since $T_{\mathbb{C}}M = T'M \oplus T''M$, this proves that $\nabla J \equiv 0$.

(h) ⇒ (i): The assumption $\nabla J \equiv 0$ implies that $\nabla_V(JW) = J\nabla_V W$ for all complex vector fields V and W. Using the fact that the Levi-Civita connection is compatible with g, we compute

$$\begin{aligned}(\nabla_V \omega)(X,Y) &= V\big(\omega(X,Y)\big) - \omega\big(\nabla_V X,\ Y\big) - \omega\big(X,\ \nabla_V Y\big)\\ &= V\big(g(JX,Y)\big) - g\big(J\nabla_V X,\ Y\big) - g\big(JX,\ \nabla_V Y\big)\\ &= g\big(\nabla_V JX,\ Y)\big) + g\big(JX,\ \nabla_V Y)\big)\\ &\qquad - g\big(J\nabla_V X,\ Y\big) - g\big(JX,\ \nabla_V Y\big)\\ &= 0.\end{aligned}$$

(i) ⇒ (a): Let $p \in M$ be arbitrary. As we did earlier in the proof, to prove that $d\omega_p = 0$, we may use any convenient coordinate chart. If we use Riemannian normal coordinates centered at p, then all of the Christoffel symbols vanish at p, and the equation $\nabla\omega = 0$ implies that all first partial derivatives of the components of ω vanish at p. It follows easily that $d\omega|_p = 0$.

(h) ⇔ (j): With X and Y arbitrary real vector fields, we compute

$$\begin{aligned}(\nabla_X \xi)Y &= \nabla_X(\xi Y) - \xi(\nabla_X Y) = \nabla_X(Y - iJY) - (\nabla_X Y - iJ\nabla_X Y)\\ &= -i\big(\nabla_X(JY) - J\nabla_X Y\big) = -i(\nabla_X J)Y.\end{aligned}$$

Thus $\nabla_X J = 0$ for all X if and only if $\nabla_X \xi = 0$ for all X.

(j) ⇒ (k): Note that the previous steps showed that (j) ⇒ (h) ⇒ ⋯ ⇒ (f), so assuming $\nabla\xi \equiv 0$ implies that ∇ is compatible with the holomorphic structure on $T'M$. The way we have defined the holomorphic structure on $T_J M$ is via ξ, so a section V of $T_J M$ is holomorphic if and only if $\xi(V)$ is a holomorphic section of $T'M$. Thus if V is a holomorphic section of $T_J M$ and $\overline{Z}$ is a smooth section of $T''M$, we have

$$\xi(\nabla_{\overline{Z}} V) = \nabla_{\overline{Z}}(\xi(V)) - (\nabla_{\overline{Z}}\xi)V = 0 - 0.$$

Since ξ is injective, this proves that $\nabla_{\overline{Z}} V = 0$.

(k) ⇒ (h): Suppose ∇ is compatible with the holomorphic structure on $T_J M$. Let $z^j = x^j + iy^j$ be holomorphic local coordinates on M, and define smooth sections $(X_1, \dots, X_{2n})$ of TM by

$$X_j = \begin{cases} \dfrac{\partial}{\partial x^j}, & j \le n,\\[2mm] \dfrac{\partial}{\partial y^j}, & j > n.\end{cases}$$

These form a commuting local frame for TM (as a real vector bundle). A computation shows that

$$\xi(X_j) = \begin{cases} 2\dfrac{\partial}{\partial z^j}, & j \le n, \\ 2i\dfrac{\partial}{\partial z^j}, & j > n, \end{cases}$$

so by the way we have defined the holomorphic structure on $T_J M$, these are all holomorphic sections of $T_J M$. Now for each j, note that $X_j + iJX_j$ is a section of $T''M$, so the assumption that ∇ is compatible with the holomorphic structure on $T_J M$ implies

$$0 = \nabla_{X_j + iJX_j} X_k = \nabla_{X_j} X_k + J\nabla_{JX_j} X_k = \nabla_{X_k} X_j + J\nabla_{X_k}(JX_j),$$

where in the second equality we used the fact that multiplication by i in $T_J M$ is accomplished by applying J; and in the third we used the fact that $\nabla_V W = \nabla_W V$ when V and W commute because ∇ is torsion-free. It follows that

$$(\nabla_{X_k} J)X_j = \nabla_{X_k}(JX_j) - J\nabla_{X_k} X_j = -J\big(\nabla_{X_k} X_j + J\nabla_{X_k}(JX_j)\big) = 0,$$

which shows that J is parallel with respect to ∇.

(k) $\Leftrightarrow$ (l): The previous steps showed that (k) $\Rightarrow$ (h) $\Rightarrow$ (i), so assumption (k) implies $\nabla\omega = 0$. Therefore, $\nabla h = \nabla(g - i\omega) = 0$, so ∇ is also compatible with the Hermitian fiber metric on $T_J M$. By uniqueness of the Chern connection, this implies $\nabla = \nabla^{(h)}$. Conversely, if $\nabla = \nabla^{(h)}$, then ∇ is compatible with the holomorphic structure on $T_J M$ by definition of the Chern connection.

(l) $\Leftrightarrow$ (m): If $\nabla^{(h)} = \nabla$, then $\nabla^{(h)}$ is torsion-free because ∇ is. Conversely, suppose $\nabla^{(h)}$ is torsion-free. By definition of the Chern connection, for any real vector field X, we have $0 = \nabla^{(h)}_X h = \nabla^{(h)}_X (g - i\omega)$. The real part of this equation is $\nabla^{(h)}_X g = 0$. Thus $\nabla^{(h)}$ is torsion-free and compatible with the metric g, so it is equal to the Levi-Civita connection of g. □

If g is a Kähler metric with Kähler form ω, a real-valued function u such that $\omega = i\partial\bar\partial u$ is called a ***Kähler potential for g***. Part (b) of the preceding theorem shows that every Kähler metric admits a Kähler potential in a neighborhood of each point; but as we will see in Theorem 8.18 below, on a compact Kähler manifold there is never a global Kähler potential because ω is never exact.

Examples of Kähler Metrics

There are plenty of examples of Kähler metrics.

Example 8.11 (The Standard Metric on $\mathbb{C}^n$). Let g_E be the standard Hermitian metric on $\mathbb{C}^n$ (see Example 8.6). Its fundamental 2-form is given by (8.2), which is closed. Thus g_E is Kähler. //

Example 8.12 (Complex Tori). Recall that a complex torus is a quotient space of the form $\mathbb{C}^n/\Lambda$, where $\Lambda \subseteq \mathbb{C}^n$ is a lattice (see Example 1.18). Because the action of Λ on $\mathbb{C}^n$ preserves the holomorphic structure, the Euclidean metric g_{E}, and the standard Kähler form $\omega_{\mathrm{E}} = g_{\mathrm{E}}(\cdot, J\cdot)$, it follows that g_{E} descends to a Kähler metric on $\mathbb{C}^n/\Lambda$. Thus every complex torus has a Kähler metric. //

Example 8.13 (Riemann Surfaces). Every Riemann surface (like every complex manifold) has Hermitian metrics. Because every 2-form on a real 2-manifold is closed for dimensional reasons, every Hermitian metric on a Riemann surface is Kähler. //

Example 8.14 (The Fubini–Study Metric on $\mathbb{CP}^n$). Recall the Hermitian fiber metric h we defined on the hyperplane bundle $H \to \mathbb{CP}^n$ (Example 7.26). The curvature of its Chern connection is the global 2-form Θ_H whose expression in each set U_α where $w^\alpha \neq 0$ is

$$\Theta_H|_{U_\alpha} = \bar\partial\partial \log \frac{|w^\alpha|^2}{|w|^2}. \tag{8.10}$$

In affine coordinates $(z^1, \dots, z^n) \leftrightarrow [z^1, \dots, 1, \dots, z^n]$ on U_α, Example 7.26 showed that

$$\Theta_H = \frac{\sum_j dz^j \wedge d\bar z^j}{1 + |z|^2} - \frac{\sum_{j,k} \bar z^j z^k \, dz^j \wedge d\bar z^k}{(1 + |z|^2)^2}.$$

Let ω_{FS} be the following closed real $(1,1)$-form:

$$\omega_{\mathrm{FS}} = \frac{i}{2}\Theta_H. \tag{8.11}$$

We will show that ω_{FS} is positive, and therefore is a Kähler form. The corresponding Kähler metric $g_{\mathrm{FS}} = \omega_{\mathrm{FS}}(\cdot, J\cdot)$ is called the ***Fubini–Study metric***, after the mathematicians who first described it in the early twentieth century, Guido Fubini and Eduard Study [**Fub04**, **Stu05**]. It follows from the computation above that in each affine coordinate chart, it is given by the formula $g_{\mathrm{FS}} = 2g_{j\bar k} dz^j d\bar z^k$, where

$$g_{j\bar k} = \frac{1}{2}\left(\frac{\delta_{jk}}{1 + |z|^2} - \frac{\bar z^j z^k}{(1 + |z|^2)^2} \right). \tag{8.12}$$

(Some authors use a different normalization. Our definition of g_{FS} is motivated primarily by the relationship with the standard metric on the sphere, described in Problem 8-2. Others define the Kähler form to be equal to the Chern form of H, which is $1/\pi$ times our ω_{FS}; the motivation for that choice is described in the proof of Theorem 10.13.)

To see that ω_{FS} is a positive $(1,1)$-form, or equivalently that g_{FS} is positive definite, let X be a nonvanishing smooth real vector field on an open subset of one of the affine charts U_α, written as $X = Z + \overline{Z}$ with Z a local section of $T'\mathbb{CP}^n$, and compute

$$\begin{aligned}
\omega_{\mathrm{FS}}(X, JX) &= \omega_{\mathrm{FS}}(Z + \overline{Z},\ iZ - i\overline{Z}) = -2i\omega_{\mathrm{FS}}(Z, \overline{Z}) = \Theta_H(Z, \overline{Z}) \\
&= \frac{(1 + |z|^2)\sum_j Z^j \overline{Z}^j - \sum_{j,k} \bar{z}^j z^k Z^j \overline{Z}^k}{(1 + |z|^2)^2} \\
&= \frac{(1 + |z|^2)|Z|^2 - |Z \cdot \bar{z}|^2}{(1 + |z|^2)^2} \\
&\geq \frac{(1 + |z|^2)|Z|^2 - |Z|^2|z|^2}{(1 + |z|^2)^2} = \frac{|Z|^2}{(1 + |z|^2)^2} > 0,
\end{aligned}$$

where we have written $Z \cdot \bar{z} = \sum_j Z^j \bar{z}^j$ and $|Z|^2 = Z \cdot \overline{Z}$, and the fourth line follows from the Cauchy–Schwarz inequality. This shows that ω_{FS} is positive. //

One of the most important features of the Fubini–Study metric is that it is homogeneous. Because the unitary group $\mathrm{U}(n+1) \subseteq \mathrm{GL}(n+1, \mathbb{C})$ acts transitively on 1-dimensional subspaces of $\mathbb{C}^{n+1}$, it acts transitively on $\mathbb{CP}^n$ by projective transformations.

Proposition 8.15 (Homogeneity of the Fubini–Study Metric). *The Fubini–Study metric on $\mathbb{CP}^n$ is invariant under the action of* $\mathrm{U}(n+1)$.

Proof. By (8.10) and (8.11), on $U_\alpha \subseteq \mathbb{CP}^n$ we can write the Kähler form of g_{FS} as $\omega_{\mathrm{FS}} = \frac{i}{2}\bar{\partial}\partial \log u_\alpha$, where

$$u_\alpha([w]) = \frac{|w^\alpha|^2}{|w|^2}.$$

Suppose $A \in \mathrm{U}(n+1)$, and let $\widetilde{A}\colon \mathbb{CP}^n \to \mathbb{CP}^n$ denote the corresponding projective transformation given by $\widetilde{A}([w]) = [Aw]$. Given $p \in \mathbb{CP}^n$, choose α, β such that $p \in U_\alpha$ and $\widetilde{A}(p) \in U_\beta$. Let $f = (Aw)^\beta / w^\alpha$, where $(Aw)^\beta$ denotes the β component of Aw. Then f is a nonvanishing holomorphic function on a neighborhood of p, so

$$u_\beta \circ \widetilde{A}([w]) = \frac{|(Aw)^\beta|^2}{|Aw|^2} = \frac{|f|^2|w^\alpha|^2}{|w|^2},$$

where in the last equality we used the fact that $|Aw|^2 = |w|^2$ because A is unitary. Thus in a small enough neighborhood of p where f has a complex logarithm, we have

$$\widetilde{A}^*\omega_{\mathrm{FS}} = \frac{i}{2}\bar{\partial}\partial \log\left(u_\beta \circ \widetilde{A}\right) = \frac{i}{2}\bar{\partial}\partial\left(\log f + \log \overline{f} + \log \frac{|w^\alpha|^2}{|w|^2}\right) = \omega_{\mathrm{FS}},$$

since $\bar{\partial}\partial \log f = \bar{\partial}\partial \log \overline{f} = 0$. Therefore, $\widetilde{A}$ preserves ω_{FS}, and since it also preserves J, it preserves g_{FS}. □

Here is one more class of examples of Kähler manifolds.

Example 8.16 (Kähler Submanifolds). Suppose M is a Kähler manifold and $N \subseteq M$ is a complex submanifold. Let g_M, ω_M denote the Kähler metric and Kähler form of M and set $g_N = \iota^* g_M$ and $\omega_N = \iota^* \omega_M$, where $\iota\colon N \hookrightarrow M$ is the inclusion map. Because d commutes with pullbacks, ω_N is closed; and Proposition 4.10 shows that it is a $(1,1)$-form. Since N is a complex submanifold, the almost complex structure map J of M maps TN to itself. Thus for any vector field X that is tangent to N,

$$g_N(X,X) = g_M(X,X) = \omega_M(X,JX) = \omega_N(X,JX),$$

so g_N is a Kähler metric with Kähler form ω_N. Thus every complex submanifold of a Kähler manifold is a Kähler manifold in a natural way.

In particular, this shows that all projective complex manifolds and all Stein manifolds have Kähler metrics. //

On the other hand, there are complex manifolds that do not admit any Kähler metrics. To exhibit some examples, we need the following lemma.

Lemma 8.17. *Let (M,g) be an n-dimensional Kähler manifold and ω its Kähler form. The Riemannian volume form of g is given explicitly by*

$$dV_g = \frac{1}{n!}\omega^n = \frac{1}{n!}\omega \wedge \cdots \wedge \omega \quad (\textit{n-fold wedge product}).$$

Proof. Given $p \in M$, by Theorem 8.10(d) we can choose holomorphic coordinates $(z^j = x^j + iy^j)$ centered at p such that $g_{j\bar{k}}(p) = \frac{1}{2}\delta_{jk}$, which implies

$$\omega_p = \frac{i}{2}\sum_{j=1}^{n} dz^j \wedge d\bar{z}^j = \sum_{j=1}^{n} dx^j \wedge dy^j.$$

For $1 \le k \le n$, let $\omega_{(k)}$ be the 2-form

$$\omega_{(k)} = \sum_{j=1}^{k} dx^j \wedge dy^j.$$

Because the frame

$$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\right)$$

is orthonormal at p and positively oriented, it follows that

$$dV_g\big|_p = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n\big|_p.$$

We will prove by induction on k that the following formula holds at p:

$$(\omega_{(k)})^k = k!\,dx^1 \wedge dy^1 \wedge \cdots \wedge dx^k \wedge dy^k. \tag{8.13}$$

For $k = 1$ this is obvious, and for $k = n$ it is the result we are trying to prove. Thus suppose (8.13) holds for some $k < n$, and write

$$\omega_{(k+1)} = \alpha + \beta, \quad \text{where } \alpha = \omega_{(k)} \text{ and } \beta = dx^{k+1} \wedge dy^{k+1}.$$

Note that α and β commute under wedge product because they are 2-forms. Also, $\beta \wedge \beta = 0$ because $dx^{k+1} \wedge dx^{k+1} = 0$, and the induction hypothesis shows that

$$\alpha^k = k!\, dx^1 \wedge dy^1 \wedge \cdots \wedge dx^k \wedge dy^k.$$

It follows that $\alpha^{k+1} = 0$ because it is a sum of terms each of which has a repeated 1-form.

Using the binomial theorem, we compute

$$\begin{aligned}(\omega_{(k+1)})^{k+1} &= (\alpha + \beta)^{k+1} \\ &= \cancel{\alpha^{k+1}} + (k+1)\alpha^k \wedge \beta + \frac{(k+1)k}{2}\cancel{\alpha^{k-1} \wedge \beta^2} + \cdots \\ &\quad + (k+1)\cancel{\alpha \wedge \beta^k} + \cancel{\beta^{k+1}} \\ &= (k+1)k!\, dx^1 \wedge dy^1 \wedge \cdots \wedge dx^k \wedge dy^k \wedge dx^{k+1} \wedge dy^{k+1},\end{aligned}$$

thus completing the induction. □

Here is our first topological obstruction to the existence of Kähler metrics.

Theorem 8.18. *If M is a compact n-dimensional Kähler manifold with Kähler form ω, then $\omega^k = \omega \wedge \cdots \wedge \omega$ represents a nonzero element of $H^{2k}_{\mathrm{dR}}(M;\mathbb{R})$ for $k = 1, \dots, n$. Thus a $2n$-dimensional compact manifold with $b^{2k}(M) = 0$ for some $k \in \{1, \dots, n\}$ cannot admit Kähler metrics.*

Proof. For each $k = 1, \dots, n$, we have $d(\omega^k) = k\, d\omega \wedge \omega^{k-1} = 0$, so ω^k represents a cohomology class in $H^{2k}_{\mathrm{dR}}(M;\mathbb{R})$. From the preceding lemma, we see that $\int_M \omega^n = n! \int_M dV_g > 0$, so ω^n is not exact. Suppose for the sake of contradiction that ω^k is exact for some $1 \le k < n$, so there is some η such that $\omega^k = d\eta$. Then

$$\omega^n = \omega^k \wedge \omega^{n-k} = d\eta \wedge \omega^{n-k} = d(\eta \wedge \omega^{n-k}),$$

showing that ω^n is exact, a contradiction. Thus ω^k represents a nonzero element of $H^{2k}_{\mathrm{dR}}(M;\mathbb{R})$ for each $k = 1, \dots, n$. □

Example 8.19 (The 6-sphere). Problem 1-13 showed that there is a nonintegrable almost complex structure on $\mathbb{S}^6$, and noted that it is still not known whether there is an integrable one. One thing we do know, however, is that there is no Kähler structure, because $H^2_{\mathrm{dR}}(\mathbb{S}^6;\mathbb{R}) = H^4_{\mathrm{dR}}(\mathbb{S}^6;\mathbb{R}) = 0$. //

Example 8.20 (Hopf Manifolds). The Hopf manifolds (Example 1.19) are complex n-manifolds diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. It follows from the Künneth formula of algebraic topology [**Hat02**, Thm. 3.15] together with the de Rham theorem that

$$\begin{aligned} H^2_{\text{dR}}(\mathbb{S}^{2n-1} \times \mathbb{S}^1; \mathbb{R}) &\cong \big(H^2_{\text{dR}}(\mathbb{S}^{2n-1}; \mathbb{R}) \otimes H^0_{\text{dR}}(\mathbb{S}^1; \mathbb{R})\big) \\ &\oplus \big(H^1_{\text{dR}}(\mathbb{S}^{2n-1}; \mathbb{R}) \otimes H^1_{\text{dR}}(\mathbb{S}^1; \mathbb{R})\big) \oplus \big(H^0_{\text{dR}}(\mathbb{S}^{2n-1}; \mathbb{R}) \otimes H^2_{\text{dR}}(\mathbb{S}^1; \mathbb{R})\big), \end{aligned}$$

which is zero when $n > 1$ because all three terms in the direct sum are zero. Thus by Theorem 8.18, Hopf manifolds of complex dimension greater than 1 do not admit Kähler metrics. //

Curvature of Kähler Metrics

Suppose (M, g) is a Kähler manifold, and let Rm denote its ***Riemann curvature tensor***, a covariant 4-tensor field defined by

$$Rm(W, X, Y, Z) = g(R(W, X)Y,\ Z),$$

where $R\colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is the ***curvature endomorphism field*** defined by

$$R(W, X)Y = \nabla_W \nabla_X Y - \nabla_X \nabla_W Y - \nabla_{[W,X]} Y.$$

Initially defined only for real vector fields, R and Rm can be extended to act on complex vector fields by the same formulas. The Riemann curvature is multilinear over $C^\infty(M; \mathbb{C})$, and satisfies the standard Riemannian symmetries for all complex vector fields W, X, Y, Z:

$$Rm(W, X, Y, Z) = -Rm(X, W, Y, Z), \tag{8.14}$$

$$Rm(W, X, Y, Z) = -Rm(W, X, Z, Y), \tag{8.15}$$

$$Rm(W, X, Y, Z) = Rm(Y, Z, W, X), \tag{8.16}$$

$$Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0. \tag{8.17}$$

(Equation (8.17) is called the ***algebraic Bianchi identity***.) These are proved for real vector fields in [**LeeRM**, Prop. 7.12]. To see that they also hold for complex vector fields, just note that we can choose a local frame for $T_{\mathbb{C}}M$ consisting of real vector fields, so the identities for complex vector fields follow from the real ones by complex multilinearity. In addition, since the action of Rm on complex vector fields is obtained from its action on real ones by complex multilinearity, we have

$$\overline{Rm(W, \overline{X}, Y, \overline{Z})} = Rm(\overline{W}, X, \overline{Y}, Z). \tag{8.18}$$

On a Kähler manifold, there are additional symmetries.

Theorem 8.21 (Kähler Curvature Symmetries). *In addition to the standard symmetries, the curvature tensor of a Kähler metric satisfies the following symmetries for all $W, X, Y, Z \in \Gamma(T'M)$:*

$$Rm(W, X, \cdot, \cdot) = Rm(\cdot, \cdot, W, X) = 0, \tag{8.19}$$

$$Rm(\overline{W}, \overline{X}, \cdot, \cdot) = Rm(\cdot, \cdot, \overline{W}, \overline{X}) = 0 \tag{8.20}$$

$$Rm(W, \overline{X}, Y, \overline{Z}) = Rm(Y, \overline{X}, W, \overline{Z}), \tag{8.21}$$

$$Rm(W, \overline{X}, Y, \overline{Z}) = Rm(W, \overline{Z}, Y, \overline{X}), \tag{8.22}$$

Proof. Suppose Z, W are sections of $T'M$ and U, V are arbitrary complex vector fields. Then $R(U,V)Z = \nabla_U \nabla_V Z - \nabla_V \nabla_U Z - \nabla_{[U,V]} Z \in \Gamma(T'M)$ because the Levi-Civita connection maps $\Gamma(T'M)$ to itself (Thm. 8.10(g)). Hence

$$Rm(U, V, Z, W) = g(R(U, V)Z,\ W) = 0,$$

because g annihilates pairs of sections of $T'M$. This proves (8.19), and then (8.20) follows by conjugation.

For sections W, X, Y, Z of $T'M$, the algebraic Bianchi identity (8.17) gives

$$\begin{aligned} 0 &= Rm(W, \overline{X}, Y, \overline{Z}) + Rm(\overline{X}, Y, W, \overline{Z}) + \cancel{Rm(Y, W, \overline{X}, \overline{Z})} \\ &= Rm(W, \overline{X}, Y, \overline{Z}) - Rm(Y, \overline{X}, W, \overline{Z}), \end{aligned}$$

which is (8.21), and then (8.22) follows from (8.16). □

► **Exercise 8.22.** Prove that the curvature tensor on a Kähler manifold satisfies the following additional symmetries for all (real or complex) vector fields:

$$Rm(W, X, JY, JZ) = Rm(W, X, Y, Z) = Rm(JW, JX, Y, Z).$$

Let us work out the formulas for the connection and curvature in holomorphic coordinates. Using the facts that $g_{lp} = 0$ and $\partial_j g_{l\bar{q}} = \partial_l g_{j\bar{q}}$, we compute

$$\begin{aligned} \Gamma_{jl}^{p} &= \tfrac{1}{2} g^{p\bar{q}} \big(\partial_j g_{l\bar{q}} + \partial_l g_{j\bar{q}} - \cancel{\partial_{\bar{q}} g_{lp}} \big) \\ &= g^{p\bar{q}} \partial_j g_{l\bar{q}}. \end{aligned} \tag{8.23}$$

(This can also be viewed as a reflection of the fact that ∇ restricts to the Chern connection on $T'M$ by virtue of Theorem 8.10(f), so its connection 1-forms are determined by (7.17).) The only other nonzero Christoffel symbols are the ones obtained from these by conjugation.

Using this formula, we can compute the components of the curvature endomorphism field and the Riemann curvature tensor.

$$\begin{aligned} R(\partial_j, \partial_{\bar{k}})\partial_l &= \cancel{\nabla_{\partial_j} \nabla_{\partial_{\bar{k}}} \partial_l} - \nabla_{\partial_{\bar{k}}} \nabla_{\partial_j} \partial_l - \cancel{\nabla_{[\partial_j, \partial_{\bar{k}}]} \partial_l} \\ &= -\partial_{\bar{k}} \big(\Gamma_{jl}^{p} \big) \partial_p - \cancel{\Gamma_{jl}^{p} \nabla_{\partial_{\bar{k}}} \partial_p}, \end{aligned}$$

so

$$R_{j\bar{k}l}{}^{p} = -\partial_{\bar{k}}(\Gamma^{p}_{jl}) = -\partial_{\bar{k}}\big(g^{p\bar{q}}\partial_j g_{l\bar{q}}\big), \tag{8.24}$$

$$R_{j\bar{k}l\bar{m}} = -g_{p\bar{m}}\partial_{\bar{k}}(\Gamma^{p}_{jl}) = -g_{p\bar{m}}\partial_{\bar{k}}\big(g^{p\bar{q}}\partial_j g_{l\bar{q}}\big). \tag{8.25}$$

We will use these formulas later in the chapter.

Holomorphic Sectional Curvature

A Kähler metric allows us to define a new curvature invariant, closely related to the sectional curvature. For each nonzero vector $Z \in T'M$, we define the ***holomorphic sectional curvature*** in the direction Z by

$$H(Z) = \frac{Rm\big(Z, \overline{Z}, Z, \overline{Z}\big)}{|Z|^4}.$$

Lemma 8.23. *For every nonzero vector $Z \in T'M$, $H(Z)$ is the ordinary sectional curvature of the plane spanned by* $\{\operatorname{Re} Z, \operatorname{Im} Z\}$.

Proof. Given $Z \in T'M$, let us write $X = \operatorname{Re} Z$ and $Y = \operatorname{Im} Z$. The sectional curvature of the plane spanned by $\{X, Y\}$ is

$$\sec(X, Y) = \frac{Rm(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2}.$$

(See [**LeeRM**, Prop. 8.29].) We will prove the lemma by expanding this formula using $X = \frac{1}{2}(Z + \overline{Z})$ and $Y = \frac{1}{2i}(Z - \overline{Z})$, and interpreting the inner product in the denominator as the Hermitian fiber metric on $T_{\mathbb{C}}M$. The Kähler symmetries of the curvature tensor allow us to simplify the numerator:

$$\begin{aligned} Rm(X, Y, Y, X) &= -\tfrac{1}{8}(Rm(Z + \overline{Z}, Z - \overline{Z}, Z - \overline{Z}, Z + \overline{Z}) \\ &= -\tfrac{1}{8}\big(Rm(Z, -\overline{Z}, Z, \overline{Z}) + Rm(Z, -\overline{Z}, -\overline{Z}, Z) \\ &\qquad + Rm(\overline{Z}, Z, Z, \overline{Z}) + Rm(\overline{Z}, Z, -\overline{Z}, Z)\big) \\ &= \tfrac{1}{2} Rm(Z, \overline{Z}, Z, \overline{Z}). \end{aligned}$$

On the other hand, for the denominator, we use the facts that $T'M$ and $T''M$ are orthogonal and $|Z|^2 = |\overline{Z}|^2$ to compute

$$\begin{aligned} |X|^2|Y|^2 - \langle X, Y\rangle^2 &= \tfrac{1}{8}\big(|Z + \overline{Z}|^2|Z - \overline{Z}|^2 - \langle Z + \overline{Z}, Z - \overline{Z}\rangle^2\big) \\ &= \tfrac{1}{8}\Big(\big(|Z|^2 + |\overline{Z}|^2\big)\big(|Z|^2 + |\overline{Z}|^2\big) - \big(|Z|^2 - |\overline{Z}|^2\big)^2\Big) \\ &= \tfrac{1}{2}|Z|^4. \end{aligned}$$

□

Although this lemma suggests that the holomorphic sectional curvatures contain only part of the information encoded by the curvature tensor (namely, sectional curvatures of planes spanned by the real and imaginary parts of holomorphic vectors, or equivalently by pairs of real vectors of the form $\{X, JX\}$), it turns out that they actually determine the full curvature tensor thanks to the Kähler symmetries.

Lemma 8.24 (Holomorphic Sectional Curvature Determines the Curvature). *Suppose R_1 and R_2 are covariant 4-tensor fields on a complex manifold M that satisfy the Riemann curvature symmetries* (8.14)–(8.17) *as well as the Kähler symmetries* (8.19)–(8.22). *If the following equality holds for every nonzero vector $Z \in T'M$,*

$$\frac{R_1(Z,\overline{Z},Z,\overline{Z})}{|Z|^4} = \frac{R_2(Z,\overline{Z},Z,\overline{Z})}{|Z|^4},$$

then $R_1 = R_2$.

Proof. Define $G = R_1 - R_2$, so that G has the same symmetries as R_1 and R_2 and satisfies $G(Z,\overline{Z},Z,\overline{Z}) = 0$ for all $Z \in T'M$. Given a point $p \in M$ and vectors $Z, W \in T'_pM$, consider the following smooth function of a complex variable z:

$$u(z) = G\big(W + zZ, \overline{W + zZ}, W + zZ, \overline{W + zZ}\big).$$

Our hypothesis guarantees that u is identically zero, so using the Kähler symmetries we conclude

$$0 = \frac{\partial^2 u}{\partial z \partial \bar{z}}(0) = 4G\big(W, \overline{W}, Z, \overline{Z}\big).$$

Now let U, V be two more elements of T'_pM and consider the smooth function $v(w, z)$ defined by

$$v(w,z) = G\big(W + wU, \overline{W + wU}, Z + zV, \overline{Z + zV}\big).$$

The first part of the proof showed that v is identically zero, so

$$0 = \frac{\partial^2 v}{\partial w \partial z}(0,0) = G\big(U, \overline{W}, V, \overline{Z}\big).$$

It then follows from the Riemann and Kähler symmetries that $G = 0$. □

A Kähler manifold is said to have ***constant holomorphic sectional curvature*** if there is a constant c such that $H(Z) = c$ for every $p \in M$ and every nonzero $Z \in T'_pM$.

Lemma 8.25. *A Kähler metric g has constant holomorphic sectional curvature c if and only if in each holomorphic coordinate chart, the coefficients of the Riemann curvature tensor satisfy*

$$R_{j\bar{k}l\bar{m}} = \tfrac{1}{2}c(g_{j\bar{k}}g_{l\bar{m}} + g_{l\bar{k}}g_{j\bar{m}}). \tag{8.26}$$

Proof. A simple computation shows that if the curvature tensor satisfies (8.26), then $Rm(Z,\overline{Z},Z,\overline{Z}) = c|Z|^4$ for all $Z \in T'M$, and therefore g has constant holomorphic sectional curvature c. Conversely, if g has constant holomorphic sectional curvature c, then the curvature tensor of g and the one given by the right-hand side of (8.26) both give the same holomorphic sectional curvatures, so they must be equal by Lemma 8.24. □

Example 8.26 (Constant Holomorphic Sectional Curvature Manifolds).

(a) The standard Kähler metric on $\mathbb{C}^n$ (which is just the Euclidean metric) is flat, and thus has constant holomorphic sectional curvature zero.

(b) We will show that the Fubini–Study metric on $\mathbb{CP}^n$ has constant holomorphic sectional curvature equal to 4. To verify this, because the metric is homogeneous, it suffices to check it at one point. We choose the point $[1,0,\dots,0]$, which corresponds to the origin in affine coordinates $(z^1,\dots,z^n) \leftrightarrow [1,z^1,\dots,z^n]$. It follows from (8.12) that

$$\partial_j g_{l\bar{m}} = \frac{1}{2}\left(-\frac{\delta_{lm}\bar{z}^j}{(1+|z|^2)^2} - \frac{\delta_{jm}\bar{z}^l}{(1+|z|^2)^2} + \frac{2\bar{z}^l z^m \bar{z}^j}{(1+|z|^2)^3}\right).$$

Therefore, at the origin, $g_{l\bar{m}} = \frac{1}{2}\delta_{lm}$, $\partial_j g_{l\bar{m}} = \partial_{\bar{k}} g_{l\bar{m}} = 0$, and

$$\partial_{\bar{k}}\partial_j g_{l\bar{m}} = -\tfrac{1}{2}\left(\delta_{lm}\delta_{jk} + \delta_{jm}\delta_{lk}\right).$$

Thus it follows from the result of Problem 8-11 that

$$R_{j\bar{k}l\bar{m}} = \tfrac{1}{2}\left(\delta_{lm}\delta_{jk} + \delta_{jm}\delta_{lk}\right) = 2\left(g_{j\bar{k}}g_{l\bar{m}} + g_{l\bar{k}}g_{j\bar{m}}\right),$$

and Lemma 8.25 shows that the holomorphic sectional curvatures at the origin are all equal to 4. The result then follows by homogeneity.

(c) The ***complex hyperbolic metric*** is the Kähler metric g_{CH} on $\mathbb{B}^{2n}$ whose Kähler form is

$$\omega_{\mathrm{CH}} = -\frac{i}{2}\partial\bar{\partial}\log(1-|z|^2).$$

Problem 8-5 asks you to show that g_{CH} is homogeneous and geodesically complete, and has constant holomorphic sectional curvature -4. //

Thus we have examples of Kähler manifolds in every dimension with constant holomorphic sectional curvature that is positive (the Fubini–Study metric), zero (the Euclidean metric), and negative (the complex hyperbolic metric). By multiplying these metrics by constants, we can obtain a Kähler metric with holomorphic sectional curvature equal to any real constant. The next theorem shows that, in a sense, these are all the possibilities. It is based on the theory of analytic continuations of Riemannian isometries described in [**LeeRM**, Chap. 12]. The basic definition is the following: Suppose (M,g) and $\left(\widehat{M},\hat{g}\right)$ are Riemannian manifolds of the same dimension and $\varphi\colon U \to \widehat{M}$ is a ***local isometry*** (that is, a smooth map such that

$\varphi^*\hat{g} = g$) defined on a connected open subset $U \subseteq M$. For any continuous path $\gamma : [0,1] \to M$ such that $\gamma(0) \in U$, an ***analytic continuation of φ along γ*** is a family of pairs $\{(U_t, \varphi_t) : t \in [0,1]\}$, where U_t is a connected neighborhood of $\gamma(t)$ and $\varphi_t : U_t \to \widehat{M}$ is a local isometry, such that $\varphi_0 = \varphi$ on $U_0 \cap U$, and for each $t \in [0,1]$ there exists $\delta > 0$ such that $|t - t_1| < \delta$ implies that $\gamma(t_1) \in U_t$ and that φ_t agrees with φ_{t_1} on $U_t \cap U_{t_1}$.

If (M, g) and $\big(\widehat{M}, \hat{g}\big)$ are Riemannian manifolds, a smooth covering map $\pi : M \to \widehat{M}$ is called a ***Riemannian covering*** if it is also a local isometry.

Theorem 8.27. *Suppose (M, g) is an n-dimensional Kähler manifold with constant holomorphic sectional curvature c, and assume M is connected and geodesically complete. Then M admits a holomorphic Riemannian covering by one of the following:*

- $\mathbb{CP}^n$ *with a multiple of the Fubini–Study metric,*
- $\mathbb{C}^n$ *with the Euclidean metric, or*
- $\mathbb{B}^{2n}$ *with a multiple of the complex hyperbolic metric.*

Proof. First consider the case in which M is simply connected. Given (M, g) satisfying the hypothesis, note that because the Riemann curvature tensor can be expressed purely in terms of the metric tensor by formula (8.25), it follows that $\nabla Rm \equiv 0$ since the metric is parallel. Let $(\widehat{M}, \hat{g})$ be the model Kähler manifold ($\mathbb{CP}^n$, $\mathbb{C}^n$, or $\mathbb{B}^{2n}$) endowed with a Kähler metric $\hat{g}$ with the same constant holomorphic sectional curvature as M. Then $(\widehat{M}, \hat{g})$ also has a parallel curvature tensor by the same reasoning, and is geodesically complete and simply connected.

Given any points $p \in M$ and $\hat{p} \in \widehat{M}$, let $A : T_pM \to T_{\hat{p}}\widehat{M}$ be a complex-linear isometry. Because M and $\widehat{M}$ both have the same constant holomorphic sectional curvature, it follows from (8.25) that $A^*(Rm_{\hat{p}}) = Rm_p$. Then [**LeeRM**, Lemma 10.18] shows that there is a Riemannian isometry $\varphi : U \to \widehat{U}$ from a neighborhood U of p to a neighborhood $\widehat{U}$ of $\hat{p}$ that satisfies $\varphi(p) = \hat{p}$ and $D\varphi(p) = A$ (where $D\varphi(p)$ is the total derivative). (That lemma is stated for two points in the same Riemannian manifold, but it applies in the present case by taking the manifold to be the disjoint union of M and $\widehat{M}$.) Corollary 12.3 of [**LeeRM**] shows that if φ can be analytically continued along every path starting at p, then there is a global Riemannian isometry from M to $\widehat{M}$ that agrees with φ on a neighborhood of p.

To see that analytic continuation is always possible, we use the fact that every isometry between open subsets of $\widehat{M}$ is the restriction of a global isometry (Problem 8-6). Let $\gamma : [0,1] \to M$ be a path starting at p. Each point in the image of γ has a geodesically convex neighborhood that is isometric to an open subset of $\widehat{M}$. Cover the image of γ with finitely many such neighborhoods $U_0, \dots, U_k$, with $p \in U_0 \subseteq U$, $U_i \cap U_{i+i} \neq \varnothing$ for each i, and $\gamma(1) \in U_k$. For each i, let $\varphi_i : U_i \to \widehat{U}_i \subseteq \widehat{M}$ be an isometry, chosen so that $\varphi_0 = \varphi|_{U_0}$. For each i, the intersection $U_i \cap U_{i+1}$ is connected, and the composition $\varphi_i \circ \varphi_{i+1}^{-1}$ is an isometry between the connected open sets $\varphi_{i+1}(U_i \cap U_{i+1})$ and $\varphi_i(U_i \cap U_{i+1})$ in $\widehat{M}$, so by Problem 8-6 it is the restriction of a global isometry Ψ. By replacing φ_{i+1} by $\Psi \circ \varphi_{i+1}$, we can ensure that φ_{i+1} agrees with φ_i on the overlap. Thus by induction, we can continue φ all the way to $\gamma(1)$.

Consequently, there exists a global Riemannian isometry $\Phi : M \to \widehat{M}$. To see that it is holomorphic, we argue as follows. Let J and $\widehat{J}$ be the almost complex structures of M and $\widehat{M}$, respectively, and define $\Phi^*\widehat{J} : TM \to TM$ by

$$(\Phi^*\widehat{J})(X_q) = D\Phi(q)^{-1}\widehat{J}(D\Phi(q)X_q).$$

Because Φ is an isometry, it pulls back the Levi-Civita connection of $\widehat{M}$ to that of M, and because $\widehat{J}$ is parallel, it follows that $\Phi^*\widehat{J}$ is also parallel. At the original point $p \in M$, we have $D\Phi(p) = D\varphi(p) = A$, which was chosen to be complex-linear; thus $\Phi^*\widehat{J}$ agrees with J at p. Since both $\Phi^*\widehat{J}$ and J are parallel and M is connected, it follows that $\Phi^*\widehat{J} \equiv J$, and therefore Φ is holomorphic. That completes the proof in case M is simply connected.

For the general case, just apply the above argument to the universal covering space of M with the pullback metric. □

Ricci and Scalar Curvatures

On any Riemannian manifold, the ***Ricci curvature*** is the covariant 2-tensor field defined by

$$Rc(X,Y) = \operatorname{tr}\left(Z \mapsto R(Z,X)Y\right).$$

Because the trace of a linear map is well defined, independent of choice of basis, this is a globally defined tensor field, and it follows from the symmetries of the Riemann curvature tensor that it is symmetric [**LeeRM**, Lemma 7.15]). In terms of any local frame, it has components $R_{ab} = R_{cab}{}^{c}$. In addition, the ***scalar curvature*** is the real-valued function S defined by raising an index of Rc and taking the trace; in a local frame, it is $S = g^{ab}R_{ab}$.

Lemma 8.28. *On a Kähler manifold in holomorphic coordinates, the Ricci and scalar curvatures have the coordinate expressions*

$$Rc = 2R_{j\bar{k}}dz^j\,d\bar{z}^k, \qquad S = 2g^{j\bar{k}}R_{j\bar{k}}, \tag{8.27}$$

where the coefficients $R_{j\bar{k}}$ are given by any of the following expressions:

$$R_{j\bar{k}} = R_{\bar{m}j\bar{k}}{}^{\bar{m}} = R_{j\bar{k}l}{}^{l} = -\partial_j\partial_{\bar{k}}\log(\det g). \tag{8.28}$$

(*Here* $\det g$ *denotes the determinant of the $n \times n$ matrix $(g_{j\bar{k}})$ in coordinates.*)

Proof. The formula for the scalar curvature is an easy consequence of the one for the Ricci curvature, so we focus on the latter.

A priori, as a symmetric 2-tensor field, the Ricci curvature might have components of the forms R_{jk}, $R_{j\bar{k}} = R_{\bar{k}j}$, and $R_{\bar{j}\bar{k}}$. However, on a Kähler manifold,

$$R_{jk} = R_{ljk}{}^{l} + R_{\bar{l}jk}{}^{\bar{l}} = g^{l\bar{m}} R_{ljk\bar{m}} + g^{m\bar{l}} R_{\bar{l}jkm} = 0 + 0,$$

and by conjugation $R_{\bar{j}\bar{k}} = 0$ as well. Thus the only nontrivial components are those of the form $R_{j\bar{k}}$, so we can write Rc as in (8.27) with $R_{j\bar{k}} = Rc(\partial_j, \partial_{\bar{k}})$.

For the components of the Ricci tensor, we use the Kähler curvature symmetries to compute

$$\begin{aligned} R_{j\bar{k}} &= \cancel{R_{mj\bar{k}}{}^{m}} + R_{\bar{m}j\bar{k}}{}^{\bar{m}} \\ &= g^{l\bar{m}} R_{\bar{m}j\bar{k}l} = g^{l\bar{m}} R_{j\bar{m}l\bar{k}} = g^{l\bar{m}} R_{j\bar{k}l\bar{m}} \\ &= R_{j\bar{k}l}{}^{l}. \end{aligned}$$

This proves the first two equalities in (8.28).

The total derivative of the determinant function is given by

$$D(\det)_X(B) = (\det X) \operatorname{tr}\left(X^{-1}B\right)$$

for $X \in \mathrm{GL}(n,\mathbb{C})$ and any $n \times n$ matrix B. (See [**LeeSM**, Problem 7-4, p. 172], where the proof is sketched for $\mathrm{GL}(n,\mathbb{R})$; the same argument works for complex matrices.) Thus by the chain rule,

$$\partial_j(\det g) = D(\det)_g(\partial_j g) = (\det g) \operatorname{tr}\left(g^{-1}\partial_j g\right) = (\det g) g^{l\bar{q}} \partial_j g_{l\bar{q}}. \tag{8.29}$$

Now, (8.23) shows that

$$\Gamma^{l}_{jl} = g^{l\bar{q}} \partial_j g_{l\bar{q}} = \partial_j \log(\det g). \tag{8.30}$$

On the other hand, (8.24) shows that

$$R_{j\bar{k}} = R_{j\bar{k}l}{}^{l} = -\partial_{\bar{k}}(\Gamma^{l}_{jl}). \tag{8.31}$$

The third equality in (8.28) follows from (8.29) and (8.30), together with the fact that ∂_j and $\partial_{\bar{k}}$ commute. □

This leads to an important feature of the Ricci curvature.

Lemma 8.29. *On a Kähler manifold, the Ricci tensor is invariant under J: for all complex vector fields X and Y,*

$$Rc(JX, JY) = Rc(X, Y).$$

Proof. This follows straightforwardly in coordinates by applying (8.27) with the roles of (X, Y) played by (∂_j, ∂_k), $(\partial_j, \partial_{\bar{k}})$ and $(\partial_{\bar{j}}, \partial_{\bar{k}})$. □

As a consequence of this lemma, we can "twist" the Ricci tensor with J to produce a 2-form, in the same way we did with the metric to produce the fundamental 2-form.

Proposition 8.30 (The Ricci Form). *Let* (M, g) *be a Kähler manifold and* Rc *its Ricci curvature. Define a* 2*-tensor field* ρ *by*

$$\rho(X, Y) = Rc(JX, Y).$$

Then ρ *is a closed* $(1,1)$*-form, called the* ***Ricci form of g****.*

Proof. To see that ρ is antisymmetric, we use Lemma 8.29:

$$\begin{aligned}\rho(X, Y) = Rc(JX, Y) = Rc(J^2X, JY) = -Rc(X, JY) = -Rc(JY, X)\\ = -\rho(Y, X).\end{aligned}$$

To prove the other claims, note that Lemma 8.28 implies ρ has the following expressions in local holomorphic coordinates:

$$\rho = iR_{j\bar{k}}dz^j \wedge d\bar{z}^k = -i\partial_j\partial_{\bar{k}} \log(\det g)dz^j \wedge d\bar{z}^k = -i\partial\bar{\partial} \log(\det g). \tag{8.32}$$

This is a $(1,1)$-form, and because $\partial \circ \bar{\partial} = d \circ \bar{\partial}$, this shows that ρ is locally exact and thus closed. □

The significance of the Ricci form is based on the following theorem.

Theorem 8.31. *On a Kähler manifold* M*, the Ricci form is equal to* 2π *times the first Chern form of the Chern connection on* $T'M$*.*

Proof. Let ∇' denote the Chern connection on $T'M$ with respect to the Hermitian fiber metric $\langle \cdot, \cdot \rangle$ defined by (8.4). Recall from Theorem 8.10(f) that ∇' is equal to the restriction of the Levi-Civita connection ∇. To compute its first Chern form, we work in holomorphic coordinates $(z^1, \dots, z^n)$ and note that the connection forms θ_j^k are determined by

$$\theta_k^l(X)\partial_l = \nabla'_X\partial_k = \Gamma_{jk}^l X^j \partial_l,$$

which implies

$$\theta_k^l = \Gamma_{jk}^l dz^j.$$

From formula (7.12), the Chern form is determined in this coordinate domain by

$$c_1(\nabla') = \frac{i}{2\pi} d\theta_l^l = \frac{i}{2\pi} d\left(\Gamma_{jl}^l dz^j\right),$$

and (8.30) and (8.32) show that we can rewrite this as

$$c_1(\nabla') = \frac{i}{2\pi} d\partial \log(\det g) = \frac{i}{2\pi} \bar{\partial}\partial \log(\det g) = \frac{1}{2\pi}\rho.$$ □

The Italian-American mathematician Eugenio Calabi conjectured in the 1950s [**Cal57**] that if (M, g) is a compact Kähler manifold and $\tilde{\rho}$ is any closed real $(1,1)$-form representing the cohomology class $2\pi c_1^{\mathbb{R}}(T'M)$, then there is a Kähler metric in the same Kähler class whose Ricci form is equal to $\tilde{\rho}$. The conjecture was proved in 1977 by Shing-Tung Yau [**Yau78**].

Theorem 8.32 (Calabi–Yau). *Let (M, g) be a compact Kähler manifold with Kähler form ω. If $\tilde{\rho}$ is any closed real $(1,1)$-form representing $2\pi c_1^{\mathbb{R}}(T'M)$, then there is a unique Kähler metric on M whose Kähler form is cohomologous to ω and whose Ricci form is equal to $\tilde{\rho}$.*

The proof of this theorem (which was one of the main accomplishments for which Yau was awarded the Fields Medal) was based on deep ideas in nonlinear partial differential equations. We do not have the tools to prove the theorem, but it is worthwhile outlining how to set it up as a PDE problem.

A Kähler metric in the same Kähler class as ω is given by a Kähler form $\widetilde{\omega} = \omega + \gamma$, where γ is an exact real $(1,1)$-form. In fact, the global $\partial\bar{\partial}$-lemma (which we will prove in Chapter 9) shows that every exact real $(1,1)$-form can be expressed as $\gamma = i\partial\bar{\partial}u$ for some smooth real-valued function u. Thus we seek a function u such that $\widetilde{\omega} = \omega + i\partial\bar{\partial}u$ is a Kähler form with the prescribed Ricci form.

To see how the Ricci form changes under such a change in metric, we work in local holomorphic coordinates. The coefficients of the new metric $\tilde{g}$ are given by

$$\tilde{g}_{j\bar{k}} = g_{j\bar{k}} + \partial_j \partial_{\bar{k}} u.$$

We note also that since the given form $\tilde{\rho}$ and the Ricci form ρ of g both represent 2π times the first Chern class of $T'M$, they are cohomologous, so the global $\partial\bar{\partial}$-lemma shows that there is smooth real function v such that $\tilde{\rho} = \rho + i\partial\bar{\partial}v$.

Using (8.32), we see that in coordinates we have

$$\begin{aligned} \rho &= -i\partial\bar{\partial}\log(\det g_{j\bar{k}}), \\ \tilde{\rho} &= -i\partial\bar{\partial}\log(\det(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}u)), \end{aligned}$$

so we need to solve the following system of PDEs for u:

$$-i\partial\bar{\partial}\log(\det(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}u)) = -i\partial\bar{\partial}\log(\det g_{j\bar{k}}) + i\partial\bar{\partial}v. \tag{8.33}$$

Calabi realized that this can be hugely simplified by just looking for a function u that satisfies

$$\log(\det(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}u)) = \log(\det g_{j\bar{k}}) - v,$$

which obviously implies (8.33). This last equation is equivalent to

$$\frac{\det(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}u)}{\det(g_{j\bar{k}})} = e^{-v}, \tag{8.34}$$

and in this form the equation actually makes sense globally, because the left-hand side is the ratio of the two globally defined volume forms: $dV_{\tilde{g}}/dV_g$. Thus the problem has been reduced to solving a single equation for a single unknown function. Equation (8.34) is of a type called a ***complex Monge–Ampère equation***: a real Monge–Ampère equation is any equation involving the determinant of the Hessian of an unknown function, and the complex version is similar but uses only the $(1,1)$-part of the Hessian (or, as in this case, the Hessian plus some known matrix). It is a fully nonlinear second-order partial differential equation, for which Yau had to develop a substantial array of new techniques.

There is another natural problem on compact Kähler manifolds that is closely related to the Calabi–Yau theorem. For Riemann surfaces, there is a powerful result called the ***uniformization theorem*** (see, for example, [**FK92**, p. 191]), which says every Riemann surface admits a holomorphic covering by $\mathbb{CP}^1$, $\mathbb{C}^1$, or the unit disk. An important geometric consequence of this is that every Riemann surface possesses a complete Kähler metric with constant Gaussian curvature [**FK92**, Thm. IV.8.6]. Much effort has been expended in attempting to generalize this theorem to higher dimensions. The most obvious generalization—finding a Kähler metric of constant holomorphic sectional curvature—is hopeless in higher dimensions because Theorem 8.27 shows that only certain manifolds (those covered by $\mathbb{CP}^n$, $\mathbb{C}^n$, or $\mathbb{B}^{2n}$) admit such metrics.

A plausible candidate for generalizing the uniformization theorem to higher dimensions would be to search for a ***Kähler–Einstein metric***, which is a Kähler metric whose Ricci tensor is a constant multiple of the metric, or equivalently whose Ricci form is a constant multiple of the Kähler form. This condition is satisfied by every metric of constant holomorphic sectional curvature, but also by many other Kähler metrics as well, as we will see below.

Certainly a necessary condition for M to possess a Kähler–Einstein metric is that it must possess a Kähler metric. Beyond that, since a Kähler–Einstein metric has Ricci form $\rho = \lambda\omega$ for some constant λ, and ρ represents a positive multiple of the first real Chern class of $T'M$, another necessary condition is that the first Chern class must contain a representative $(1,1)$-form that is positive (which would be the case if $\lambda > 0$), zero (for $\lambda = 0$), or negative (for $\lambda < 0$).

The zero case is a direct consequence of the Calabi–Yau theorem: if $c_1^{\mathbb{R}}(T'M)$ contains the zero form, then there is a Kähler–Einstein metric whose Ricci form is zero. Around the same time as Yau proved the Calabi conjecture, Yau [**Yau78**] and Thierry Aubin [**Aub78**] independently solved the negative case, using techniques very similar to those used in the proof of the Calabi–Yau theorem.

Theorem 8.33 (Existence of Negative Kähler–Einstein Metrics). *If (M, g) is a compact Kähler manifold and $c_1^{\mathbb{R}}(T'M)$ is represented by a negative $(1,1)$-form, then M admits a Kähler–Einstein metric with the same Kähler class as g.*

The positive case is more complicated, because there are obstructions to the existence of Kähler–Einstein metrics even when $c_1^{\mathbb{R}}(T'M)$ contains a positive $(1,1)$-form.

A connected compact Kähler manifold M with holomorphically trivial canonical bundle is called a ***Calabi–Yau manifold***. Problem 8-10 shows that the assumption of trivial canonical bundle implies $c_1^{\mathbb{R}}(T'M) = 0$, and therefore the Calabi–Yau theorem implies that M carries a unique Kähler–Einstein metric with zero Ricci curvature in each Kähler class (such a metric is said to be ***Ricci flat***).

Here are some examples of Calabi–Yau manifolds.

Example 8.34 (Tori as Calabi–Yau Manifolds). Let $\Lambda \subseteq \mathbb{C}^n$ be a lattice and $M = \mathbb{C}^n/\Lambda$ the associated n-dimensional complex torus. The holomorphic $(n, 0)$-form $dz^1 \wedge \cdots \wedge dz^n \in \Omega^n(\mathbb{C}^n)$ is invariant under the action of Λ, and thus descends to a nonvanishing holomorphic section of the canonical bundle of M. Therefore, every complex torus is a Calabi–Yau manifold. The corresponding Ricci-flat Kähler metrics on M are the flat metrics obtained from Hermitian inner products on $\mathbb{C}^n$. //

Example 8.35 (K3 Surfaces). A simply connected compact complex 2-manifold with trivial canonical bundle is called a ***K3 surface***. (The term was coined by André Weil [**Wei14**, p. 546] as a play on the name of the mountain peak K2 in Kashmir, in honor of Ernst Kummer, Erich Kähler, and Kunihiko Kodaira.) Kodaira [**Kod64**] proved in 1964 that all K3 surfaces are diffeomorphic to each other, but there is a 20-parameter family of inequivalent holomorphic structures. It was proved in 1983 by Yum-Tong Siu [**Siu83**] that every K3 surface has a Kähler metric, and thus is a Calabi–Yau manifold. //

Example 8.36 (Projective Calabi–Yau Hypersurfaces). Projective hypersurfaces provide a rich source of examples of Calabi–Yau manifolds. Suppose $S \subseteq \mathbb{CP}^n$ is a nonsingular projective algebraic hypersurface defined by a homogeneous polynomial of degree d. The adjunction formula (Problem 4-10) shows that the canonical bundle K_S is isomorphic to the restriction to S of $K_{\mathbb{CP}^n} \otimes L_S$. On the other hand, Example 3.40 showed that $L_S \cong H^d$, while Proposition 4.17 shows that $K_{\mathbb{CP}^n} \cong H^{-(n+1)}$. Thus K_S is trivial exactly when $d = n + 1$, and every nonsingular projective algebraic hypersurface in $\mathbb{CP}^n$ of degree $n + 1$ is a Calabi–Yau manifold. A simple example in each dimension is provided by the Fermat hypersurface of degree $n + 1$ in $\mathbb{CP}^n$ (Example 2.39). //

Calabi–Yau manifolds play a central role in the approach to elementary particle theory known as *superstring theory*, because the theory seems to be consistent only if spacetime is viewed as the 10-dimensional total space of a fiber bundle whose base has the familiar 4 dimensions and whose model fiber is a Calabi–Yau threefold (which is, in particular, a 6-dimensional Ricci-flat Riemannian manifold). The interaction between physicists and mathematicians has led to striking new insights into the structure of Calabi–Yau manifolds, which are worth learning about if you want to pursue this field. (See [**YN11**] for an informal description of this interaction.)

It should be noted that except for the trivial case of tori, there is no known Calabi–Yau manifold whose Ricci-flat Kähler metric can be written down explicitly. Even for Calabi–Yau manifolds that are defined algebraically such as the projective hypersurfaces of Example 8.36, there is no known algebraic construction of their associated Ricci-flat Kähler metrics.

The definition of Calabi–Yau manifolds we have given is the simplest and perhaps the most common one, but you will find a number of closely related but inequivalent definitions in the literature. Some definitions require in addition that the manifold be simply connected or have finite fundamental group (which rules out tori). Other definitions relax one or more of the conditions, such as allowing M to be noncompact or requiring only trivial first real Chern class instead of trivial canonical bundle. See also Problem 10-11 for another condition that is sometimes imposed.

Problems

8-1. Suppose (M, g) is a Kähler manifold of dimension $n \geq 2$, and $\tilde{g}$ is a metric conformal to g (i.e., $\tilde{g} = fg$ for some smooth positive function f). Prove that $\tilde{g}$ is Kähler if and only if f is constant.

8-2. If $\left(\widetilde{M}, \tilde{g}\right)$ and (M, g) are Riemannian manifolds, a smooth submersion $\pi : \widetilde{M} \to M$ is called a ***Riemannian submersion*** if for each $x \in \widetilde{M}$, the total derivative $D\pi(x) : T_x\widetilde{M} \to T_{\pi(x)}M$ restricts to a linear isometry from H_x to $T_{\pi(x)}M$, where $H_x \subseteq T_x\widetilde{M}$, called the ***horizontal tangent space***, denotes the orthogonal complement of $\operatorname{Ker} D\pi(x)$. Let $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ denote the restriction of the canonical quotient map $\mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$. Show that π is a Riemannian submersion when $\mathbb{S}^{2n+1}$ is given the standard round metric and $\mathbb{CP}^n$ is given the Fubini–Study metric. [Hint: Since the metrics on both $\mathbb{S}^{2n+1}$ and $\mathbb{CP}^n$ are invariant under $\mathrm{U}(n+1)$, it suffices to check the condition at one point of $\mathbb{S}^{2n+1}$.]

8-3. Let g be a Kähler metric on a Riemann surface M. Show that the holomorphic sectional curvature of g is equal to its Gaussian curvature, and in terms of any local holomorphic coordinate z, both are given by the formula

$$-\frac{1}{u}\frac{\partial^2}{\partial z \partial \bar{z}} \log u,$$

where $u = g(\partial/\partial z, \partial/\partial \bar{z})$.

8-4. Let $Q \subseteq \mathbb{CP}^2$ be the quadric curve defined by the homogeneous polynomial $w^1w^2 - (w^0)^2$. Compute the Gaussian curvature of Q in the metric obtained by restricting the Fubini-Study metric to Q.

8-5. Let g_{CH} be the complex hyperbolic metric on $\mathbb{B}^{2n}$, defined by the Kähler form $\omega_{\mathrm{CH}} = -\frac{i}{2}\partial\bar{\partial}\log(1 - |z|^2)$ (see Example 8.26(c)). Let $\mathrm{U}(n,1)$ be the subgroup of $\mathrm{GL}(n+1,\mathbb{C})$ leaving invariant the following sesquilinear form:

$$H(v,w) = v^1\overline{w}^1 + \cdots + v^n\overline{w}^n - v^0\overline{w}^0.$$

(a) Considering the unit ball $\mathbb{B}^{2n}$ as a subset of $\mathbb{CP}^n$ via the embedding $(z^1,\dots,z^n) \mapsto [1, z^1, \dots, z^n]$, show that $\mathrm{U}(n,1)$ acts transitively on $\mathbb{B}^{2n}$ by projective transformations.
(b) Show that g_{CH} is, up to a constant multiple, the unique $\mathrm{U}(n,1)$-invariant Riemannian metric on $\mathbb{B}^{2n}$.
(c) Show that g_{CH} is geodesically complete.
(d) Show that g_{CH} has constant holomorphic sectional curvature equal to -4.

8-6. Let (M,g) be one of the following Kähler manifolds with constant holomorphic sectional curvature: $(\mathbb{CP}^n, g_{\mathrm{FS}})$, $(\mathbb{C}^n, g_{\mathrm{E}})$, or $(\mathbb{B}^{2n}, g_{\mathrm{CH}})$. Prove that if $U \subseteq M$ is a connected open set and $\varphi\colon U \to M$ is a holomorphic local isometry, then φ is the restriction of a global holomorphic isometry. [Hint: Choose $p \in U$ and show that there is a global holomorphic isometry $\psi\colon M \to M$ such that $\psi \circ \varphi$ fixes p and an orthonormal basis for T'_pM. Then use the exponential map to show that $\psi \circ \varphi$ is the identity on U.]

8-7. Let M be a complex manifold of dimension n, and let g be a Kähler metric on M with constant holomorphic sectional curvature c.
(a) Let $v, w \in T_xM$ be a pair of orthonormal vectors. Show that the (ordinary) sectional curvature of g in the direction of the plane spanned by (v,w) is given by

$$\sec(v,w) = \tfrac{1}{4}c\left(1 + 3\langle v, Jw\rangle^2\right).$$

(b) Show that if $n \geq 2$, then at each point of M, the (ordinary) sectional curvatures of g take on all values between $\frac{1}{4}c$ and c, inclusive. Conclude that M cannot have constant sectional curvature unless it is flat.

8-8. Calculate the volume of $\mathbb{CP}^n$ with the Fubini–Study metric.

8-9. Let $m > n$ and let $F\colon \mathbb{CP}^n \to \mathbb{CP}^m$ be the holomorphic embedding of Problem 3-2. Show that F pulls back the Fubini–Study metric of $\mathbb{CP}^m$ to that of $\mathbb{CP}^n$.

8-10. Suppose (M,g) is a Kähler manifold and K is its canonical bundle. Show that $c_1^{\mathbb{R}}(T'M) = -c_1^{\mathbb{R}}(K)$.

8-11. Let (M, g) be a Kähler manifold, and let (z^j) be holomorphic coordinates satisfying (8.8) at a point $a \in M$. Prove that the components of the curvature tensor at a are given by

$$R_{j\bar{k}l\bar{m}} = -\partial_{\bar{k}}\partial_j g_{l\bar{m}}.$$

8-12. Suppose M is a smooth manifold endowed with a Riemannian metric g and an almost complex structure J satisfying $g(JX, JY) = g(X, Y)$ for all vector fields X and Y. Suppose in addition that J is parallel, that is, $\nabla J \equiv 0$. Prove that J is integrable and g is Kähler.

8-13. Let (M, g) be a Kähler manifold with constant holomorphic sectional curvature c, and let $N \subseteq M$ be a complex submanifold endowed with the induced metric. Show that all holomorphic sectional curvatures of N are less than or equal to c. [Hint: Use the Gauss equation, [**LeeRM**, Thm. 8.5].]

8-14. Here is another curvature quantity that can be defined on Kähler manifolds. Given a Kähler manifold (M, g), for any two linearly independent vectors $Z, W \in T'_x M$ at the same point $x \in M$, define the ***holomorphic bisectional curvature*** associated with Z, W by

$$B(Z, W) = \frac{Rm\big(Z, \overline{Z}, W, \overline{W}\big)}{|Z|^2 |W|^2}.$$

(a) Show that $B(Z, W)$ is equal to the sum of two sectional curvatures:

$$B(Z, W) = \sec(\operatorname{Re} Z, \operatorname{Im} Z) + \sec(\operatorname{Re} Z, \operatorname{Im} W).$$

(b) Show that if g has constant holomorphic sectional curvature c, then its holomorphic bisectional curvatures at each point take on all values between $\frac{1}{2}c$ and c, inclusive.

(c) Show that if g has positive holomorphic bisectional curvature, then it has positive Ricci curvature, and the same is true with "positive" replaced by "nonpositive," "negative," or "nonnegative."

8-15. Let M be a compact, connected Riemann surface endowed with a Kähler metric, and let ∇' be the Chern connection on $T'M$.

(a) Show that the Chern form $c_1(\nabla')$ is equal to $\frac{1}{2\pi} k\, dA$, where k is the Gaussian curvature (see Problem 8-3) and dA is the Riemannian area form.

(b) Use the Gauss–Bonnet theorem to show that the degree of the line bundle $T'M$ is $2 - 2g$, and the degree of the canonical bundle K_M is $2g - 2$, where g is the genus of M.

8-16. Suppose $M \subseteq \mathbb{CP}^2$ is a connected nonsingular algebraic curve of genus g defined by a single polynomial of degree d. Prove the ***genus-degree formula***:

$$g = \frac{(d-1)(d-2)}{2}.$$

[Hint: Use the results of Propositions 7.30 and 4.17, and Problems 4-10 and 8-15.]

8-17. Let $M \subseteq \mathbb{CP}^2$ be a 1-dimensional complex submanifold defined by a homogeneous polynomial of degree d, endowed with the restriction of the Fubini–Study metric. Show that the area of M is equal to πd.

8-18. A ***Fano manifold*** is a compact complex manifold whose anticanonical bundle is positive. Use the Calabi–Yau theorem to prove that a compact complex manifold is a Fano manifold if and only if it admits a Kähler metric with strictly positive Ricci curvature.

8-19. Let (M, g) be a connected Riemannian N-manifold and $p \in M$. For any piecewise smooth loop $\gamma : [a, b] \to M$ based at p (meaning $\gamma(a) = \gamma(b) = p$), let $P^\gamma : T_pM \to T_pM$ denote the parallel transport operator along γ (see [**LeeRM**, Chap. 4]). The set of all linear maps from T_pM to itself obtained in this way is a group, denoted by $\mathrm{Hol}(p)$ and called the ***holonomy group at p***. Because parallel transport preserves inner products, a choice of orthonormal basis for T_pM yields a representation of $\mathrm{Hol}(p)$ as a subgroup of $O(N)$.

(a) Show that if $N = 2n$ and g is a Kähler metric, then $\mathrm{Hol}(p) \subseteq \mathrm{U}(n)$ with respect to an appropriate basis, where we view $\mathrm{U}(n)$ as the subgroup of $O(2n)$ that preserves the inner product and the complex structure on $\mathbb{C}^n$, identified with $\mathbb{R}^{2n}$ in the usual way.

(b) Conversely, suppose g is Riemannian and $\mathrm{Hol}(p) \subseteq \mathrm{U}(n)$ for some $p \in M$ and some choice of orthonormal basis for T_pM. Show that M has an integrable almost complex structure with respect to which g is Kähler.

10.1090/gsm/244/09

Chapter 9

Hodge Theory

We have seen in Chapter 6 that sheaf cohomology groups can play the role of obstructions to surjectivity of certain maps of global sections of sheaves. And thanks to the de Rham–Weil theorem, we can often identify sheaf cohomology groups with the cohomology groups of certain complexes of global sections, such as the de Rham complex or the Dolbeault complex. But these cohomology groups are still typically quotients of infinite-dimensional spaces by infinite-dimensional subspaces, so by themselves they are not very practical for computations. Of course, the de Rham groups are isomorphic to singular cohomology groups, for which there are a multitude of computational methods coming from algebraic topology. But for the Dolbeault groups we do not have such topological tools.

The computations would become more tractable if we could single out a particular representative for each cohomology class, one that has special properties that might lead to computational simplifications or new insights. A plausible approach to finding such a representative would be to seek a representative that is "smallest" or "most efficient" according to some scheme for measuring sizes.

In this chapter, we introduce an inner product and associated norm on the space of global differential forms on a compact Riemannian manifold, and show that a closed form minimizes the norm within its cohomology class if and only if it satisfies a certain differential equation; forms satisfying this equation are called *harmonic forms*. The main result about harmonic forms is the Hodge theorem, which says that on a compact Riemannian manifold, every cohomology class has a unique harmonic representative. The proof of this important theorem is based on a fundamental result about elliptic partial differential operators on compact manifolds; the proof of that result would take us too far afield into the realm of PDE theory, so we merely state it and give several references where proofs can be found.

Then we turn to the case of Hermitian complex manifolds, and prove a similar Hodge theorem for the Dolbeault complex. The theory takes its most powerful form on a compact Kähler manifold, where there is a close relationship between harmonic representatives of de Rham cohomology classes and harmonic representatives of Dolbeault classes, leading to deep consequences for de Rham cohomology and sheaf cohomology.

The general strategy of using harmonic forms to deduce topological, geometric, or complex-analytic properties of manifolds is now known as ***Hodge theory***, after William V. D. Hodge, who developed the theory in a series of papers in the 1930s, summarized in [**Hod41**]. It is now one of the most fundamental tools in both differential and algebraic geometry.

The Hodge Inner Product

To introduce the Hodge theorem in the simplest possible context, we begin by taking a detour into the realm of Riemannian manifolds. Once the basic results of the theory have been established, we will return below to the case of complex manifolds. (The work we do in the Riemannian context will not be wasted, though, because the Riemannian Hodge theorem also has important applications to complex manifolds.)

Suppose (M, g) is a Riemannian manifold of dimension N. The Riemannian metric g together with the musical isomorphism $\sharp\colon T^*_{\mathbb{C}}M \to T_{\mathbb{C}}M$ yields a Hermitian fiber metric $\langle \cdot, \cdot \rangle$ on the space of complex-valued 1-forms defined by

$$\langle \alpha, \beta \rangle = \langle \alpha^\sharp, \beta^\sharp \rangle = g\big(\alpha^\sharp, \overline{\beta^\sharp}\big).$$

In any smooth coordinates $(x^1, \dots, x^N)$, it is given by

$$\langle dx^j, dx^k \rangle = g^{jk}.$$

If $(E_1, \dots, E_N)$ is a local orthonormal frame for $T_{\mathbb{C}}M$, then its dual coframe $(\varepsilon^1, \dots, \varepsilon^N)$ is a local orthonormal frame for $T^*_{\mathbb{C}}M$.

We will extend this fiber metric to differential forms of all degrees. For $0 \le q \le N$, we define a Hermitian fiber metric, called the ***pointwise Hodge inner product***, on the bundle of complex-valued q-forms as follows: For 0-forms (complex-valued functions), we simply set $\langle u, v \rangle = u\bar{v}$. For higher-degree forms, for any local orthonormal coframe $(\varepsilon^1, \dots, \varepsilon^N)$ for $T^*_{\mathbb{C}}M$, let $\langle \cdot, \cdot \rangle$ be the pointwise Hermitian inner product on $\Lambda^q_{\mathbb{C}}M$ for which the following collection of q-forms is orthonormal:

$$\big\{\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_q} : j_1 < \cdots < j_q\big\}.$$

Thanks to the following lemma, this yields a well-defined global fiber metric.

Lemma 9.1. *For $q \geq 1$, the pointwise Hodge inner product $\langle\cdot,\cdot\rangle$ on q-forms is uniquely determined by the following condition: for any locally defined 1-forms $\alpha^1,\ldots,\alpha^q,\beta^1,\ldots,\beta^q$, we have*

$$\langle\alpha^1\wedge\cdots\wedge\alpha^q,\ \beta^1\wedge\cdots\wedge\beta^q\rangle = \det\Big(\langle\alpha^j,\beta^k\rangle\Big). \tag{9.1}$$

Thus it is well defined, independently of the choice of local coframe.

Proof. Because both sides of (9.1) are linear over $C^\infty(M;\mathbb{C})$ in each α^i and conjugate-linear in each β^i, it suffices to check the equality for basis covectors $\alpha^i = \varepsilon^{j_i}$ and $\beta^i = \varepsilon^{k_i}$. Let J and K denote the multi-indices $(j_1,\ldots,j_q)$ and $(k_1,\ldots,k_q)$, respectively, and let S_q denote the group of permutations of the set $\{1,\ldots,q\}$.

For the left-hand side of (9.1), since $\varepsilon^{j_1}\wedge\cdots\wedge\varepsilon^{j_q}$ and $\varepsilon^{k_1}\wedge\cdots\wedge\varepsilon^{k_q}$ match the basis forms we have declared to be orthonormal up to sign, we have

$$\langle\varepsilon^{j_1}\wedge\cdots\wedge\varepsilon^{j_q},\ \varepsilon^{k_1}\wedge\cdots\wedge\varepsilon^{k_q}\rangle = \begin{cases}\operatorname{sgn}\sigma & \text{if } K=\sigma(J) \text{ for some } \sigma\in S_q;\\ 0 & \text{if } K \text{ is not a permutation of } J.\end{cases}$$

(Here $\operatorname{sgn}\sigma$ denotes the ***sign of σ***, which is equal to 1 if σ is an even permutation and -1 if it is odd.)

On the other hand, if $K=\sigma(J)$, then the matrix on the right-hand side of (9.1) is obtained from the identity by applying the permutation σ to the columns, so its determinant is equal to the sign of σ. If K is not a permutation of J, then the matrix has a column of zeros, so its determinant is zero. □

It should be noted that there is another fiber metric on the bundle of real or complex q-forms, obtained by treating it as a subbundle of a tensor bundle (see [**LeeRM**, Prop. 2.40]). These inner products are the same for 1-forms, but for $q > 1$, the pointwise Hodge inner product differs from the tensor inner product by a constant factor depending on q. (The exact constants depend on which convention for the wedge product is in use; see [**LeeRM**, Problem 2-17 on p. 49].) In this book, we use only the pointwise Hodge inner product for differential forms.

When M is compact, we can use this pointwise inner product to define a global inner product on the space $\mathscr{E}^q(M)$ of smooth complex-valued q-forms, called the ***(global) Hodge inner product***, by

$$(\alpha,\beta) = \int_M \langle\alpha,\beta\rangle dV_g,$$

where we interpret dV_g to be the Riemannian volume form of g if M is oriented, and otherwise it is the Riemannian density [**LeeSM**, Prop. 16.45]. This is always a well-defined complex number because we are assuming M is compact, and it is easy to check that it is a Hermitian inner product on the complex vector space $\mathscr{E}^q(M)$. We can also extend this definition to noncompact M if we restrict attention to compactly supported forms. (This is sometimes called the ***L^2 inner product***, because it is used to define the Hilbert space $L^2\big(M;\Lambda^q_{\mathbb{C}}M\big)$ of square-integrable forms with measurable coefficients. Note, however, that $\mathscr{E}^q(M)$ is not complete under this inner product, so the tools of Hilbert space theory, such as orthogonal projections and orthonormal bases, cannot be used here.) We will consistently denote the global Hodge inner product by (α,β) with parentheses, and its associated norm by $\|\alpha\| = (\alpha,\alpha)^{1/2}$, reserving the notations $\langle\cdot,\cdot\rangle$ and $|\cdot|$ for the pointwise inner product and norm, respectively.

Using this inner product, we can single out a candidate for a "best" representative of each cohomology class.

Proposition 9.2. *On a compact Riemannian manifold M, a closed q-form α minimizes the norm $\|\alpha\|$ within its cohomology class if and only if it is orthogonal to the space $\mathscr{B}^q(M)$ of exact forms. If so, it is the unique minimizer in its cohomology class.*

Proof. Suppose $\alpha \in \mathscr{E}^q(M)$ is closed and orthogonal to $\mathscr{B}^q(M)$. Any other representative of the same cohomology class is of the form $\widetilde{\alpha} = \alpha + d\beta$ for some $\beta \in \mathscr{E}^{q-1}(M)$. The hypothesis implies

$$\begin{aligned}\|\widetilde{\alpha}\|^2 &= (\alpha + d\beta, \alpha + d\beta) = \|\alpha\|^2 + 2\cancel{\operatorname{Re}(\alpha, d\beta)} + \|d\beta\|^2\\ &\geq \|\alpha\|^2,\end{aligned}$$

with equality if and only if $d\beta = 0$ and thus $\widetilde{\alpha} = \alpha$.

Conversely, suppose α minimizes the norm within its cohomology class. For an arbitrary $\beta \in \mathscr{E}^{q-1}(M)$, the hypothesis implies $\|\alpha + td\beta\|^2 \geq \|\alpha\|^2$ for all $t \in \mathbb{R}$. Since the smooth real-valued function $t \mapsto \|\alpha + td\beta\|^2$ takes a minimum at $t = 0$, we conclude that

$$\begin{aligned}0 &= \frac{d}{dt}\bigg|_{t=0} \|\alpha + td\beta\|^2\\ &= \frac{d}{dt}\bigg|_{t=0} \Big(\|\alpha\|^2 + 2t\operatorname{Re}(\alpha, d\beta) + t^2\|\beta\|^2\Big)\\ &= 2\operatorname{Re}(\alpha, d\beta).\end{aligned}$$

Applying the same computation with $i\beta$ in place of β shows that $\operatorname{Im}(\alpha, d\beta) = \operatorname{Re}(\alpha, id\beta) = 0$ as well, so α is orthogonal to $d\beta$. □

To make use of this observation, we need a practical way to identify which forms are orthogonal to the image of d. As a warm-up, let us examine a simpler model problem. Suppose V and W are finite-dimensional complex vector spaces endowed with Hermitian inner products, and $A\colon V \to W$ is a linear map. The ***adjoint of A*** is the unique linear map $A^*\colon W \to V$ that satisfies

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \quad \text{for all } x \in V \text{ and } y \in W.$$

In terms of orthonormal bases for V and W, it is the map whose matrix representation is the Hermitian adjoint (transposed conjugate) of that of A, as you can check.

Lemma 9.3. *In the situation described above,*

$$(\operatorname{Im} A)^\perp = \operatorname{Ker}(A^*).$$

Proof. Suppose first that $y \in (\operatorname{Im} A)^\perp$. Then

$$|A^* y|^2 = \langle A^* y, A^* y\rangle = \langle AA^* y, y\rangle = 0,$$

which implies $A^* y = 0$. Conversely if $A^* y = 0$, then for every $x \in V$ we have

$$\langle Ax, y\rangle = \langle x, A^* y\rangle = 0,$$

which implies $y \perp \operatorname{Im} A$. □

To apply this to the infinite-dimensional spaces $\mathscr{E}^q(M)$, we first need to show that operators like d have suitable adjoint operators.

Suppose M is smooth manifold, and E and F are smooth vector bundles over M. (As usual in this book, we assume complex vector bundles, but the theory works equally well for real ones.) A complex-linear map $P : \Gamma(E) \to \Gamma(F)$ is called a ***linear differential operator of order m*** if there is a covering of M by open subsets on which there exist smooth local coordinates $(x^1, \dots, x^N)$ for M and local frames (e_α) for E and (f_β) for F, such that for each such subset U, the restriction of Pu to U has a local expression of the form

$$P\Big(\sum_\alpha u^\alpha e_\alpha\Big) = \sum_{q=0}^{m} \sum_{1 \le j_i \le N} \sum_{\alpha,\beta} p_\alpha^{\beta j_1 \dots j_q} \partial_{j_1} \cdots \partial_{j_q}\big(u^\alpha\big) f_\beta \tag{9.2}$$

for some smooth functions $p_\alpha^{\beta j_1 \dots j_q}$ on U, with the $q = m$ coefficients $p_\alpha^{\beta j_1 \dots j_m}$ not all zero (to ensure that it is actually an operator of order m and not one of lower order).

It is worth noting that this definition implies that a linear differential operator is a ***local operator***, meaning that if two sections u and v agree on an open set $V \subseteq M$, then $Pu|_V = Pv|_V$.

Now let M be a Riemannian manifold. For any Hermitian vector bundle $E \to M$, we denote by $\Gamma_c(E)$ the space of smooth compactly supported sections of E, and define a global Hermitian inner product on $\Gamma_c(E)$ by

$$(u, v) = \int_M \langle u, v\rangle dV_g.$$

Suppose E and F are Hermitian vector bundles over M and $P : \Gamma(E) \to \Gamma(F)$ is a linear differential operator. Another linear differential operator $P^* : \Gamma(F) \to \Gamma(E)$

is called a ***formal adjoint of P*** if it satisfies the following formula for all $u \in \Gamma_c(E)$ and $v \in \Gamma_c(F)$:

$$(Pu, v) = (u, P^* v). \tag{9.3}$$

(The term "formal" comes from functional analysis, where there is a notion of *Hilbert space adjoint* of an operator acting on a Hilbert space of sections of a bundle, which is required to satisfy (9.3) for a much larger class of sections. By contrast, the formal adjoint is only required to satisfy it for smooth compactly supported sections. The formal adjoint is the only type of adjoint we will consider in this book.)

Proposition 9.4 (Existence and Uniqueness of Formal Adjoints). *Every linear differential operator between smooth vector bundles has a unique formal adjoint, which is a linear differential operator of the same order.*

Proof. First, a comment: The proof looks a little daunting because of the notation, but at heart it is really nothing but integration by parts.

To prove existence, let $\{U_i\}_{i\in I}$ be a cover of M by coordinate domains on which P has a coordinate expression like (9.2).

First suppose $v \in \Gamma_c(F)$ is supported in one of the sets U_i and $u \in \Gamma_c(E)$ is arbitrary. Using local coordinates to identify U_i with an open subset of $\mathbb{R}^N$, we compute

$$(Pu, v) = \sum_q \sum_{j_i} \sum_{\alpha\beta\gamma} \int_{U_i} p_\alpha^{\beta j_1 \dots j_q} \left(\partial_{j_1} \cdots \partial_{j_q} u^\alpha\right) \overline{v^\gamma} h_{\beta\gamma} \sqrt{\det g}\, dx^1 \cdots dx^N,$$

where $h_{\beta\gamma} = \langle f_\beta, f_\gamma \rangle$. Since the integrand is smooth and compactly supported in U_i, we can consider this as an integral over a large cube in $\mathbb{R}^N$ and integrate by parts with respect to each variable in turn. The boundary terms vanish, and we obtain

$$(Pu, v) = \sum_q (-1)^q \sum_{j_i} \sum_{\alpha\beta\gamma} \int_{U_i} u^\alpha \partial_{j_q} \cdots \partial_{j_1} \left(p_\alpha^{\beta j_1 \dots j_q} \overline{v^\gamma} h_{\beta\gamma} \sqrt{\det g}\right) dx^1 \cdots dx^N.$$

This is equal to the inner product $(u, Q_i v)$, where

$$Q_i v = \frac{1}{\sqrt{\det g}} \sum_q (-1)^q \sum_{j_i} \sum_{\alpha\beta\gamma\sigma} \partial_{j_q} \cdots \partial_{j_1} \left(\overline{p_\alpha^{\beta j_1 \dots j_q}} v^\gamma \overline{h_{\beta\gamma}} \sqrt{\det g}\right) \overline{H^{\sigma\alpha}} e_\sigma, \tag{9.4}$$

as you can check. (Here $H^{\sigma\alpha}$ denotes the inverse of the matrix $H_{\alpha\sigma} = \langle e_\alpha, e_\sigma \rangle$.)

For the general case, let $\{\varphi_i\}_{i\in I}$ be a partition of unity subordinate to the U_i's, and define $Q : \Gamma(F) \to \Gamma(E)$ by

$$Qv = \sum_{i\in I} Q_i(\varphi_i v). \tag{9.5}$$

Because the supports of the φ_i's are locally finite, each point of M has a neighborhood on which only finitely many terms in the sum on the right-hand side are nonzero, so this is well defined. After we expand out the derivatives using the product rule, we see that this has the same form as (9.2), with coefficients involving various derivatives and conjugates of $p_\alpha^{\beta j_1 \dots j_q}$, $h_{\beta\gamma}$, $H^{\sigma\alpha}$, $\det g$, and φ_i; thus it is a differential operator of the same order as P. Based on the argument in the first part of the proof, for u and v compactly supported we compute

$$(Pu, v) = \sum_{i \in I} (Pu, \varphi_i v) = \sum_{i \in I} (u, Q_i(\varphi_i v)) = (u, Qv)$$

(using the fact that there are now only finitely many values of i for which $\varphi_i v$ is not identically zero), so Q is a formal adjoint of P.

To prove uniqueness, suppose Q and $\widetilde{Q}$ are both formal adjoints of P. Let $\delta v = Qv - \widetilde{Q}v$ for $v \in \Gamma(F)$; then by the definition of formal adjoint, for all $u \in \Gamma_c(E)$ and $v \in \Gamma_c(F)$, we have

$$(u, \delta v) = (u, Qv) - (u, \widetilde{Q}v) = (Pu, v) - (Pu, v) = 0.$$

Applying this with $u = \delta v$, we see that $\delta v = 0$ for all compactly supported v. Since every smooth section agrees with a compactly supported section in a neighborhood of each point and differential operators act locally, this implies $Q = \widetilde{Q}$. □

Proposition 9.5 (Properties of Formal Adjoints). *Suppose E, F, and G are smooth Hermitian vector bundles over a Riemannian manifold M, and $P : \Gamma(E) \to \Gamma(F)$ and $Q : \Gamma(F) \to \Gamma(G)$ are linear differential operators with formal adjoints P^* and Q^*, respectively.*

(a) $(P^*)^* = P$.

(b) $(Q \circ P)^* = P^* \circ Q^*$.

Proof. Part (a) follows from the fact that $(P^*u, v) = \overline{(v, P^*u)} = \overline{(Pv, u)} = (u, Pv)$ for all $u \in \Gamma_c(F)$ and $v \in \Gamma_c(E)$, so P satisfies the criterion to be the adjoint of P^*.

To prove (b), for all $u \in \Gamma_c(E)$ and $v \in \Gamma_c(G)$, we have

$$(Q \circ Pu, v) = (Pu, Q^*v) = (u, P^* \circ Q^*v),$$

which shows that $P^* \circ Q^*$ is the formal adjoint of $Q \circ P$. □

Using the concept of formal adjoints, we have the following analogue of Lemma 9.3.

Proposition 9.6. *Suppose E and F are smooth Hermitian vector bundles over a compact Riemannian manifold M, and $P : \Gamma(E) \to \Gamma(F)$ is a linear differential operator. Then* $(\operatorname{Im} P)^\perp = \operatorname{Ker}(P^*)$.

Proof. Because all sections of E and F are compactly supported in this case, the relation $(Pu, v) = (u, P^*v)$ holds for all $u \in \Gamma(E)$ and $v \in \Gamma(F)$. Based on this formula, the proof of the proposition is exactly the same as the proof of Lemma 9.3. □

As a consequence of Proposition 9.4, for any linear differential operator P, we can freely refer to its unique formal adjoint P^*. In particular, d^* denotes the formal adjoint of the exterior derivative operator. The following corollary follows immediately from Propositions 9.2 and 9.6.

Corollary 9.7. *On a compact Riemannian manifold M, a closed differential form α is the unique representative of its de Rham cohomology class that minimizes the Hodge norm if and only if it satisfies $d^*\alpha = 0$.* □

Although the conclusion of Proposition 9.6 is exactly the same as that of Lemma 9.3, it should be noted that there is one crucial difference between the two situations. In the case of an operator $A\colon V \to W$ between finite-dimensional inner product spaces, the fact that $(\operatorname{Im} A)^\perp = \operatorname{Ker}(A^*)$ implies also that $\operatorname{Im} A = (\operatorname{Ker} A^*)^\perp$, just by taking orthogonal complements of both sides. In other words, given $y \in W$, the equation $Ax = y$ has a solution if and only if y is orthogonal to the kernel of A^*. But in the case of a differential operator $P\colon \Gamma(E) \to \Gamma(F)$, the infinite-dimensional inner product space $\Gamma(F)$ is not complete, so it need not be the case that $((\operatorname{Im} P)^\perp)^\perp = \operatorname{Im} P$. Thus we need to do more work to determine when an equation like $Pu = f$ has a solution.

The Hodge Star Operator

To make use of Corollary 9.7, we need to compute the formal adjoint of d explicitly. For our purposes, the computations will be easier if we assume M is oriented, which is certainly the case for every complex manifold. The main tool will be a linear operator defined on every oriented Riemannian manifold, called the *Hodge star operator*, that takes differential forms to forms of the complementary degree.

Let M be an oriented Riemannian N-manifold. The Hodge star operator will be a smooth bundle homomorphism taking q-forms to $(N-q)$-forms. We define it first in terms of local coframes. Let $(\varepsilon^1, \dots, \varepsilon^N)$ be a local *real* oriented orthonormal coframe for T^*M; thus the collection of forms $\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_q}$ with $j_1 < \cdots < j_q$ forms an orthonormal frame for $\Lambda^q_{\mathbb{C}} M$. To simplify the notation, given any ordered q-tuple of positive integers $J = (j_1, \dots, j_q)$, we will use the abbreviation ε^J for $\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_q}$.

Given any increasing multi-index J, the idea is to define

$$* \varepsilon^J = \operatorname{sgn} \sigma(J, J')\varepsilon^{J'} \tag{9.6}$$

and extend to all complex-valued q-forms by complex linearity, where J' is the unique increasing multi-index of length $N - q$ consisting of the indices complementary to $\{j_1, \dots, j_q\}$, and $\sigma(J, J')$ is the permutation that takes $(1, \dots, N)$ to $(j_1, \dots, j_q, j'_1, \dots, j'_{N-q})$. The next proposition shows that this is well defined, independently of the choice of coframe.

Proposition 9.8. *On an oriented Riemannian N-manifold (M, g), there is a unique smooth complex-linear bundle isomorphism $*: \Lambda^q_{\mathbb{C}} M \to \Lambda^{N-q}_{\mathbb{C}} M$, called the* ***Hodge star operator****, that satisfies the following formula for all $\alpha, \beta \in \Lambda^q_{\mathbb{C}} M$:*

$$\alpha \wedge * \bar{\beta} = \langle \alpha, \beta \rangle dV_g, \tag{9.7}$$

where dV_g is the Riemannian volume form. In terms of any local real oriented orthonormal coframe, $*$ is given by (9.6)*. In addition,*

$$\overline{* \alpha} = * \bar{\alpha}; \tag{9.8}$$

$$* * \alpha = (-1)^{q(N-q)} \alpha; \tag{9.9}$$

$$\langle * \alpha, * \beta \rangle = \langle \alpha, \beta \rangle. \tag{9.10}$$

Proof. We begin by showing that the operator $*$ defined locally by (9.6) in terms of an oriented real orthonormal coframe satisfies (9.7). Because both sides of (9.7) are complex-linear in α and conjugate-linear in β, it suffices to prove it for α and β equal to basis forms. Thus let $\alpha = \varepsilon^J$ and $\beta = \varepsilon^K$, where J and K are arbitrary increasing multi-indices of length q. Because we assumed the basis forms are real, we have $\bar{\beta} = \beta$. Then $* \beta = \operatorname{sgn} \sigma(K, K') \varepsilon^{K'}$, which implies that $\beta \wedge * \beta = \varepsilon^1 \wedge \cdots \wedge \varepsilon^N = dV_g$.

First assume $J = K$. Then $\alpha = \beta$, $\langle \alpha, \beta \rangle = 1$, and

$$\alpha \wedge * \bar{\beta} = \beta \wedge * \beta = dV_g = \langle \alpha, \beta \rangle dV_g,$$

which proves (9.7) in this case. On the other hand, if $J \neq K$, then $\langle \alpha, \beta \rangle = 0$, and $\alpha \wedge * \bar{\beta} = \pm \varepsilon^J \wedge \varepsilon^{K'} = 0$ because there is some index common to J and K'.

To show that $*$ is the unique operator satisfying (9.7), suppose $\tilde{*}$ also satisfies the same property, and let $\delta = * - \tilde{*}$. Then $\alpha \wedge \delta \bar{\beta} = 0$ for all α and β, and applying this with $\bar{\beta}$ in place of β shows that $\alpha \wedge \delta \beta = 0$ as well. Thus

$$|\delta \beta|^2 dV_g = \delta \beta \wedge * \overline{\delta \beta} = \pm * \overline{\delta \beta} \wedge \delta \beta = 0,$$

which shows that δ is identically zero. Therefore, the definitions given by different local coframes agree, so $*$ is globally well defined, and it is smooth because it takes a smooth local frame to a smooth local frame.

Now (9.8) follows from the fact that $*$ takes real basis forms to real forms and is extended by complex linearity.

To prove (9.9), it suffices to consider the case $\alpha = \varepsilon^J$ for some increasing multi-index J. On the one hand, $* \alpha = \pm \varepsilon^{J'}$ and thus $* * \alpha = \pm \varepsilon^J = \pm \alpha$, with the sign determined uniquely by the requirement that $* \alpha \wedge * * \alpha = dV_g$. On the other hand, since α is a q-form and $* \alpha$ is an $(N - q)$-form,

$$dV_g = \alpha \wedge * \alpha = (-1)^{q(N-q)} * \alpha \wedge \alpha,$$

so $(-1)^{q(N-q)}\alpha$ satisfies the sign requirement and thus is equal to $**\alpha$. It also follows from this formula that $*$ is a bundle isomorphism, because $\pm *$ is an inverse for it.

Finally, (9.6) shows that $*$ takes an orthonormal frame for $\Lambda^q_{\mathbb{C}} M$ to an orthonormal frame for $\Lambda^{N-q}_{\mathbb{C}} M$, so (9.10) follows. □

For forms of degrees 0 and 1, there are simple formulas for the Hodge star operator.

Proposition 9.9. *On an oriented Riemannian manifold (M, g), the following formulas hold for a 0-form (scalar function) u and a 1-form β:*

$$*u = u\, dV_g, \tag{9.11}$$

$$*\beta = \beta^\sharp \lrcorner\, dV_g. \tag{9.12}$$

Proof. We just need to verify that the given formulas satisfy condition (9.7) for all suitable forms α and β. For scalar functions u and v,

$$v \wedge \left(\bar{u}\, dV_g\right) = (v\bar{u})\, dV_g = \langle v, u\rangle dV_g,$$

which proves (9.11). For 1-forms α and β, we use the fact that interior multiplication by a vector is an antiderivation. Because $\alpha \wedge dV_g$ is an $(N+1)$-form on an N-manifold, it is zero, so

$$\begin{aligned} 0 = \overline{\beta^\sharp} \lrcorner\, (\alpha \wedge dV_g) &= \left(\overline{\beta^\sharp} \lrcorner\, \alpha\right) dV_g - \alpha \wedge \left(\overline{\beta^\sharp} \lrcorner\, dV_g\right) \\ &= \langle \alpha, \beta\rangle dV_g - \alpha \wedge \overline{\left(\beta^\sharp \lrcorner\, dV_g\right)}, \end{aligned}$$

which proves (9.12). □

Using (9.7), we can write the global Hodge inner product as

$$(\alpha, \beta) = \int_M \alpha \wedge * \bar{\beta}. \tag{9.13}$$

This is the formula we will use to determine the formal adjoint of d.

Proposition 9.10. *For an oriented Riemannian N-manifold M, the formal adjoint of the exterior derivative operator $d : \mathscr{E}^{q-1}(M) \to \mathscr{E}^q(M)$ is the differential operator $d^* : \mathscr{E}^q(M) \to \mathscr{E}^{q-1}(M)$ given by*

$$d^*\beta = (-1)^{Nq+N+1} * d * \beta. \tag{9.14}$$

Proof. Let $\delta\beta$ denote the right-hand side of (9.14). The proposition will be proved if we can show that the following holds for all compactly supported $\alpha \in \mathscr{E}^{q-1}(M)$ and $\beta \in \mathscr{E}^q(M)$:

$$(\alpha, \delta\beta) = (d\alpha, \beta). \tag{9.15}$$

Using (9.13), we have

$$\begin{aligned}
(\alpha, \delta\beta) &= \int_M \alpha \wedge * \overline{\delta\beta} \\
&= (-1)^{Nq+N+1} \int_M \alpha \wedge * \left(* d * \bar{\beta} \right) \\
&= (-1)^{Nq+N+1} (-1)^{(N-q+1)(q-1)} \int_M \alpha \wedge d * \bar{\beta} \\
&= (-1)^q \int_M \alpha \wedge d * \bar{\beta},
\end{aligned}$$

where in the third line we have used (9.9) together with the fact that $d * \bar{\beta}$ is an $(N - q + 1)$-form. Stokes's theorem gives

$$\begin{aligned}
0 &= \int_M d\left(\alpha \wedge * \bar{\beta}\right) \\
&= \int_M d\alpha \wedge * \bar{\beta} + (-1)^{q-1} \int_M \alpha \wedge d * \bar{\beta} \\
&= (d\alpha, \beta) - (\alpha, \delta\beta).
\end{aligned}$$

□

Elliptic Differential Operators

We are going to prove that every de Rham cohomology class on a compact oriented Riemannian manifold has a unique representative that minimizes the Hodge norm in its cohomology class. To get started, we show how to convert this to a problem in partial differential equations.

Starting with a closed q-form $\alpha \in \mathscr{E}^q(M)$, we seek a $(q-1)$-form η such that $\alpha - d\eta$ satisfies $d^*(\alpha - d\eta) = 0$. Thus we need to solve the partial differential equation $d^* d\eta = d^*\alpha$ for the unknown form η.

Let us see how this works in the special case $q = 1$. In this case $d^*\alpha$ is a smooth scalar function, and we are looking for a function u such that $d^* du = d^*\alpha$. This operator $d^* d$ on functions has another guise. Recall that the ***gradient*** of a function u on a Riemannian manifold is the vector field $\operatorname{grad} u$ defined by

$$\operatorname{grad} u = (du)^\sharp,$$

and the ***divergence*** of a vector field X is the unique scalar function $\operatorname{div} X$ that satisfies

$$d(X \lrcorner dV_g) = (\operatorname{div} X) dV_g.$$

The ***Laplace–Beltrami operator*** is the partial differential operator $\Delta : \mathscr{E}(M) \to \mathscr{E}(M)$ defined by

$$\Delta u = \operatorname{div} \operatorname{grad} u.$$

(Some authors define the Laplace–Beltrami operator to be the negative of this, so as to eliminate the negative sign in the following lemma. But the definition above is most common among Riemannian geometers.)

Lemma 9.11. *Let (M, g) be a Riemannian manifold. The operator d^*d acting on scalar functions is equal to the negative of the Laplace–Beltrami operator: $d^*du = -\Delta u$.*

Proof. By Proposition 9.10, $d^*du = -*d*du$. Using Proposition 9.9, we compute

$$d^*du = -*d\big((du)^\sharp \lrcorner\, dV_g\big) = -*\big((\operatorname{div}\operatorname{grad} u)\, dV_g\big) = -\operatorname{div}\operatorname{grad} u. \qquad \square$$

Thus to find a norm-minimizing representative of the cohomology class of α, we need to solve the equation $\Delta u = -d^*\alpha$ for u. There is a necessary condition for the equation $\Delta u = f$ to have a solution on a compact manifold M: if M_0 is any connected component of M, then by the divergence theorem, $\int_{M_0}(\Delta u)dV_g = \int_{M_0}\operatorname{div}(\operatorname{grad} u)dV_g = 0$ since ∂M_0 is empty. But this is the only obstruction. The essential fact about the Laplace–Beltrami operator is the following theorem.

Theorem 9.12. *On a compact oriented Riemannian manifold M, the equation $\Delta u = f$ has a solution u if and only if the integral of f over each connected component of M is zero.*

We will prove this below, as a consequence of a much more general theorem about differential operators. Accepting this for now, let M_0 be any connected component of M and let $\chi_0 \in C^\infty(M)$ be the function that is equal to 1 on M_0 and 0 on all the other components of M. Then since χ_0 is locally constant,

$$\int_{M_0}(d^*\alpha)dV_g = \int_M (d^*\alpha)\overline{\chi}_0\, dV_g = (d^*\alpha,\ \chi_0) = (\alpha,\ d\chi_0) = 0,$$

so $d^*\alpha$ satisfies the necessary and sufficient condition for the existence of a function u satisfying $\Delta u = -d^*\alpha$ and thus $d^*(\alpha - du) = 0$. The 1-form $\alpha - du$ is then the (necessarily unique) norm-minimizing representative of the cohomology class $[\alpha]$.

The feature of the Laplace–Beltrami operator that makes a characterization like that of Theorem 9.12 possible is that it is *elliptic*, a condition we will define shortly.

To define ellipticity, we need to introduce the notion of the *principal symbol* of a differential operator. To begin, for a linear differential operator P of order m, expressed in coordinates as in (9.2), we define the ***total symbol of P*** in these coordinates to be the matrix-valued function $p(x,\xi)$ of $2N$ variables $(x^1, \dots, x^N, \xi_1, \dots, \xi_N)$ obtained by replacing every occurrence of ∂_j in the formula for P with a new independent variable ξ_j:

$$p_\alpha^\beta(x,\xi) = \sum_{q=0}^{m} \sum_{1\le j_i\le N} p_\alpha^{\beta j_1\dots j_q}(x)\xi_{j_1}\cdots\xi_{j_q}.$$

The ***principal symbol*** is the matrix-valued function $\sigma_P(x,\xi)$ obtained by keeping only the terms of highest order $q = m$ in ξ:

$$(\sigma_P)^\beta_\alpha(x,\xi) = \sum_{1\le j_i\le N} p_\alpha^{\beta j_1\dots j_m}(x)\xi_{j_1}\cdots\xi_{j_m}. \tag{9.16}$$

Each matrix entry is a homogeneous polynomial of degree m in $(\xi_1,\dots,\xi_N)$, with coefficients that are smooth functions of x.

When we change to different coordinates or different local frames, the total symbol transforms by a complicated formula involving derivatives of the coefficients and of the coordinate transition functions, so it is of limited use on manifolds. However, the principal symbol has a coordinate-independent interpretation. Recall that if $(x^1,\dots,x^N)$ are smooth local coordinates for M, the ***natural coordinates for*** $\boldsymbol{T^*M}$ are the coordinates $(x,\xi) = (x^1,\dots,x^N,\xi_1,\dots,\xi_N)$ defined by the correspondence

$$\xi_j\,dx^j\big|_p \leftrightarrow (x^1(p),\dots,x^N(p),\xi_1,\dots,\xi_N).$$

(See [**LeeSM**, p. 277].)

Proposition 9.13. *Suppose E and F are smooth complex vector bundles over a smooth manifold M, and $P\colon \Gamma(E)\to\Gamma(F)$ is a linear differential operator of order m. There is a globally defined smooth map $\sigma_P\colon T^*M\to \operatorname{Hom}(E,F)$ called the* ***principal symbol of P****, such that for each $(x,\xi)\in T^*M$, $\sigma_P(x,\xi)$ is the linear map from E_x to F_x whose matrix representation with respect to local frames for E and F and natural coordinates for T^*M is given by* (9.16). *For any point $x_0\in M$ and covector $\xi_0\in T^*_{x_0}M$, the linear map $\sigma_P(x_0,\xi_0)\colon E_{x_0}\to F_{x_0}$ satisfies*

$$\sigma_P(x_0,\xi_0)u_0 = \frac{1}{m!}P(v^m u)(x_0), \tag{9.17}$$

where u is any smooth local section of E that satisfies $u(x_0) = u_0$ and v is any smooth real-valued function on a neighborhood of x_0 that satisfies $v(x_0) = 0$ and $dv_{x_0} = \xi_0$.

Proof. It suffices to show that the map whose local matrix representation is defined by (9.16) also satisfies (9.17), because then (9.17) shows it is independent of choices of coordinates and frames, and (9.16) shows that it is smooth.

Thus given $(x_0,\xi_0)\in T^*M$, let (x^j) be local coordinates for M and (e_α) and (f_β) be local frames, and choose u and v as in the statement of the proposition. We can write $u_0 = u(x_0) = \sum_\alpha u^\alpha(x_0)e_\alpha(x_0)$ and $\xi_0 = \sum_j \xi_j\,dx^j|_{x_0}$ where $\xi_j = \partial_j v(x_0)$. When we apply P to $v^m u$ and expand out the derivatives using the product rule, every term in which fewer than m derivatives fall on the v^m factor will vanish when we set $x = x_0$ because $v(x_0) = 0$, so the only terms that remain are

$$\begin{aligned}P(v^m u)(x_0) &= \sum_{1\le j_i\le N}\sum_{\alpha,\beta} p_\alpha^{\beta j_1\dots j_m}(x_0)\big(\partial_{j_1}\cdots\partial_{j_m}(v^m)\big)\big|_{x=x_0} u^\alpha(x_0) f_\beta(x_0)\\ &= m!\sum_{1\le j_i\le N}\sum_{\alpha,\beta} p_\alpha^{\beta j_1\dots j_m}(x_0)\xi_{j_1}\cdots\xi_{j_m} u^\alpha(x_0) f_\beta(x_0)\\ &= m!\,\sigma_P(x_0,\xi_0)u_0.\end{aligned}$$ □

We note that a linear differential operator of order zero is just a smooth bundle homomorphism $P\colon \Gamma(E)\to\Gamma(F)$, in which case the principal symbol of P is P itself, and does not depend on ξ.

A linear differential operator $P\colon \Gamma(E)\to\Gamma(F)$ is said to be ***elliptic*** if for every $x\in M$ and every nonzero $\xi\in T_x^*M$, the homomorphism $\sigma_P(x,\xi)\colon E_x\to F_x$ is invertible. Note that this is possible only if the bundles E and F have the same rank. In particular, if $P\colon \mathscr{E}(M)\to\mathscr{E}(M)$ is an operator on scalar functions, we can think of it as acting on sections of the trivial complex line bundle $M\times\mathbb{C}\to M$, so its principal symbol at each point $(x,\xi)\in T^*M$ is just a linear map from $\mathbb{C}$ to itself, which is multiplication by a complex number. In that case, we can treat σ_P as a complex-valued function on T^*M, and P is elliptic if and only if this function is never zero for $\xi\neq 0$.

The term "elliptic" may seem mysterious in this context, so it is worth pointing out where it comes from. The first linear partial differential operators to be studied systematically were the constant-coefficient second-order operators acting on real scalar functions on $\mathbb{R}^2$. Such an operator has the form

$$Pu = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu,$$

for some real constants A,B,C,D,E,F with A,B,C not all zero. Because P is a scalar operator, its principal symbol is the scalar-valued function given by

$$\sigma_P\big((x,y),(\xi,\eta)\big) = A\xi^2 + B\xi\eta + C\xi^2.$$

This is a homogeneous quadratic polynomial in (ξ,η). It is invertible (as a linear map on the 1-dimensional fibers) for a particular value of $\big((x,y),(\xi,\eta)\big)$ if and only $\sigma_P\big((x,y),(\xi,\eta)\big)\neq 0$. The only homogeneous quadratic polynomials in (ξ,η) that are never zero for $(\xi,\eta)\neq 0$ are the ones that are positive-definite or negative-definite, which can be detected by the discriminant condition $B^2-4AC<0$; thus P is elliptic if and only if its principal symbol is positive- or negative-definite. These are exactly the homogeneous quadratic polynomials whose level curves are ellipses, hence the name. Problem 9-1 shows that such an operator can be transformed by a linear change of coordinates to (plus or minus) the Laplace operator $\partial^2/\partial x^2+\partial^2/\partial y^2$ plus lower-order terms if and only if it is elliptic.

The fundamental fact about elliptic operators on compact manifolds is the following theorem.

Theorem 9.14 (Fredholm Theorem for Elliptic Operators). *Suppose M is a compact Riemannian manifold, E and F are Hermitian vector bundles over M, and $P\colon \Gamma(E) \to \Gamma(F)$ is an elliptic linear differential operator. Then* $\operatorname{Ker} P$ *and* $\operatorname{Ker} P^*$ *are finite-dimensional, and P restricts to a bijection from* $(\operatorname{Ker} P)^\perp$ *to* $(\operatorname{Ker} P^*)^\perp$.

Developing the tools to prove this theorem would take us too far afield from complex manifold theory, but here are some references where proofs can be found: [**Wel08**, pp. 136–141], [**GH94**, pp. 80–100], [**Bes87**, pp. 456–467], or [**War83**, pp. 220–251].

Theorem 9.12 will follow from this once we show that Δ is elliptic.

Lemma 9.15. *The Laplace–Beltrami operator Δ on scalar functions is elliptic.*

Proof. In any local coordinates (x^i), the Laplace–Beltrami operator has the coordinate representation

$$\Delta u = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\,\frac{\partial u}{\partial x^j}\right) = g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + \text{terms of order 1}$$

(see [**LeeRM**, Prop. 2.46]). Thus its principal symbol is $\sigma_\Delta(x,\xi) = g^{ij}(x)\xi_i\xi_j = |\xi|^2$, which is invertible whenever $\xi \neq 0$. □

Proof of Theorem 9.12. The Fredholm theorem shows that $\Delta u = f$ has a solution if and only if f is orthogonal to the kernel of Δ^*. Now $\Delta^* = (-d^*d)^* = -d^*d = \Delta$. Clearly any function that is constant on each connected component of M is in the kernel of Δ. Conversely, if $\Delta u = 0$, then $0 = (\Delta u, u) = -(d^*du, u) = -\|du\|^2$, which shows that u is constant on each connected component. It follows that a function is orthogonal to $\operatorname{Ker}\Delta$ if and only if its integral over each connected component of M is zero. □

Hodge Theory on Riemannian Manifolds

When we try to apply the same method to differential forms of higher degree, we run into a problem: the operator d^*d is not elliptic on forms of degree higher than 0. One easy way to see that this is true is to note that every closed form is in the kernel of d^*d, so the kernel is infinite-dimensional on manifolds of positive dimension, which would contradict the Fredholm theorem if it were elliptic.

To get around this problem, we define a new differential operator on differential forms of all degrees. The ***Hodge Laplacian*** is the operator $\Delta_d\colon \mathscr{E}^q(M) \to \mathscr{E}^q(M)$ defined by

$$\Delta_d \eta = dd^*\eta + d^*d\eta,$$

where we interpret d^* to be the zero operator when acting on 0-forms. We define a ***harmonic form*** to be a smooth differential form η that satisfies $\Delta_d \eta = 0$.

The next proposition describes some basic properties of the Hodge Laplacian.

Proposition 9.16. *Let M be a compact oriented Riemannian manifold.*

(a) *Δ_d is **formally self-adjoint** (equal to its formal adjoint): $\Delta_d^* = \Delta_d$.*

(b) *A differential form $\alpha \in \mathscr{E}^q(M)$ is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$.*

(c) *A 0-form is harmonic if and only if its restriction to each connected component of M is constant.*

Proof. Part (a) follows from the formula for the formal adjoint of a composition (Proposition 9.5(b)).

For (b), if $d\alpha = 0$ and $d^*\alpha = 0$, then it follows immediately from the definition of Δ_d that $\Delta_d \alpha = 0$. Conversely, suppose $\Delta_d \alpha = 0$. Then

$$0 = (\Delta_d \alpha, \alpha) = (dd^*\alpha, \alpha) + (d^*d\alpha, \alpha) = \|d^*\alpha\|^2 + \|d\alpha\|^2,$$

which shows that $d\alpha = 0$ and $d^*\alpha = 0$.

Finally, part (c) follows from the fact that harmonic 0-forms are exactly the functions that satisfy $du = 0$, since d^* is the zero map in this case. □

For a compact oriented Riemannian manifold M, let $\mathscr{H}^q(M)$ denote the space of harmonic complex-valued q-forms, the kernel of $\Delta_d : \mathscr{E}^q(M) \to \mathscr{E}^q(M)$. By virtue of the preceding proposition, $\mathscr{H}^q(M)$ is contained in the space $\mathscr{Z}^q(M)$ of closed q-forms.

We will show below that Δ_d is elliptic in all degrees. But first, we need some more facts about principal symbols, which will aid in computations.

Lemma 9.17. *Suppose E and F are smooth Hermitian vector bundles over a Riemannian manifold M, and $P : \Gamma(E) \to \Gamma(F)$ is a linear differential operator of order m. Then the principal symbol of its formal adjoint is given by*

$$\sigma_{P^*}(x,\xi) = (-1)^m \big(\sigma_P(x,\xi)\big)^* : F_x \to E_x.$$

Proof. The adjoint P^* is given by $P^*v = \sum_i Q_i(\varphi_i v)$, where (φ_i) is a partition of unity and Q_i is defined by (9.4). When we ignore all terms in which the argument v is differentiated fewer than m times, P^* has the local expression

$$P^*v = (-1)^m \sum_{j_i} \sum_{\alpha\beta\gamma\sigma} \overline{p_\alpha^{\beta j_1 \dots j_m}} \partial_{j_m} \cdots \partial_{j_1} v^\gamma \overline{h_{\beta\gamma}}\, \overline{H^{\sigma\alpha}} e_\sigma + \text{lower-order terms.}$$

Now if we assume that the local frames for E and F are orthonormal (so that $h_{\beta\gamma}$ and $H^{\sigma\alpha}$ are identity matrices) and compute the principal symbol of P^*, we get the matrix representation

$$\left(\sigma_{P^*}\right)_\gamma^\sigma(x,\xi) = (-1)^m \sum_{j_i} \overline{p_\sigma^{\gamma j_1\dots j_m}(x)}\xi_{j_q}\cdots\xi_{j_1},$$

which is the transposed conjugate of that of $(-1)^m\sigma_P(x,\xi)$. □

Lemma 9.18. *Suppose E, F, and G are smooth vector bundles over a smooth manifold M, and $P\colon \Gamma(E)\to\Gamma(F)$, $Q\colon\Gamma(F)\to\Gamma(G)$ are linear differential operators. Then for all $(x,\xi)\in T^*M$,*

$$\sigma_{Q\circ P}(x,\xi) = \sigma_Q(x,\xi)\circ\sigma_P(x,\xi).$$

► **Exercise 9.19.** Prove this lemma.

Proposition 9.20. *On an oriented Riemannian manifold, the principal symbols of d, d^*, and Δ_d are given by*

$$\sigma_d(x,\xi)\alpha = \xi\wedge\alpha; \tag{9.18}$$

$$\sigma_{d^*}(x,\xi)\alpha = -\xi^\sharp \lrcorner\, \alpha; \tag{9.19}$$

$$\sigma_{\Delta_d}(x,\xi)\alpha = -|\xi|^2\alpha. \tag{9.20}$$

Thus Δ_d is elliptic.

Proof. To prove (9.18), we use (9.17). Thus given $x\in M$, $\xi\in T_x^*M$, and $\alpha\in(\Lambda_x^q M)_{\mathbb{C}}$, let $\widetilde{\alpha}$ be a smooth form on a neighborhood of x whose value at x is α, and let v be a smooth real-valued function such that $v(x)=0$ and $dv_x=\xi$. Since d is a first-order differential operator, (9.17) gives

$$\sigma_d(x,\xi)\alpha = d(v\widetilde{\alpha})\big|_x = (dv\wedge\widetilde{\alpha} + \cancel{v\,d\widetilde{\alpha}})\big|_x = \xi\wedge\alpha.$$

Next, for (9.19), we use Lemma 9.17. Thus we need to show that for each real covector ξ, the map $\beta\mapsto\xi^\sharp\lrcorner\,\beta$ is the adjoint of $\alpha\mapsto\xi\wedge\alpha$, or in other words, for all $\alpha\in(\Lambda_x^{q-1}M)_{\mathbb{C}}$ and $\beta\in(\Lambda_x^q M)_{\mathbb{C}}$,

$$\langle\xi\wedge\alpha,\ \beta\rangle = \langle\alpha,\xi^\sharp\lrcorner\,\beta\rangle. \tag{9.21}$$

Choose a local real orthonormal coframe $(\varepsilon^1,\dots,\varepsilon^N)$ such that $\xi = a\varepsilon^1|_x$ for some real number a, and let $(E_1,\dots,E_N)$ be the dual orthonormal frame. Then $\xi^\sharp = aE_1$. Both sides of (9.21) are linear over $\mathbb{C}$ in α and conjugate-linear in β, so it suffices to prove it when $\alpha=\varepsilon^J$ and $\beta=\varepsilon^K$, where J and K are increasing multi-indices of lengths $q-1$ and q, respectively. There are two cases: If $(1,j_1,\dots,j_{q-1}) = (k_1,\dots,k_q)$, then both sides of (9.21) are equal to a. If not, then both sides are equal to zero.

Finally, (9.20) follows from (9.18) and (9.19) together with Lemma 9.18:

$$\sigma_{\Delta_d}(x,\xi)\alpha = -\xi \wedge (\xi^\sharp \lrcorner \alpha) - \xi^\sharp \lrcorner (\xi \wedge \alpha). \tag{9.22}$$

Because interior multiplication by $\xi^\sharp$ is an antiderivation, we obtain

$$\xi^\sharp \lrcorner (\xi \wedge \alpha) = (\xi^\sharp \lrcorner \xi) \wedge \alpha - \xi \wedge (\xi^\sharp \lrcorner \alpha) = |\xi|^2\alpha - \xi \wedge (\xi^\sharp \lrcorner \alpha),$$

where we have used the definition of the sharp operator to obtain $\xi^\sharp \lrcorner \xi = \xi(\xi^\sharp) = g(\xi^\sharp, \xi^\sharp) = g(\xi,\xi) = |\xi|^2$. Substituting this into (9.22) yields (9.20). □

Here is the main theorem of this section.

Theorem 9.21 (Hodge Theorem for Riemannian Manifolds). *Suppose M is an oriented compact Riemannian manifold. Then $\mathscr{H}^q(M)$ is finite-dimensional for each q, and the composite map*

$$\mathscr{H}^q(M) \hookrightarrow \mathscr{Z}^q(M) \to H^q_{\mathrm{dR}}(M;\mathbb{C}) \tag{9.23}$$

is an isomorphism, where $\mathscr{Z}^q(M)$ is the space of closed q-forms and the second map is projection onto the quotient space. Thus every de Rham cohomology class on M has a unique harmonic representative.

Remark. The conclusion of the Hodge theorem holds also for nonoriented Riemannian manifolds. The proof is very similar, but requires different techniques for computing the adjoint of d and showing that Δ_d is elliptic. Since complex manifolds are always canonically oriented, it is easier for our purposes to stick with the oriented case.

Proof. The finite-dimensionality of $\mathscr{H}^q(M)$ follows immediately from the Fredholm theorem.

Proposition 9.16 showed, in particular, that $\mathscr{H}^q(M)$ is contained in the space of closed forms, so the composite map (9.23) makes sense. We just need to show it is injective and surjective.

To see that it is injective, suppose α is a harmonic form. Proposition 9.2 shows that it is the unique form minimizing the Hodge norm within its cohomology class, so no other harmonic form represents the same class.

To see that it is surjective, suppose α is any closed q-form on M. If $q = 0$, then α is already harmonic. If $q \geq 1$, then $d^*\alpha$ is orthogonal to the kernel of d by Proposition 9.6, and therefore orthogonal to the kernel of $\Delta_d = \Delta_d^*$, so the Fredholm theorem for Δ_d shows that there is a $(q-1)$-form β satisfying $\Delta_d\beta = d^*\alpha$. This means $dd^*\beta + d^*d\beta = d^*\alpha$, or

$$dd^*\beta = d^*\alpha - d^*d\beta.$$

The right-hand side of this equation is an element of the image of d^*, and the left-hand side is an element of the image of d and hence also in the kernel of d. Since the image of d^* is orthogonal to the kernel of d, the two sides of this equation are orthogonal to each other and thus must both be zero. That implies $d^*(\alpha - d\beta) = 0$. Since it is also the case that $d(\alpha - d\beta) = 0$, it follows that $\alpha - d\beta$ is a harmonic representative of the cohomology class of α. □

Here is an important application of the Hodge theorem. For any vector spaces V and W, a bilinear map $B : V \times W \to \mathbb{C}$ is said to be ***nondegenerate*** if $B(v, w) = 0$ for all w implies $v = 0$, and $B(v, w) = 0$ for all v implies $w = 0$.

Corollary 9.22 (Poincaré Duality). *Let M be a compact oriented smooth N-dimensional anifold. There is a well-defined nondegenerate bilinear map from $H^k(M;\mathbb{C}) \times H^{N-k}(M;\mathbb{C})$ to $\mathbb{C}$ given by*

$$(\eta, \zeta) \mapsto \int_M \eta \wedge \zeta \tag{9.24}$$

for any smooth forms η and ζ representing their de Rham cohomology classes. Thus for each k there is a complex-linear isomorphism

$$H^k(M;\mathbb{C}) \cong H^{N-k}(M;\mathbb{C})^*, \tag{9.25}$$

and the Betti numbers of M satisfy $b^k(M) = b^{N-k}(M)$. Analogous results hold for cohomology with real coefficients.

Proof. If η and ζ are closed and either one is exact, then $\eta \wedge \zeta$ is exact, so the expression $\int_M \eta \wedge \zeta$ depends only on the cohomology classes of η and ζ, and thus descends to a well-defined bilinear map on cohomology.

To show it is nondegenerate, choose any Riemannian metric for M, so that $H^k(M;\mathbb{C})$ is isomorphic to $\mathscr{H}^k(M;\mathbb{C})$, which is finite dimensional for each k. Suppose η is a closed k-form such that $\int_M \eta \wedge \zeta = 0$ for all closed $(N-k)$-forms ζ. Without loss of generality, we can take η to be the harmonic representative of its cohomology class. We wish to apply this with $\zeta = *\bar{\eta}$. To see that this is a closed form, note that $\bar{\eta}$ is also harmonic because $\Delta_d \bar{\eta} = \overline{\Delta_d \eta} = 0$, so $0 = d^*\bar{\eta} = \pm * d * \bar{\eta}$, and because $*$ is an isomorphism, this implies $d * \bar{\eta} = 0$. Thus $0 = \int_M \eta \wedge (*\bar{\eta}) = \|\eta\|^2$, so $\eta = 0$. The same argument shows that $\int_M \eta \wedge \zeta = 0$ for all closed η implies $\zeta = 0$, thus proving that the bilinear form is nondegenerate.

The isomorphism in (9.25) is defined by sending $[\eta] \in H^k(M;\mathbb{C})$ to the linear functional $[\zeta] \mapsto \int_M \eta \wedge \zeta$ in $H^{N-k}(M;\mathbb{C})^*$. This is a well-defined complex-linear map, and the nondegeneracy of (9.24) implies it is injective. On the other hand, we also have an injective linear map from $H^{N-k}(M;\mathbb{C})$ to $H^k(M;\mathbb{C})^*$ given by the same formula, which shows that the dimension of $H^{N-k}(M;E)$ is less than or equal to that of $H^k(M;E^*)$. Thus both maps are isomorphisms for dimensional reasons, and the statement about Betti numbers follows from this. The argument for real cohomology is essentially identical. □

Hodge Theory on Complex Manifolds

Now suppose M is a compact complex manifold of dimension n endowed with a Hermitian metric g. Then M is, in particular, a smooth Riemannian manifold, so the considerations of the previous section all apply. Since the real dimension $N = 2n$ is even, some of the formulas simplify. Here are the main properties of the Hodge star operator in this case.

Lemma 9.23. *Let M be a compact Hermitian n-manifold.*

(a) *For $\alpha \in \mathscr{E}^{p+q}(M)$, $**\alpha = (-1)^{p+q}\alpha$ and $d^*\alpha = -*d*\alpha$.*

(b) *$*$ maps $\mathscr{E}^{p,q}(M)$ isomorphically onto $\mathscr{E}^{n-q,n-p}(M)$.*

Proof. Part (a) follows immediately from (9.9) and (9.14). To prove (b), note first that the fact that $T'M$ is orthogonal to $T''M$ under the fiber metric $\langle\cdot,\cdot\rangle$ implies that $\mathscr{E}^{p,q}(M)$ is orthogonal to $\mathscr{E}^{p',q'}(M)$ under the Hodge inner product unless $(p,q) = (p',q')$. For $\alpha \in \mathscr{E}^{p,q}(M)$, we will show that $*\alpha \in \mathscr{E}^{n-q,n-p}(M)$ by showing that it is orthogonal to $\mathscr{E}^{r,s}(M)$ unless $(r+q, s+p) = (n,n)$.

Thus let $\beta \in \mathscr{E}^{r,s}(M)$ be arbitrary, and compute

$$\langle\beta, *\alpha\rangle dV_g = \beta \wedge ** \bar{\alpha} = (-1)^{p+q}\beta \wedge \bar{\alpha}.$$

Since $\bar{\alpha}$ is a (q,p)-form, this last expression is a $2n$-form of bidegree $(r+q, s+p)$, which is zero unless $(r+q, s+p) = (n,n)$. □

Example 9.24 (Hodge Star on Riemann Surfaces). If M is a Riemann surface with a Kähler metric g, for any point $p \in M$ we can choose a holomorphic coordinate $z = x+iy$ such that (∂_x, ∂_y) is an oriented orthonormal basis at p. It follows that $*dx = dy$ and $*dy = -dx$, so $*dz = -idz$ and $*d\bar{z} = id\bar{z}$. Since dz and $d\bar{z}$ span the spaces of $(1,0)$-forms and $(0,1)$-forms, respectively, this implies $*\alpha = -i\alpha$ for all $(1,0)$-forms and $*\beta = i\beta$ for all $(0,1)$-forms. //

We define two new Laplace-type operators mapping (p,q)-forms to (p,q)-forms on every complex manifold: the ***Dolbeault Laplacian***

$$\Delta_{\bar{\partial}}\alpha = \bar{\partial}\bar{\partial}^*\alpha + \bar{\partial}^*\bar{\partial}\alpha,$$

and its conjugate

$$\Delta_{\partial}\alpha = \partial\partial^*\alpha + \partial^*\partial\alpha.$$

(We interpret $\bar{\partial}^*$ to be zero on $(p,0)$-forms, and ∂^* to be zero on $(0,q)$-forms.) We will mostly be concerned with the Dolbeault Laplacian $\Delta_{\bar{\partial}}$ because of its relation with Dolbeault cohomology. The next result is the analogue of Proposition 9.16 for the Dolbeault Laplacian and its conjugate.

Proposition 9.25. *Let (M, g) be a compact Hermitian manifold.*

(a) *$\Delta_{\bar\partial}$ and Δ_∂ are formally self-adjoint: $\Delta_{\bar\partial}^* = \Delta_{\bar\partial}$ and $\Delta_\partial^* = \Delta_\partial$.*

(b) *A differential form α is in the kernel of $\Delta_{\bar\partial}$ if and only $\bar\partial\alpha = 0$ and $\bar\partial^*\alpha = 0$, and in the kernel of Δ_∂ if and only $\partial\alpha = 0$ and $\partial^*\alpha = 0$.*

(c) *A $(p,0)$-form is in the kernel of $\Delta_{\bar\partial}$ if and only if it is holomorphic.*

► **Exercise 9.26.** Prove this proposition.

Our next task is to prove that $\Delta_{\bar\partial}$ is elliptic, for which we need to compute the formal adjoint of $\bar\partial$.

Proposition 9.27. *For $\alpha \in \mathscr{E}^{p,q}(M)$, $\bar\partial^*\alpha = - * \partial * \alpha$.*

Proof. This is just like the proof of Proposition 9.10, with a little more attention paid to the types of forms. Let $\alpha \in \mathscr{E}^{p,q}(M)$ and $\beta \in \mathscr{E}^{p,q-1}(M)$, and note that $\beta \wedge * \bar\alpha$ is an $(n, n-1)$-form, so $d(\beta \wedge * \bar\alpha) = \bar\partial(\beta \wedge * \bar\alpha)$. Stokes's theorem gives

$$\begin{aligned}
0 &= \int_M d(\beta \wedge * \bar\alpha)\\
&= \int_M \bar\partial(\beta \wedge * \bar\alpha)\\
&= \int_M \bar\partial\beta \wedge * \bar\alpha + (-1)^{p+q-1} \int_M \beta \wedge \bar\partial * \bar\alpha\\
&= \int_M \bar\partial\beta \wedge * \bar\alpha + \int_M \beta \wedge * * \bar\partial * \bar\alpha\\
&= (\bar\partial\beta, \alpha) + (\beta, * \partial * \alpha).
\end{aligned}$$

□

Proposition 9.28. *The principal symbols of $\bar\partial$, $\bar\partial^*$, and $\Delta_{\bar\partial}$ are*

$$\begin{aligned}
\sigma_{\bar\partial}(x,\xi)\alpha &= \xi^{0,1} \wedge \alpha,\\
\sigma_{\bar\partial^*}(x,\xi)\alpha &= -\left(\xi^{1,0}\right)^\sharp \lrcorner\, \alpha,\\
\sigma_{\Delta_{\bar\partial}}(x,\xi)\alpha &= -\left|\xi^{0,1}\right|^2\alpha = -\tfrac{1}{2}|\xi|^2\alpha,
\end{aligned}$$

where the real covector ξ is decomposed as $\xi = \xi^{0,1} + \xi^{1,0} = \xi^{0,1} + \overline{\xi^{0,1}}$. Thus $\Delta_{\bar\partial}$ is elliptic.

Proof. Problem 9-2. □

For a compact Hermitian manifold M, we define $\mathscr{H}^{p,q}(M)$ to be the kernel of $\Delta_{\bar\partial} : \mathscr{E}^{p,q}(M) \to \mathscr{E}^{p,q}(M)$. Elements of $\mathscr{H}^{p,q}(M)$ are called ***$\bar\partial$-harmonic forms***; by Proposition 9.25, they are the forms that satisfy $\bar\partial\alpha = 0$ and $\bar\partial^*\alpha = 0$. Similarly

α is ***∂-harmonic*** if $\Delta_\partial \alpha = 0$. To avoid confusion, sometimes we will refer to forms that are *harmonic* in the sense defined earlier (i.e., in the kernel of Δ_d) as ***d-harmonic forms***.

The next proposition is proved exactly like its counterpart for d.

Proposition 9.29. *A $\bar\partial$-closed (p,q)-form α is the unique minimizer of the Hodge norm within its Dolbeault cohomology class if and only if it is $\bar\partial$-harmonic.*

The next theorem, like its Riemannian counterpart, was also proved by Hodge [**Hod41**]. Its proof is just like that of Theorem 9.21.

Theorem 9.30 (Hodge–Dolbeault Theorem). *Let M be a compact Hermitian manifold. Then $\mathscr{H}^{p,q}(M)$ is finite-dimensional for each p and q, and the composite map*

$$\mathscr{H}^{p,q}(M) \hookrightarrow \mathscr{Z}^{p,q}(M) \to H^{p,q}(M)$$

is an isomorphism, where $\mathscr{Z}^{p,q}(M)$ is the space of $\bar\partial$-closed (p,q)-forms and the second map is projection onto the quotient space. Thus every Dolbeault cohomology class on M has a unique $\bar\partial$-harmonic representative.

► **Exercise 9.31.** Prove Proposition 9.29 and Theorem 9.30.

For a compact complex manifold M, recall that the ***Hodge numbers of M*** are

$$h^{p,q}(M) = \dim H^{p,q}(M) = \dim H^q(M;\Omega^p).$$

The Hodge–Dolbeault theorem shows that $h^{p,q}(M) = \dim \mathscr{H}^{p,q}(M)$ and therefore the Hodge numbers are finite for all p and q.

Bundle-Valued Forms

There is an important generalizaton of the Hodge–Dolbeault theorem to bundle-valued forms. Suppose M is a complex manifold and $E \to M$ is a holomorphic vector bundle. Recall that the Cauchy–Riemann operator $\bar\partial_E : \mathscr{E}^{p,q}(M;E) \to \mathscr{E}^{p,q+1}(M;E)$ is used to define the Dolbeault cohomology groups $H^{p,q}(M;E)$ with coefficients in E, and the Dolbeault theorem (Thm. 6.19) shows that these are isomorphic to the sheaf cohomology groups $H^q(M;\Omega^p(E))$.

To apply Hodge theory to these groups, we start by choosing a Hermitian metric g on M and a Hermitian fiber metric $\langle\cdot,\cdot\rangle$ on E. We use these to define a Hermitian fiber metric on the bundle $\Lambda^{p,q} \otimes E$ of E-valued forms by setting

$$\langle \alpha\otimes\sigma,\ \beta\otimes\tau\rangle = \langle\alpha,\beta\rangle\,\langle\sigma,\tau\rangle \tag{9.26}$$

and extending by complex linearity in the first argument and conjugate linearity in the second. Here $\langle\alpha,\beta\rangle$ is the pointwise Hodge inner product of the ordinary (p,q)-forms α and β, and $\langle\sigma,\tau\rangle$ is the inner product of σ and τ defined by the chosen

Hermitian fiber metric on E. This defines as usual a global Hermitian inner product on the space of compactly supported E-valued forms:

$$(\eta_1, \eta_2) = \int_M \langle \eta_1, \eta_2 \rangle \, dV_g. \tag{9.27}$$

Using this inner product, we define the adjoint operator $\bar{\partial}_E^*$ and the ***E-valued Dolbeault Laplacian*** $\Delta_{\bar{\partial}_E} : \mathscr{E}^{p,q}(M;E) \to \mathscr{E}^{p,q}(M;E)$ given by

$$\Delta_{\bar{\partial}_E} = \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E. \tag{9.28}$$

An E-valued differential form in the kernel of $\Delta_{\bar{\partial}_E}$ is said to be ***$\bar{\partial}_E$-harmonic***.

Lemma 9.32. *If E is a holomorphic vector bundle, an E-valued differential form α is $\bar{\partial}_E$-harmonic if and only if $\bar{\partial}_E \alpha = 0$ and $\bar{\partial}_E^* \alpha = 0$.*

► **Exercise 9.33.** Prove this lemma.

Proposition 9.34. *If E is a holomorphic vector bundle, the operator $\Delta_{\bar{\partial}_E}$ is elliptic.*

Proof. Let (s_j) be a holomorphic local frame for E over an open set $U \subseteq M$. From (4.20), in terms of this frame $\bar{\partial}_E$ has the local expression

$$\bar{\partial}_E\left(\alpha^j \otimes s_j\right) = \left(\bar{\partial}\alpha^j\right) \otimes s_j.$$

Since the principal symbol of $\bar{\partial}_E$ is independent of the choice of frame, this shows that the principal symbol of $\bar{\partial}_E$ is the same as that of $\bar{\partial}$ on scalar-valued forms, namely

$$\sigma_{\bar{\partial}_E}(x,\xi)\eta = \xi^{0,1} \wedge \eta.$$

It then follows from Lemma 9.17 that the principal symbol of $\bar{\partial}_E^*$ is the same as that of $\bar{\partial}^*$, and therefore

$$\sigma_{\Delta_{\bar{\partial}_E}}(x,\xi)\eta = \sigma_{\Delta_{\bar{\partial}}}(x,\xi)\eta = -\tfrac{1}{2}|\xi|^2\eta,$$

which proves the lemma. □

Theorem 9.35 (Hodge–Dolbeault Theorem for Bundle-Valued Forms). *Suppose M is a compact Hermitian manifold and $E \to M$ is a Hermitian holomorphic vector bundle. Then the space $\mathscr{H}^{p,q}(M;E)$ of $\bar{\partial}_E$-harmonic E-valued (p,q)-forms is finite-dimensional for each p and q, and the composite map*

$$\mathscr{H}^{p,q}(M;E) \hookrightarrow \mathscr{Z}^{p,q}(M;E) \to H^{p,q}(M;E)$$

is an isomorphism. Thus every E-valued Dolbeault cohomology class has a unique $\bar{\partial}_E$-harmonic representative.

Proof. Just like the proof of Theorem 9.30. □

One application of this theorem is to provide another proof that the space of holomorphic sections of a holomorphic vector bundle on a compact manifold is finite-dimensional.

Another Proof of Theorem 3.13. Suppose M is a compact manifold and $E \to M$ is a holomorphic vector bundle. Smooth sections of E are elements of $\mathscr{E}^{0,0}(M; E)$, and they are holomorphic if and only if they are in the kernel of $\bar{\partial}_E$. Because $\bar{\partial}_E^*$ acts trivially on $(0,0)$-forms, the holomorphic sections are exactly the elements of $\mathscr{H}^{0,0}(M; E)$. Finite-dimensionality then follows from the Hodge–Dolbeault theorem. □

Another application is the following finiteness theorem for sheaf cohomology groups.

Proposition 9.36 (Finiteness Theorem). *Suppose M is a compact complex manifold and $E \to M$ is a holomorphic vector bundle. Then the sheaf cohomology groups $H^q(M; \Omega^p(E))$ are all finite-dimensional, and are zero unless* $0 \le p, q \le \dim M$.

Proof. The Dolbeault theorem shows that $H^q(M; \Omega^p(E)) \cong H^{p,q}(M; E)$ for each p and q. Once we endow M with a Hermitian metric, the Hodge–Dolbeault theorem then shows that $H^{p,q}(M; E) \cong \mathscr{H}^{p,q}(M; E)$, which is finite-dimensional. Moreover since there are no nontrivial (p, q) forms unless $0 \le p, q \le \dim M$, those are the only values of p and q for which the groups $H^q(M; \Omega^p(E))$ can be nontrivial. □

Using different techniques, one can prove a much more general version of this finiteness theorem, which says that every coherent analytic sheaf on a compact complex manifold has finite-dimensional cohomology in all degrees. For Stein manifolds, there is an even stronger result: the French mathematician Henri Cartan proved in 1953 [**Car53**] that if $\mathscr{S}$ is any coherent analytic sheaf on a Stein manifold M, then $H^q(M; \mathscr{S}) = 0$ for all $q > 0$. This fact is now known as *Cartan's theorem B*. (*Theorem A*, proved in the same paper, showed that for every such sheaf $\mathscr{S}$, each stalk $\mathscr{S}_x$ is generated as an $\mathscr{O}_x$-module by global sections of $\mathscr{S}$.) Proofs of the general finiteness theorem and Cartan's theorems A and B can be found in [**GR09**, pp. 243–246].

Serre Duality

There is an analogue of Poincaré duality for Dolbeault cohomology, and more generally for Dolbeault cohomology with coefficients in a holomorphic vector bundle $E \to M$. Recall the wedge product operation between E-valued forms and E^*-valued forms defined by (4.16). When applied to sections $\eta \in \mathscr{E}^{p,q}(M; E)$ and $\zeta \in \mathscr{E}^{n-p,n-q}(M; E^*)$, it yields a scalar-valued (n, n)-form $\eta \wedge \zeta$, which can be integrated over M (assuming $n = \dim_{\mathbb{C}} M$). The following theorem was proved in 1955 by Jean-Pierre Serre [**Ser55a**].

Theorem 9.37 (Serre Duality). *Let M be a compact complex n-manifold and let $E \to M$ be a holomorphic vector bundle. There is a well-defined nondegenerate* *bilinear map from* $H^{p,q}(M;E)\times H^{n-p,n-q}(M;E^*)$ *to* $\mathbb{C}$ *given by*

$$(\eta,\zeta)\mapsto\int_M \eta\wedge\zeta \tag{9.29}$$

for any smooth bundle-valued forms η *and* ζ *representing their Dolbeault cohomology classes. Consequently, we have the following complex-linear isomorphisms for all* p *and* q*:*

$$H^{p,q}(M;E)^*\cong H^{n-p,n-q}(M;E^*); \tag{9.30}$$

$$H^{q}(M;\Omega^p(E))^*\cong H^{n-q}(M;\Omega^{n-p}(E^*)); \tag{9.31}$$

$$H^{q}(M;\mathscr{O}(E))^*\cong H^{n-q}(M;\mathscr{O}(K_M\otimes E^*)). \tag{9.32}$$

In addition, the Hodge numbers of M *satisfy*

$$h^{p,q}(M)=h^{n-p,n-q}(M). \tag{9.33}$$

Proof. First we show that (9.29) is well defined on cohomology classes. Suppose $\eta\in\mathscr{E}^{p,q}(M;E)$ is $\bar\partial_E$-closed and $\zeta\in\mathscr{E}^{n-p,n-q}(M;E^*)$ is $\bar\partial_{E^*}$-closed. If $\eta=\bar\partial_E\gamma$, then it follows from Proposition 4.16(iii) and Stokes's theorem that

$$\int_M\eta\wedge\zeta=\int_M\bar\partial_E\gamma\wedge\eta=\int_M\bar\partial(\gamma\wedge\eta)=\int_M d(\gamma\wedge\eta)=0.$$

Similarly, the integral is zero when ζ is $\bar\partial_{E^*}$-exact.

Choose any Hermitian metric g on M and a Hermitian fiber metric $\langle\cdot,\cdot\rangle_h$ on E, and let $\langle\cdot,\cdot\rangle_{h^*}$ be the dual metric on E^* (Problem 7-3). Use these metrics to define $\bar\partial_E$, $\bar\partial_E^*$, $\Delta_{\bar\partial_E}$, and $\mathscr{H}^{p,q}(M;E)$, and let $\hat h\colon E\to E^*$ be the conjugate-linear bundle isomorphism defined in Problem 7-3. We also define a conjugate-linear bundle map

$$\bar *_E\colon \mathscr{E}^{p,q}(M;E)\to\mathscr{E}^{n-p,n-q}(M;E^*)$$

by

$$\bar *_E(\alpha\otimes\sigma)=\overline{*\alpha}\otimes\hat h(\sigma)\quad\text{for }\alpha\in\mathscr{E}^{p,q}(M),\ \sigma\in\Gamma(E),$$

extended bilinearly. The map $\bar *_{E^*}\colon\mathscr{E}^{p,q}(M;E^*)\to\mathscr{E}^{n-p,n-q}(M;E)$ is defined similarly, using the conjugate-linear isomorphism $\hat h^*\colon E^*\to E^{**}$, which is equal to $\hat h^{-1}$ under the canonical identification $E^{**}\cong E$. Straightforward computations using Lemma 9.23 show that

$$\bar *_{E^*}\bar *_E\eta=(-1)^{p+q}\eta\quad\text{for }\eta\in\mathscr{E}^{p,q}(M;E), \tag{9.34}$$

$$\bar *_E\bar *_{E^*}\zeta=(-1)^{p+q}\zeta\quad\text{for }\zeta\in\mathscr{E}^{p,q}(M;E^*). \tag{9.35}$$

We also have an analogue of formula (9.7) for $\bar *_E$ acting on $\mathscr{E}^{p,q}(M;E)$:

$$\begin{aligned}(\alpha\otimes\sigma)\wedge\bar *_E(\beta\otimes\tau)&=(\alpha\otimes\sigma)\wedge\left(\overline{*\beta}\otimes\hat h(\tau)\right)\\&=\hat h(\tau)(\sigma)\alpha\wedge\overline{*\beta}\\&=\langle\sigma,\tau\rangle_h\langle\alpha,\beta\rangle dV_g\\&=\langle\alpha\otimes\sigma,\beta\otimes\tau\rangle\,dV_g,\end{aligned}$$

where the expression in the last line is the Hermitian fiber metric on E-valued forms defined by (9.26).

We need to compute an explicit expression for the formal adjoint $\bar\partial_E^*$. Let $\eta \in \mathscr{E}^{p,q-1}(M;E)$ and $\zeta \in \mathscr{E}^{p,q}(M;E)$. As in the proof of Proposition 9.27, we compute

$$\begin{aligned}
0 &= \int_M d\left(\eta \wedge \bar{*}_E\zeta\right)\\
&= \int_M \bar\partial\left(\eta \wedge \bar{*}_E\zeta\right)\\
&= \int_M \bar\partial_E\eta \wedge \bar{*}_E\zeta + (-1)^{p+q-1}\int_M \eta \wedge \bar\partial_{E^*}\left(\bar{*}_E\zeta\right)\\
&= \int_M \bar\partial_E\eta \wedge \bar{*}_E\zeta + \int_M \eta \wedge \bar{*}_E\bar{*}_{E^*}\bar\partial_{E^*}\left(\bar{*}_E\zeta\right)\\
&= \left(\bar\partial_E\eta, \zeta\right) + \left(\eta, \bar{*}_{E^*}\bar\partial_{E^*}\bar{*}_E\zeta\right),
\end{aligned}$$

which shows that

$$\bar\partial_E^* = -\bar{*}_{E^*}\bar\partial_{E^*}\bar{*}_E \quad \text{on } \mathscr{E}^{p,q}(M;E).$$

To show that (9.29) is nondegenerate, suppose the cohomology class $[\eta] \in H^{p,q}(M;E)$ satisfies $\int_M \eta\wedge\zeta = 0$ for all $[\zeta] \in H^{n-p,n-q}(M;E^*)$. We can take η to be the $\Delta_{\bar\partial_E}$-harmonic representative of its cohomology class, so that $\bar\partial_E\eta = 0$ and $\bar\partial_E^*\eta = 0$. Note that $\bar{*}_E\eta \in \mathscr{E}^{n-p,n-q}(M;E^*)$ satisfies

$$\bar\partial_{E^*}(\bar{*}_E\eta) = \pm\bar{*}_E\bar{*}_{E^*}\bar\partial_{E^*}(\bar{*}_E\eta) = \pm\bar{*}_E(-\bar\partial_E^*\eta) = 0.$$

Thus $\bar{*}_E\eta$ represents a cohomology class in $H^{n-p,n-q}(M;E^*)$, and our hypothesis implies

$$0 = \int_M \eta\wedge\bar{*}_E\eta = \int_M \langle\eta,\eta\rangle\, dV_g = \|\eta\|^2,$$

so $[\eta] = 0$. A nearly identical argument shows that $\int_M \eta\wedge\zeta = 0$ for all η implies $[\zeta] = 0$.

The isomorphism in (9.30) is defined by sending $[\zeta] \in H^{n-p,n-q}(M;E^*)$ to the linear functional $[\eta] \mapsto \int_M \eta\wedge\zeta$ in $H^{p,q}(M;E)^*$. This is a well-defined complex-linear map, and the Serre duality theorem implies it is injective.

On the other hand, we also have an injective linear map from $H^{p,q}(M;E)$ to $H^{n-p,n-q}(M;E^*)^*$ that sends $[\eta]$ to $([\zeta] \mapsto \int_M \eta\wedge\zeta)$, which shows that the dimension of $H^{p,q}(M;E)$ is less than or equal to that of $H^{n-p,n-q}(M;E^*)$. Thus both maps are isomorphisms for dimensional reasons.

The isomorphism (9.31) follows directly from (9.30) and the Dolbeault theorem, and (9.32) is the $p = 0$ case of (9.30), noting that the canonical bundle K_M is equal to $\Lambda^{n,0}M$. Finally, (9.33) follows from (9.30) in the special case in which E is the trivial line bundle. □

Hodge Theory on Kähler Manifolds

In general on complex manifolds, there is no direct relation between Hodge numbers and Betti numbers. But in the special case of a Kähler manifold, it turns out that there is a very close relationship between them. This should not be surprising, because the whole idea of Kähler manifolds is that they are the Hermitian manifolds in which the relationship between the Riemannian metric structure and the holomorphic structure is as close as it can be.

The key is a set of technical identities that hold on every Kähler manifold. Suppose (M, g) is a Kähler manifold with Kähler form ω. Define a complex-linear bundle homomorphism $L_\omega : \mathscr{E}^{p,q}(M) \to \mathscr{E}^{p+1,q+1}(M)$, called the ***Lefschetz operator***, by

$$L_\omega \eta = \omega \wedge \eta,$$

and let $L_\omega^* : \mathscr{E}^{p+1,q+1}(M) \to \mathscr{E}^{p,q}(M)$ be its adjoint, so that $\langle L_\omega \alpha, \beta\rangle = \langle \alpha, L_\omega^* \beta\rangle$ for all forms α, β. The Lefschetz operator is named after Solomon Lefschetz, who introduced it as part of his statement of the hard Lefschetz theorem (Thm. 9.46 below). The following identities were first proved by Hodge [**Hod41**]. In this proposition, we use square brackets to denote commutators: for two operators P and Q, the notation $[P, Q]$ represents $P \circ Q - Q \circ P$.

Proposition 9.38 (The Kähler Identities). *On every Kähler manifold, the following identities hold:*

(a) $[\bar\partial^*, L_\omega] = i\partial$.
(b) $[\partial^*, L_\omega] = -i\bar\partial$.
(c) $[L_\omega^*, \bar\partial] = -i\partial^*$.
(d) $[L_\omega^*, \partial] = i\bar\partial^*$.

Proof. We need only prove (a), for then (b) follows by conjugation, and (c) and (d) follow by taking adjoints, noting that $(iA)^* = -iA^*$ and $[A, B]^* = [B^*, A^*]$ for any operators A and B.

To prove (a), we begin by working on $\mathbb{C}^n$ with its Euclidean metric. In standard holomorphic coordinates, the Kähler metric is $g = \sum_j dz^j\, d\bar z^j$ and its associated Kähler form is $\omega = \frac{i}{2} \sum_j dz^j \wedge d\bar z^j$ (see (8.7)). We will derive a simple expression for $\bar\partial^*$ on Euclidean space.

For a differential form $\alpha = \sum'_{J,K} \alpha_{JK} dz^J \wedge d\bar z^K$, we define $\partial_j \alpha$ to be the form obtained by applying $\partial/\partial z^j$ to the coefficients of α in standard coordinates, and $\partial_{\bar j}\alpha$ is defined similarly:

$$\partial_j \alpha = \sum_{J,K}{}' \frac{\partial \alpha_{JK}}{\partial z^j} dz^J \wedge d\bar z^K, \qquad \partial_{\bar j} \alpha = \sum_{J,K}{}' \frac{\partial \alpha_{JK}}{\partial \bar z^j} dz^J \wedge d\bar z^K.$$

The following identities are straightforward computations once everything is expanded in standard coordinates, using the fact that the coefficients of the metric are constants:

$$\bar\partial\alpha = \sum_j d\bar z^j \wedge \partial_{\bar j}\alpha, \tag{9.36}$$

$$\partial\alpha = \sum_j dz^j \wedge \partial_j\alpha, \tag{9.37}$$

$$\partial_j(\partial_{\bar k} \lrcorner\, \alpha) = \partial_{\bar k} \lrcorner\, (\partial_j\alpha), \tag{9.38}$$

$$\partial_j\langle\alpha, \beta\rangle = \langle\partial_j\alpha, \beta\rangle + \langle\alpha, \partial_{\bar j}\beta\rangle. \tag{9.39}$$

One more identity we will need is

$$\langle d\bar z^j \wedge \alpha, \beta\rangle = 2\langle\alpha, \partial_{\bar j} \lrcorner\, \beta\rangle. \tag{9.40}$$

We proved in (9.21) that $\langle\xi\wedge\alpha, \beta\rangle = \langle\alpha, \xi^\sharp \lrcorner\, \beta\rangle$ for a real covector ξ; if ξ is complex, it follows by splitting it into its real and imaginary parts that $\langle\xi\wedge\alpha, \beta\rangle = \langle\alpha, \bar\xi^\sharp \lrcorner\, \beta\rangle$. Applying this to $\xi = d\bar z^j$ and noting that $\bar\xi^\sharp = 2\partial_j$ because the metric coefficients are $g_{j\bar k} = \frac{1}{2}\delta_{jk}$, we obtain (9.40).

Using these identities, we will show that the formal adjoint of $\bar\partial$ on $\mathbb{C}^n$ is given by

$$\bar\partial^*\beta = -\sum_j 2\partial_{\bar j} \lrcorner\, (\partial_j\beta). \tag{9.41}$$

First note that for any smooth, compactly supported function f on $\mathbb{C}^n$,

$$\int_{\mathbb{C}^n} \partial_{\bar j} f\, dV_g = 0,$$

as can be seen easily by applying the fundamental theorem of calculus to the real and imaginary parts. Therefore if α is a smooth q-form and β is a smooth $(q-1)$-form, both compactly supported,

$$\begin{aligned}
0 &= \sum_j \int_{\mathbb{C}^n} \partial_{\bar j}\langle d\bar z^j \wedge \alpha,\ \beta\rangle\, dV_g \\
&= \sum_j \int_{\mathbb{C}^n} \langle\partial_{\bar j}\big(d\bar z^j \wedge \alpha\big),\ \beta\rangle\, dV_g + \sum_j \int_{\mathbb{C}^n} \langle d\bar z^j \wedge \alpha,\ \partial_j\beta\rangle\, dV_g \\
&= \sum_j \int_{\mathbb{C}^n} \langle d\bar z^j \wedge (\partial_{\bar j}\alpha),\ \beta\rangle\, dV_g + \sum_j \int_{\mathbb{C}^n} \langle\alpha,\ 2\partial_{\bar j} \lrcorner\, (\partial_j\beta)\rangle\, dV_g \\
&= \big(\bar\partial\alpha,\ \beta\big) + \Big(\alpha,\ \sum_j 2\partial_{\bar j} \lrcorner\, (\partial_j\beta)\Big).
\end{aligned}$$

This proves (9.41).

Using these identities together with the fact that interior multiplication is an antiderivation, we compute

$$\begin{aligned}
\bar\partial^*(\omega \wedge \beta) &= -\sum_j 2\partial_{\bar j} \lrcorner\, \partial_j \Big(\tfrac{i}{2}\sum_k dz^k \wedge d\bar z^k \wedge \beta\Big)\\
&= -i\sum_{j,k} \partial_{\bar j} \lrcorner \left(dz^k \wedge d\bar z^k \wedge \partial_j\beta\right)\\
&= -i\sum_{j,k} \cancel{dz^k(\partial_{\bar j}) d\bar z^k \wedge \partial_j\beta} + i\sum_{j,k} d\bar z^k(\partial_{\bar j}) dz^k \wedge \partial_j\beta\\
&\qquad - i\sum_{j,k} dz^k \wedge d\bar z^k \wedge \left(\partial_{\bar j} \lrcorner\, \partial_j\beta\right)\\
&= i\sum_j dz^j \wedge \partial_j\beta + \Big(\sum_k \tfrac{i}{2} dz^k \wedge d\bar z^k\Big) \wedge \Big(-\sum_j 2\partial_{\bar j} \lrcorner\, \partial_j\beta\Big)\\
&= i\partial\beta + \omega \wedge (\bar\partial^*\beta),
\end{aligned}$$

which proves (a) in the case of $\mathbb{C}^n$ with its Euclidean metric.

Now let (M, g) be an arbitrary Kähler manifold with Kähler form ω. Since both operators $\bar\partial^*$ and L_ω are defined independently of coordinates, we may compute $[\bar\partial^*, L_\omega]$ at a point $x \in M$ in any coordinates we choose. Because g is Kähler, we can choose holomorphic coordinates such that the metric coefficients and their first derivatives at x match those of the Euclidean metric (Thm. 8.10(e)). Then the above computation shows that $[\bar\partial^*, L_\omega] = i\partial$ at x, because $\bar\partial^* = -*\partial*$ involves only one derivative of the metric coefficients, and L_ω involves no differentiation at all. □

For later use, we also note the following two facts about the Lefschetz operator.

Lemma 9.39. *Let (M, g) be an n-dimensional Kähler manifold, let ω be its Kähler form, and let $L_\omega \colon \mathscr{E}^{p,q}(M) \to \mathscr{E}^{p+1,q+1}(M)$ be the corresponding Lefschetz operator.*

(a) *For $\alpha \in \mathscr{E}^{p,q}(M)$, $[L_\omega, L_\omega^*]\alpha = (p+q-n)\alpha$.*

(b) *For every $0 \le k \le n$, the operator $L_\omega^k = L_\omega \circ \cdots \circ L_\omega \colon \mathscr{E}^{n-k}(M) \to \mathscr{E}^{n+k}(M)$ is an isomorphism.*

Proof. Given $x_0 \in M$, Theorem 8.10(d) shows that we can find holomorphic coordinates (z^j) on a neighborhood of x_0 such that $g_{j\bar k}(x_0) = \frac{1}{2}\delta_{jk}$ and thus

$$\omega_{x_0} = \frac{i}{2}\sum_j dz^j \wedge d\bar z^j\big|_{x_0}.$$

Equation (9.40) shows that the adjoint of the operator $\alpha \mapsto d\bar z^j \wedge \alpha$ is $\beta \mapsto 2\partial_{\bar j} \lrcorner\, \beta$. Using this together with its conjugate, we conclude that the following formula holds at x_0:

$$L^*_\omega \alpha = -\frac{i}{2}\sum_j (d\bar z^j \wedge)^* (dz^j\wedge)^* \alpha = -2i \sum_j \partial_{\bar j} \lrcorner \partial_j \lrcorner \alpha.$$

Using the fact that interior multiplication is an antiderivation, we get

$$\begin{aligned}
L^*_\omega L_\omega \alpha &= -2i \sum_j \partial_{\bar j} \lrcorner \partial_j \lrcorner \left(\frac{i}{2} \sum_k dz^k \wedge d\bar z^k \wedge \alpha \right) \\
&= \sum_{j,k} \partial_{\bar j} \lrcorner \left(\delta^k_j d\bar z^k \wedge \alpha + dz^k \wedge d\bar z^k \wedge (\partial_j \lrcorner \alpha) \right) \\
&= \sum_{j,k} \Big(\delta^k_j \delta^k_j \alpha - \delta^k_j d\bar z^k \wedge (\partial_{\bar j} \lrcorner \alpha) - \delta^k_j dz^k \wedge (\partial_j \lrcorner \alpha) \\
&\qquad + dz^k \wedge d\bar z^k \wedge (\partial_{\bar j} \lrcorner \partial_j \lrcorner \alpha) \Big) \\
&= n\alpha - \sum_k d\bar z^k \wedge (\partial_{\bar k} \lrcorner \alpha) - \sum_j dz^j \wedge (\partial_j \lrcorner \alpha) + L_\omega L^*_\omega \alpha.
\end{aligned} \tag{9.42}$$

To interpret the two intermediate terms, note that for α of the form

$$\alpha = dz^{j_1} \wedge \dots \wedge dz^{j_p} \wedge d\bar z^{k_1} \wedge \dots \wedge d\bar z^{k_q},$$

and for a particular index j, we have $\partial_j \lrcorner \alpha = 0$ if j is not one of the indices $\{j_1, \dots, j_p\}$; while if $j = j_s$, then

$$\begin{aligned}
dz^{j_s} \wedge (\partial_{j_s} \lrcorner \alpha) &= (-1)^{s-1} dz^{j_s} \wedge \left(dz^{j_1} \wedge \dots \wedge \widehat{dz^{j_s}} \wedge \dots \wedge d\bar z^{k_q} \right) \\
&= \alpha.
\end{aligned}$$

Since α is a (p,q)-form, there are p indices for which this expression is equal to α and the rest are zero, so $\sum_j dz^j \wedge (\partial_j \lrcorner \alpha) = p\alpha$ for α of this form; and since every (p,q)-form is a linear combination of forms of this type, this holds for all $\alpha \in \mathscr{E}^{p,q}(M)$. Similarly $\sum_k d\bar z^k \wedge (\partial_{\bar k} \lrcorner \alpha) = q\alpha$. Inserting these formulas into (9.42) finishes the proof of (a).

To prove (b), we note that L^k_ω is a smooth bundle homomorphism and $\Lambda^{n-k}_{\mathbb{C}}(M)$ and $\Lambda^{n+k}_{\mathbb{C}}(M)$ are vector bundles with the same rank, so it suffices to show L^k_ω is injective. Writing $k = n - j$, we will show by induction on j that $L^{n-j}_\omega : \Lambda^j_{\mathbb{C}}(M) \to \Lambda^{2n-j}_{\mathbb{C}}(M)$ is injective.

The case $j = 0$ is immediate, because L^n_ω is just multiplication by $\omega^n = n! dV_g$. Assume that L^{n-j+1}_ω is injective on $(j-1)$-forms, and suppose α is a j-form such that $L^{n-j}_\omega \alpha = 0$. Then $L^{n-j+1}_\omega \alpha = 0$, and the fact that interior multiplication is an antiderivation implies that for every complex vector field X,

$$\begin{aligned}
0 &= X \lrcorner L_\omega^{n-j+1}\alpha \\
&= X \lrcorner (\omega \wedge \cdots \wedge \omega \wedge \alpha) \\
&= (n-j+1)(X \lrcorner \omega) \wedge L_\omega^{n-j}\alpha + L_\omega^{n-j+1}(X \lrcorner \alpha) \\
&= L_\omega^{n-j+1}(X \lrcorner \alpha),
\end{aligned}$$

so the inductive hypothesis shows that $X \lrcorner \alpha = 0$. Since this is true for every complex vector field X, it follows that $\alpha = 0$. □

Remark. We proved the preceding lemma for Kähler manifolds because that is the only case we will need. But the proof actually works when (M, g) is merely Hermitian and ω is its fundamental 2-form, because the computations were all carried out pointwise and did not involve any derivatives of ω.

The next three lemmas and the theorem following them are the main applications of the Kähler identities.

Lemma 9.40. *On every Kähler manifold, the following identities hold:*

$$\begin{aligned}
\bar\partial^*\partial + \partial\bar\partial^* &= 0, \\
\partial^*\bar\partial + \bar\partial\partial^* &= 0.
\end{aligned}$$

Proof. Proposition 9.38 shows that $\bar\partial^* = -i[L_\omega^*, \partial]$. Using this and the fact that $\partial \circ \partial = 0$, we compute

$$\begin{aligned}
\bar\partial^*\partial + \partial\bar\partial^* &= -i[L_\omega^*, \partial]\partial - i\partial[L_\omega^*, \partial] \\
&= -iL_\omega^*\partial\partial + i\partial L_\omega^*\partial - i\partial L_\omega^*\partial + i\partial\partial L_\omega^* = 0.
\end{aligned}$$

The second identity follows from the first by conjugation. □

Lemma 9.41. *On every Kähler manifold,* $\Delta_d = \Delta_\partial + \Delta_{\bar\partial}$.

Proof. Lemma 9.40 yields

$$\begin{aligned}
\Delta_d &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) \\
&= \partial\partial^* + \partial\bar\partial^* + \bar\partial\partial^* + \bar\partial\bar\partial^* + \partial^*\partial + \partial^*\bar\partial + \bar\partial^*\partial + \bar\partial^*\bar\partial \\
&= (\partial\partial^* + \partial^*\partial) + \cancel{(\partial\bar\partial^* + \bar\partial^*\partial)} + \cancel{(\bar\partial\partial^* + \partial^*\bar\partial)} + (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) \\
&= \Delta_\partial + \Delta_{\bar\partial}.
\end{aligned}$$

□

Lemma 9.42. *On every Kähler manifold,* $\Delta_\partial = \Delta_{\bar\partial}$.

Proof. From Proposition 9.38 again we get $\partial^* = i[L_\omega^*, \bar\partial]$. Thus

$$\begin{aligned}
\Delta_\partial &= \partial\partial^* + \partial^*\partial \\
&= i\partial[L_\omega^*, \bar\partial] + i[L_\omega^*, \bar\partial]\partial \\
&= i\partial L_\omega^*\bar\partial - i\partial\bar\partial L_\omega^* + iL_\omega^*\bar\partial\partial - i\bar\partial L_\omega^*\partial.
\end{aligned}$$

On the other hand, since $\bar\partial^* = -i[L_\omega^*, \partial]$,

$$\begin{aligned}\Delta_{\bar\partial} &= \bar\partial\bar\partial^* + \bar\partial^*\bar\partial\\ &= -i\bar\partial[L_\omega^*, \partial] - i[L_\omega^*, \partial]\bar\partial\\ &= -i\bar\partial L_\omega^*\partial + i\bar\partial\partial L_\omega^* - iL_\omega^*\partial\bar\partial + i\partial L_\omega^*\bar\partial.\end{aligned}$$

Because $\partial\bar\partial = -\bar\partial\partial$, these two expressions are equal. □

Theorem 9.43. *On every Kähler manifold,* $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$.

Proof. Just combine Lemmas 9.41 and 9.42. □

Here is the main theorem in this section.

Theorem 9.44 (Hodge Theorem for Kähler Manifolds). *Let M be a compact n-dimensional Kähler manifold.*

(a) HODGE DECOMPOSITION: *For each $k = 0, \dots, 2n$, the harmonic forms of degree k have the following direct sum decomposition:*

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M).$$

Therefore, the de Rham groups have a corresponding decomposition

$$H^k_{\mathrm{dR}}(M;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M),$$

and the Betti numbers and Hodge numbers of M are related by

$$b^k(M) = h^{k,0}(M) + h^{k-1,1}(M) + \cdots + h^{1,k-1}(M) + h^{0,k}(M).$$

(b) HODGE DUALITY: *For each p and q, conjugation gives a conjugate-linear bijection from $\mathcal{H}^{p,q}(M)$ to $\mathcal{H}^{q,p}(M)$. Thus $h^{p,q}(M) = h^{q,p}(M)$.*

Proof. Let α be a harmonic k-form, and for each $p + q = k$ let $\alpha^{p,q}$ be its projection onto $\mathcal{E}^{p,q}(M)$. Because $\Delta_d = 2\Delta_{\bar\partial}$, which maps $\mathcal{E}^{p,q}$ to itself, it follows that each component $\alpha^{p,q}$ is $\bar\partial$-harmonic and therefore lies in $\mathcal{H}^{p,q}(M)$. This shows that $\mathcal{H}^k(M)$ is the sum of the spaces $\mathcal{H}^{p,q}(M)$ for $p + q = k$, and since any two such spaces have trivial intersection, the sum is direct. Hodge duality follows from the fact that conjugation maps $\bar\partial$-harmonic (p, q)-forms to ∂-harmonic (q, p)-forms; but since $\Delta_\partial = \Delta_{\bar\partial}$, these are also $\bar\partial$-harmonic. □

Applications of Hodge Theory

The Hodge numbers of a compact complex n-manifold can be arranged in the following array, called the ***Hodge diamond***:

$$\begin{array}{ccccccc} & & & h^{n,n} & & & \\ & & h^{n,n-1} & & h^{n-1,n} & & \\ & \iddots & & \vdots & & \ddots & \\ h^{n,0} & & \cdots & & \cdots & & h^{0,n} \\ & \ddots & & \vdots & & \iddots & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0}. & & & \end{array}$$

Serre duality shows that the diamond is symmetric under 180° rotation: $h^{p,q} = h^{n-p,n-q}$. If the manifold is Kähler, then Hodge duality shows that the diamond is symmetric about its vertical axis ($h^{p,q} = h^{q,p}$), and combining these two symmetries shows that it is also symmetric about its horizontal axis: $h^{p,q} = h^{n-q,n-p}$. In addition, in the Kähler case, the sum of the Hodge numbers in each horizontal row equals the corresponding Betti number. This leads to additional topological obstructions to the existence of a Kähler metric, described in the next corollary.

Corollary 9.45. *On a compact Kähler manifold, the odd Betti numbers are even and the even Betti numbers are positive.*

Proof. The statement about the even Betti numbers is Theorem 8.18. If $k = 2m+1$ is odd, the Hodge decomposition theorem gives

$$b^k = \left(h^{2m+1,0} + h^{2m,1} + \cdots + h^{m+1,1}\right) + \left(h^{m,m+1} + \cdots + h^{1,2m} + h^{0,2m+1}\right),$$

and Hodge duality shows that the two sums in parentheses are equal. (If $k > \dim_{\mathbb{C}} M$, some of the numbers in this expression are automatically zero, but the result still holds.) □

We have already seen examples of compact complex manifolds (the Hopf manifolds) that admit no Kähler structure because their second Betti numbers are zero (Example 8.20). An application of the requirement that odd Betti numbers are even was discovered in 1976 by William Thurston. If (M, ω) is a symplectic manifold (a smooth manifold M endowed with a symplectic form ω), an almost complex structure J on M is said to be ***compatible with*** $\boldsymbol{\omega}$ if $\omega(JX, JY) = \omega(X, Y)$ and $\omega(X, JX) > 0$ for all X and Y. If in addition J is integrable, then $g(X, Y) = \omega(X, JY)$ is a Kähler metric. It can be shown that every symplectic manifold admits a compatible almost complex structure, and it had been conjectured for some time that perhaps every compact symplectic manifold had a compatible Kähler structure. A counterexample, called the *Kodaira–Thurston manifold*, was discovered in 1976 by William Thurston: it is a compact symplectic 4-manifold whose first Betti number is 3, so it does not have any Kähler structure. See [**CdS01**, p. 121] for details.

Here is another important application.

Theorem 9.46 (Hard Lefschetz Theorem). *Let M be a compact n-dimensional Kähler manifold with Kähler form ω, and let L_ω be its Lefschetz operator. Then for each k, L_ω^k descends to an isomorphism from $H^{n-k}_{\mathrm{dR}}(M;\mathbb{C})$ to $H^{n+k}_{\mathrm{dR}}(M;\mathbb{C})$.*

Proof. Because ω is closed, $d(L_\omega\alpha) = d(\omega\wedge\alpha) = \omega\wedge d\alpha = L_\omega(d\alpha)$ for every differential form α, so L_ω commutes with d. The same argument shows that L_ω commutes with $\bar\partial$. Using the Kähler identities together with the fact that $\bar\partial\partial+\partial\bar\partial = 0$, we can show that L_ω also commutes with the Dolbeault Laplacian:

$$\begin{aligned}\Delta_{\bar\partial}L_\omega\alpha &= \bar\partial\bar\partial^* L_\omega\alpha + \bar\partial^*\bar\partial L_\omega\alpha\\ &= \bar\partial(L_\omega\bar\partial^*\alpha + i\partial\alpha) + \bar\partial^* L_\omega(\bar\partial\alpha)\\ &= L_\omega\bar\partial\bar\partial^*\alpha + i\bar\partial\partial\alpha + L_\omega\bar\partial^*\bar\partial\alpha + i\partial\bar\partial\alpha\\ &= L_\omega\Delta_{\bar\partial}\alpha.\end{aligned}$$

Since $\Delta_d = 2\Delta_{\bar\partial}$, L_ω also commutes with the Hodge Laplacian Δ_d.

The fact that L_ω commutes with d implies that it descends to a linear map from $H^k_{\mathrm{dR}}(M;\mathbb{C})$ to $H^{k+2}_{\mathrm{dR}}(M;\mathbb{C})$ (still denoted by L_ω), defined by $L_\omega([\alpha]) = [\omega\wedge\alpha]$. To show that $L_\omega^k : H^{n-k}_{\mathrm{dR}}(M;\mathbb{C}) \to H^{n+k}_{\mathrm{dR}}(M;\mathbb{C})$ is injective, suppose $L_\omega^k[\alpha] = 0$ for $[\alpha]\in H^{n-k}_{\mathrm{dR}}(M)$. By the Hodge theorem, we can choose α to be the harmonic representative of its cohomology class. Then $L_\omega^k\alpha = d\beta$ for some $(n+k-1)$-form β. Because L_ω commutes with Δ_d, we have $\Delta_d(d\beta) = \Delta_d(L_\omega^k\alpha) = L_\omega^k(\Delta_d\alpha) = 0$. Thus $d\beta$ is harmonic. But all harmonic forms are in the kernel of d^*, which is orthogonal to the image of d, so $d\beta$ is both harmonic and orthogonal to the space of harmonic forms; thus it is zero. Because $L_\omega^k\alpha = 0$, injectivity of L_ω^k on $(n-k)$-forms (Lemma 9.39(b)) implies $\alpha = 0$.

To show that L_ω^k is surjective, let $[\beta]\in H^{n+k}_{\mathrm{dR}}(M;\mathbb{C})$ be arbitrary. Again, we can assume β is the harmonic representative of its cohomology class. Lemma 9.39(b) shows that there is some $(n-k)$-form α such that $L_\omega^k\alpha = \beta$. Then $L_\omega^k(\Delta_d\alpha) = \Delta_d(L_\omega^k\alpha) = \Delta_d\beta = 0$, and injectivity of L_ω^k shows that $\Delta_d\alpha = 0$. Thus α represents a cohomology class such that $L_\omega^k[\alpha] = [\beta]$. □

This result was claimed by Solomon Lefschetz in 1924 [**Lef24**], with a proof that is generally considered to be incorrect, so the theorem was considered “hard” by Lefschetz’s contemporaries. (According to the Italian-American mathematician Gian-Carlo Rota, who was an undergraduate at Princeton when Lefschetz was chair of the Mathematics Department there, it was said that Lefschetz “had never given a completely correct proof, but had never made a wrong guess either" [**Rot08**, p. 19].) As you can see, with our modern machinery of Hodge theory and Kähler identities the proof is quite straightforward; but the nickname of the theorem has stuck anyway.

One immediate consequence of the hard Lefschetz theorem is the following additional topological constraint on Kähler manifolds.

Corollary 9.47. *Let M be a compact Kähler n-manifold. The Lefschetz operator $L_\omega\colon H^k_{\mathrm{dR}}(M;\mathbb{C}) \to H^{k+2}_{\mathrm{dR}}(M;\mathbb{C})$ is injective for $k \le n-1$ and surjective for $k \ge n-1$. Thus the Betti numbers of M satisfy $b_k(M) \le b_{k+2}(M)$ for $k \le n-1$, and $b_k(M) \ge b_{k+2}(M)$ for $k \ge n-1$.* □

Because the Kähler form is of type $(1,1)$, it follows that L_ω maps $H^{p,q}(M)$ to $H^{p+1,q+1}(M)$, so the following corollary is also immediate.

Corollary 9.48. *Let M be a compact Kähler n-manifold and let L_ω be its Lefschetz operator. For each p and q, the operator L_ω^{n-p-q} restricts to an isomorphism from $H^{p,q}(M)$ to $H^{n-q,n-p}(M)$. Thus $L_\omega\colon H^{p,q}(M) \to H^{p+1,q+1}(M)$ is injective for $p+q \le n-1$ and surjective for $p+q \ge n-1$; and the Hodge numbers satisfy $h^{p,q}(M) \le h^{p+1,q+1}(M)$ for $p+q \le n-1$ and $h^{p,q}(M) \ge h^{p+1,q+1}(M)$ for $p+q \ge n-1$.* □

Note that this corollary gives another proof that the Hodge diamond is symmetric about its horizontal axis (which we previously deduced from Serre duality and Hodge symmetry). In addition, it says that in order for a compact complex manifold to admit a Kähler metric, its Hodge diamond must have the property that the Hodge numbers in each vertical column are nondecreasing below the horizontal axis and nonincreasing above it.

As another application of Hodge theory, we can determine all the Hodge numbers of compact Riemann surfaces and complex projective spaces.

Proposition 9.49 (Hodge Numbers of a Riemann Surface). *Let M be a connected compact Riemann surface of genus g. The Hodge numbers of M are $h^{0,0} = h^{1,1} = 1$ and $h^{0,1} = h^{1,0} = g$.*

Proof. Example 8.13 showed that M admits a Kähler metric. Since M is connected and compact, $H^0_{\mathrm{dR}}(M;\mathbb{C}) \cong H^2_{\mathrm{dR}}(M;\mathbb{C}) \cong \mathbb{C}$ [**LeeSM**, Prop. 17.6 and Thm. 17.31]. Thus the Hodge decomposition theorem gives $h^{0,0} = b^0 = 1$ and $h^{1,1} = b^2 = 1$.

On the other hand, $H_1(M)$ is a free abelian group of rank $2g$ [**LeeTM**, Cor. 13.15], so by the universal coefficient theorem, $H^1_{\mathrm{dR}}(M;\mathbb{C}) \cong \mathrm{Hom}(H_1(M),\mathbb{C}) \cong \mathbb{C}^{2g}$. The statement about $h^{0,1}$ and $h^{1,0}$ then follows from the Hodge decomposition theorem. □

Proposition 9.50 (Hodge Numbers of Projective Spaces). *The Hodge numbers of* $\mathbb{CP}^n$ *are* $h^{p,q} = 0$ *for* $p \neq q$*, and* $h^{p,p} = 1$ *for* $0 \leq p \leq n$*.*

Proof. From algebraic topology, $H_k(\mathbb{CP}^n) \cong \mathbb{Z}$ when k is even and $0 \leq k \leq 2n$, and it is zero otherwise (see, for example, [**LeeTM**, Example 13.35(a)] or [**Hat02**, p. 140]). Thus $H^k_{\mathrm{dR}}(\mathbb{CP}^n; \mathbb{C}) \cong \mathrm{Hom}(H_k(\mathbb{CP}^n), \mathbb{C})$ is 1-dimensional for k even and $0 \leq k \leq 2n$, and zero otherwise.

If any one of the Hodge numbers $h^{p,q}$ were nonzero for $p \neq q$, then the corresponding Betti number would satisfy $b^{p+q} \geq h^{p,q} + h^{q,p} \geq 2$, so only $h^{p,p}$ can be nonzero. The Hodge decomposition theorem then shows that $h^{p,p} = b^{2p} = 1$ for $0 \leq p \leq n$. □

For our next application, recall that smooth complex line bundles on a complex manifold are classified up to smooth isomorphism by their sheaf-theoretic Chern classes, $c(L) \in H^2_{\mathrm{Sing}}(M; \mathbb{Z})$. In most cases, the classification of holomorphic line bundles is different—there may be smooth line bundles that have no holomorphic structure, or ones that have two or more inequivalent holomorphic structures. The next proposition says, though, that for projective spaces, the holomorphic classification is the same as the smooth classification.

Proposition 9.51 (Classification of Line Bundles on Projective Space). *For each projective space* $\mathbb{CP}^n$*, the sheaf-theoretic Chern class map*

$$c\colon \mathrm{Pic}(\mathbb{CP}^n) \to H^2_{\mathrm{Sing}}(\mathbb{CP}^n; \mathbb{Z})$$

is an isomorphism. Thus $\mathrm{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$ *and* $\mathrm{Pic}^0(\mathbb{CP}^n) = \{0\}$*. It follows that every smooth complex line bundle on* $\mathbb{CP}^n$ *has a unique holomorphic structure up to isomorphism, and every holomorphic line bundle is isomorphic to* H^d *for some integer* d*.*

Proof. The long exact sequence associated with the exponential sheaf sequence (5.20) contains the following portion:

$$H^1(\mathbb{CP}^n; \mathcal{O}) \to H^1(\mathbb{CP}^n; \mathcal{O}^*) \xrightarrow{\delta_*} H^2(\mathbb{CP}^n; \underline{\mathbb{Z}}) \to H^2(\mathbb{CP}^n; \mathcal{O}).$$

By the Dolbeault theorem, the groups on the left and right ends are isomorphic to $H^{0,1}(\mathbb{CP}^n)$ and $H^{0,2}(\mathbb{CP}^n)$, respectively, which are both zero by Proposition 9.50. Since $H^1(\mathbb{CP}^n; \mathcal{O}^*)$ is isomorphic to the Picard group of $\mathbb{CP}^n$ and the Chern class map is $c = -\delta_*$, it follows that c is an isomorphism from $\mathrm{Pic}(\mathbb{CP}^n)$ to $H^2(\mathbb{CP}^n; \underline{\mathbb{Z}}) \cong H^2_{\mathrm{Sing}}(\mathbb{CP}^n; \mathbb{Z})$.

As noted in the proof of Proposition 9.50, $H_2(\mathbb{CP}^n) \cong \mathbb{Z}$ and $H_1(\mathbb{CP}^n) = 0$, so it follows from the universal coefficient theorem that $H^2_{\mathrm{Sing}}(\mathbb{CP}^n; \mathbb{Z}) \cong \mathrm{Hom}(H_2(\mathbb{CP}^n), \mathbb{Z}) \cong \mathbb{Z}$. Thus $\mathrm{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$, and since the Chern class map is injective, its kernel $\mathrm{Pic}^0(\mathbb{CP}^n)$ is trivial. Thus holomorphic line bundles on $\mathbb{CP}^n$

are classified by their Chern classes. Since smooth line bundles are also classified by their Chern classes (Thm. 6.29), it follows that every smooth line bundle has exactly one holomorphic structure up to isomorphism.

It remains only to prove that every holomorphic bundle is isomorphic to H^d for some d. In the case of $\mathbb{CP}^1$, Proposition 6.30 shows that smooth line bundles are classified by their degrees, and since we now know that the smooth classification is the same as the holomorphic classification, it follows that there is exactly one holomorphic line bundle on $\mathbb{CP}^1$ of each degree (up to isomorphism). Since H^d has degree d by Example 7.27, this shows that every line bundle on $\mathbb{CP}^1$ is isomorphic to H^d for some d.

For $\mathbb{CP}^n$ with $n > 1$, we know from the first part of the proof that the Picard group is isomorphic to $\mathbb{Z}$, so there is a holomorphic line bundle $L \to \mathbb{CP}^n$ that represents a generator of the group. The hyperplane bundle H is thus isomorphic to L^k for some integer k. Problem 3-2 shows that H pulls back to the hyperplane bundle of $\mathbb{CP}^1$ under an embedding $F\colon \mathbb{CP}^1 \to \mathbb{CP}^n$; and L pulls back to a bundle $F^*L \to \mathbb{CP}^1$ whose kth tensor power is isomorphic to F^*H. Since F^*H generates $\operatorname{Pic}(\mathbb{CP}^1)$ by the argument in the preceding paragraph, it follows that k must be ± 1, which shows that H generates $\operatorname{Pic}(\mathbb{CP}^n)$. □

Now we can prove Proposition 2.31, which asserts that every nonsingular projective algebraic hypersurface is defined by a single homogeneous polynomial. It is a consequence of the following slightly more general result.

Corollary 9.52. *If $S \subseteq \mathbb{CP}^n$ is a closed codimension-1 complex submanifold, then S is the variety defined by a single homogeneous polynomial $p\colon \mathbb{C}^{n+1} \to \mathbb{C}$.*

Proof. Given such an S, Theorem 3.39 shows that there is a holomorphic line bundle $L_S \to \mathbb{CP}^n$ and a holomorphic section $\sigma\colon \mathbb{CP}^n \to L_S$ that vanishes simply on S and nowhere else. By Proposition 9.51, L_S is isomorphic to H^d for some integer d, which must be positive because L_S has a nontrivial holomorphic section that vanishes somewhere. Theorem 3.36 showed that σ is the section determined by a homogeneous polynomial p of degree d on $\mathbb{C}^{n+1}$, and thus S is the variety determined by p. □

Recall Chow's theorem (Thm. 2.29), which says that every closed complex submanifold of $\mathbb{CP}^n$ is algebraic. We can now prove it in the special case of hypersurfaces.

Corollary 9.53 (Chow's Theorem for Hypersurfaces). *Suppose $S \subseteq \mathbb{CP}^n$ is a closed complex submanifold of codimension 1. Then S is algebraic.*

Proof. This follows immediately from Corollary 9.52. □

Corollary 9.54 (Projective Hypersurfaces are Connected). *Suppose $S \subseteq \mathbb{CP}^n$ is a closed complex submanifold of codimension* 1*. Then S is connected.*

Proof. Assume for the sake of contradiction that S is disconnected, and write $S = S_1 \cup S_2$, where S_1, S_2 are open and closed in S and disjoint. Since S_1 and S_2 are closed complex hypersurfaces in $\mathbb{CP}^n$, Corollary 9.52 shows that each is cut out by a single homogeneous polynomial. But then Lemma 2.40 implies they must intersect, which is a contradiction. □

Corollary 9.55 (Holomorphic Forms are Closed and Harmonic). *Let η be a smooth $(p,0)$-form on a compact Kähler manifold. The following are equivalent:*

(a) *η is holomorphic.*

(b) *η is harmonic.*

(c) *η is closed.*

Proof. First assume η is holomorphic, that is, $\bar\partial\eta = 0$. Proposition 9.25(c) shows that η is $\bar\partial$-harmonic, and therefore also d-harmonic by Theorem 9.43. Next, if η is harmonic, it satisfies $d\eta = 0$ by Proposition 9.16(b). Finally, if η is closed, then $0 = d\eta = \partial\eta + \bar\partial\eta$, and taking the $(p,1)$ part of this equation shows that η is $\bar\partial$-closed and thus holomorphic. □

Example 9.56 (Iwasawa Manifolds are Not Kähler). Let $M = G/\Gamma$ be an Iwasawa manifold as described in Example 1.20. The holomorphic 1-form $dz^3 - z^2\, dz^1$ is right-invariant on G and therefore preserved by the right action of Γ, so it descends to a holomorphic form on G/Γ which is not closed. Thus G/Γ has no Kähler metric. //

The local $\partial\bar\partial$-lemma (Cor. 4.15) showed that every closed (p,q)-form is *locally* in the image of $i\partial\bar\partial$. The next corollary shows that on a compact Kähler manifold, the same is true globally provided the form is exact.

Corollary 9.57 (Global $\partial\bar\partial$-Lemma). *Suppose M is a compact Kähler manifold and θ is a d-exact (p,q)-form on M, with p and q both positive. Then there exists a smooth $(p-1,q-1)$-form α defined on all of M such that $\theta = i\partial\bar\partial\alpha$. If θ is a real (p,p)-form, then α can be chosen to be real.*

Proof. Because θ is exact, there is a complex-valued $(p+q-1)$-form η such that $\theta = d\eta$. As in the proof of the local $\partial\bar\partial$-lemma (Corollary 4.15), we can assume η decomposes as $\eta = \eta^{p,q-1} + \eta^{p-1,q}$. We can also decompose $\theta = d\eta$ into bidegrees as

$$\theta = \bar\partial\eta^{p-1,q} + \left(\partial\eta^{p-1,q} + \bar\partial\eta^{p,q-1}\right) + \partial\eta^{p,q-1},$$

and the fact that θ is of type (p,q) implies that the first and last terms on the right-hand side are zero. In particular, both $\eta^{p-1,q}$ and $\overline{\eta^{p,q-1}}$ are $\bar\partial$-closed. By the Hodge–Dolbeault theorem, there are $\bar\partial$-harmonic representatives σ and ρ for the same Dolbeault cohomology classes as $\eta^{p-1,q}$ and $\overline{\eta^{p,q-1}}$, respectively, which means there exist a $(p-1,q-1)$-form β and a $(q-1,p-1)$-form γ such that

$$\begin{aligned} \eta^{p-1,q} &= \sigma + \bar\partial\beta, \\ \overline{\eta^{p,q-1}} &= \rho + \bar\partial\gamma. \end{aligned}$$

Since $\bar\partial$-harmonic forms on a Kähler manifold are also ∂-harmonic, it follows that σ and ρ are both ∂-closed and $\bar\partial$-closed. Therefore, if we set $\alpha = i\bar\gamma - i\beta$, we have

$$\begin{aligned} i\partial\bar\partial\alpha &= i\partial\bar\partial(i\bar\gamma) - i\partial\bar\partial(i\beta) \\ &= \bar\partial(\partial\bar\gamma) + \partial(\bar\partial\beta) \\ &= \bar\partial\big(\eta^{p,q-1} - \bar\rho\big) + \partial\big(\eta^{p-1,q} - \sigma\big) \\ &= \theta. \end{aligned}$$

As in the local case, if θ is a real (p,p)-form, we can choose η to be real and then take $\gamma = \beta$, so that $\alpha = i\bar\beta - i\beta$ is real. □

Corollary 9.58. *Let M be a compact Kähler manifold and $L \to M$ a holomorphic line bundle. If η is any closed real $(1,1)$-form representing the cohomology class $c_1^{\mathbb{R}}(L)$, there is a Hermitian fiber metric on L whose Chern connection has curvature Θ satisfying $\frac{i}{2\pi}\Theta = \eta$.*

Proof. Begin by choosing an arbitrary Hermitian fiber metric h on L, and let Θ be the curvature form of its Chern connection. Both $\frac{i}{2\pi}\Theta$ and η are real $(1,1)$-forms representing the first real Chern class of L, so they differ by an exact form. Therefore, by the global $\partial\bar\partial$-lemma, we can find a smooth real function u that satisfies

$$i\partial\bar\partial\big(\tfrac{1}{2\pi}u\big) = \tfrac{i}{2\pi}\Theta - \eta.$$

Define a new fiber metric by $\widetilde{h} = e^u h$, and let $\widetilde{\Theta}$ be the curvature form of its Chern connection. In terms of any holomorphic local frame s, (7.20) shows that the curvature forms Θ and $\widetilde{\Theta}$ can be expressed as

$$\begin{aligned} \Theta &= \bar\partial\partial \log |s|_h^2, \\ \widetilde{\Theta} &= \bar\partial\partial \log |s|_{\widetilde{h}}^2 = \bar\partial\partial \log(e^u |s|_h^2) = \bar\partial\partial u + \Theta = \tfrac{2\pi}{i}\eta. \end{aligned}$$

Thus $\widetilde{h}$ satisfies the conclusion of the corollary. □

The Lefschetz Theorem on $(1,1)$*-Classes*

For our next application of Hodge theory, we return to the question of classifying holomorphic line bundles. Let M be a complex manifold and $L \to M$ be a holomorphic line bundle, and let $c_1^{\mathbb{R}}(L) \in H^k_{\mathrm{dR}}(M;\mathbb{R}) \subseteq H^k_{\mathrm{dR}}(M;\mathbb{C})$ be its first real Chern class.

By Theorem 7.14, when $c_1^{\mathbb{R}}(L)$ is identified with an element of $H^2(M;\underline{\mathbb{C}})$ under the de Rham isomorphism, it is the image of the sheaf-theoretic Chern class under the coefficient homomorphism $H^2(M;\underline{\mathbb{Z}}) \to H^2(M;\underline{\mathbb{C}})$. Thus it is *integral* in the sense defined in Chapter 6. In addition, it is always represented by a form $\frac{i}{2\pi}\Theta$ of type $(1,1)$. If M is a compact Kähler manifold, let us say that a cohomology class in $H^k(M;\underline{\mathbb{C}}) \cong H^k_{\mathrm{dR}}(M;\mathbb{C})$ is of ***type*** $\boldsymbol{(p,q)}$ if it lies in the direct summand $H^{p,q}(M)$ under the Hodge decomposition.

Theorem 9.59 (Lefschetz Theorem on (1, 1)-Classes). *Let M be a compact Kähler manifold. Every integral cohomology class of type* $(1,1)$ *is the first real Chern class of a holomorphic line bundle.*

Proof. Let $[\eta] \in H^2_{\mathrm{dR}}(M;\mathbb{C})$ be both integral and of type $(1,1)$. The hypothesis that $[\eta]$ is integral means there is a class $\gamma_0 \in H^2(M;\underline{\mathbb{Z}})$ that maps to the image of $[\eta]$ in $H^2(M;\underline{\mathbb{C}})$ under the coefficient homomorphism. From the long exact sequence associated to the exponential sheaf sequence, we have

$$\mathrm{Pic}(M) \to H^2(M;\underline{\mathbb{Z}}) \xrightarrow{i_*} H^2(M;\mathscr{O}),$$

where $i : \underline{\mathbb{Z}} \hookrightarrow \mathscr{O}$ is inclusion. We will show that $i_*(\gamma_0) = 0$; by exactness of the sequence above, this implies that γ_0 is the sheaf-theoretic Chern class of a holomorphic line bundle, and then Theorem 7.14 shows that $[\eta]$ is its first real Chern class.

Consider the following commutative diagram of sheaf inclusions:

$$\begin{array}{ccc} & & \underline{\mathbb{C}} \\ & {\scriptstyle j}\nearrow & \big\downarrow{\scriptstyle k} \\ \underline{\mathbb{Z}} & \xrightarrow[i]{} & \mathscr{O}. \end{array}$$

This yields the following diagram of cohomology groups:

$$\begin{array}{ccccc} & & H^2(M;\underline{\mathbb{C}}) & \xleftarrow[\mathscr{R}]{} & H^2_{\mathrm{dR}}(M;\mathbb{C}) \\ & {\scriptstyle j_*}\nearrow & \big\downarrow{\scriptstyle k_*} & & \big\downarrow{\scriptstyle \pi^{0,2}} \\ H^2(M;\underline{\mathbb{Z}}) & \xrightarrow[i_*]{} & H^2(M;\mathscr{O}) & \xleftarrow[\mathscr{D}]{} & H^{0,2}(M), \end{array} \tag{9.43}$$

where $\mathscr{R}$ and $\mathscr{D}$ are the isomorphisms given by the de Rham–Weil theorem. The left-hand triangle commutes by functoriality of sheaf cohomology, and the right-hand square commutes by Proposition 6.21.

In our current situation, we are given $\gamma_0 \in H^2(M;\underline{\mathbb{Z}})$ and $[\eta] \in H^2_{\mathrm{dR}}(M;\mathbb{C})$ such that $j_*\gamma_0 = \mathscr{R}[\eta]$ and η is of type $(1,1)$. This implies $\pi^{0,2}[\eta] = 0$, and therefore

$$i_*\gamma_0 = k_*j_*\gamma_0 = k_*\mathscr{R}[\eta] = \mathscr{D}\pi^{0,2}[\eta] = 0. \qquad \square$$

The Riemann–Roch Theorem

As another application of Hodge theory, we will prove two important theorems regarding holomorphic line bundles on Riemann surfaces. The first is a proof that the mapping from divisors to the Picard group described in Theorem 3.41 is surjective; the second is the *Riemann–Roch theorem*, which gives a formula for the dimension of the space of sections of a holomorphic line bundle.

Both proofs make use of the following lemma about Euler characteristics of sheaves of sections. Recall the definition of the Euler characteristic of a sheaf $\mathscr{S}$ of vector spaces: $\chi(\mathscr{S}) = \sum_k (-1)^k \dim H^k(M;\mathscr{S})$, provided $H^k(M;\mathscr{S})$ is finite-dimensional for all k and zero for for all but finitely many values of k.

Lemma 9.60. *Let M be a compact Riemann surface and $L \to M$ be a holomorphic line bundle. For any $p \in M$, let $L_{\{p\}}$ denote the point bundle associated with the hypersurface $\{p\}$. Then*

$$\chi(\mathscr{O}(L \otimes L_{\{p\}})) = \chi(\mathscr{O}(L)) + 1, \tag{9.44}$$

$$\chi(\mathscr{O}(L \otimes L^*_{\{p\}})) = \chi(\mathscr{O}(L)) - 1. \tag{9.45}$$

Proof. Note that these Euler characteristics are all well defined due to Proposition 9.36. We begin by proving (9.45). Consider the following exact sheaf sequence:

$$0 \to \mathscr{I}_{\{p\}}(L) \hookrightarrow \mathscr{O}(L) \xrightarrow{e} (L_p)_p \to 0, \tag{9.46}$$

where $\mathscr{I}_{\{p\}}(L)$ is the sheaf of holomorphic sections of L that vanish at p; $(L_p)_p$ is the skyscraper sheaf whose stalk at p is the fiber L_p, with all other stalks zero; and e is the sheaf morphism defined be evaluating a section at p: for any open set $U \subseteq M$, if $p \in U$, then $e_U(\sigma) = \sigma(p)$, and otherwise e_U is the zero map.

Proposition 5.16 showed that $\mathscr{I}_{\{p\}}(L) \cong \mathscr{O}(L\otimes L^*_{\{p\}})$, so its Euler characteristic is defined. Example 6.13(c) showed that $(L_p)_p$ is a fine sheaf, so its cohomology groups are all zero except for $H^0(M;(L_p)_p) \cong L_p$; thus $\chi((L_p)_p) = \dim L_p = 1$.

Proposition 6.10 then implies

$$\chi(\mathscr{O}(L)) = \chi(\mathscr{I}_{\{p\}}(L)) + \chi((L_p)_p) = \chi(\mathscr{O}(L \otimes L^*_{\{p\}})) + 1, \tag{9.47}$$

which is equivalent to (9.45). Then (9.44) follows by applying (9.45) to the sheaf $L' = L \otimes L_{\{p\}}$ and noting that $L' \otimes L^*_{\{p\}} \cong L$. $\square$

Here is our first main result about Riemann surfaces.

Theorem 9.61. *Let M be a compact Riemann surface. The mapping from divisors to the Picard group of M that sends a divisor to the isomorphism class of its associated line bundle is surjective.*

Proof. Let $L \to M$ be a holomorphic line bundle. The Dolbeault theorem shows that for any nonnegative integer q, $H^q(M; \mathcal{O}(L)) \cong H^{0,q}(M; L)$. Since there are no nontrivial $(0, q)$ forms on a Riemann surface for $q \geq 2$, this implies $H^q(M; \mathcal{O}(L)) = 0$ for $q \geq 2$, and thus $\chi(\mathcal{O}(L)) = \dim H^0(M; \mathcal{O}(L)) - \dim H^1(M; \mathcal{O}(L))$.

Let $p \in M$ be arbitrary. An easy induction based on Lemma 9.60 shows that $\chi(L \otimes L^k_{\{p\}}) = \chi(L) + k$ for any positive integer k. Thus

$$\begin{aligned}\dim \mathcal{O}(M; L \otimes L^k_{\{p\}}) &= \dim H^0(M; L \otimes L^k_{\{p\}}) \\ &= \chi(\mathcal{O}(L \otimes L^k_{\{p\}})) + \dim H^1(M; L \otimes L^k_{\{p\}}) \\ &\geq \chi(\mathcal{O}(L \otimes L^k_{\{p\}})) = \chi(\mathcal{O}(L)) + k,\end{aligned}$$

which is strictly positive for k large enough. This implies that $L \otimes L^k_{\{p\}}$ has a nontrivial holomorphic section σ. The uniqueness assertion of Theorem 3.41 shows that $L \otimes L^k_{\{p\}} \cong L_{D_1}$, where D_1 is the divisor of σ.

If we let D_2 be the divisor kp, the fact that the mapping from divisors to the Picard group is a homomorphism implies that $L^k_{\{p\}} \cong L_{D_2}$ and thus $(L^k_{\{p\}})^* \cong L_{-D_2}$. It follows that

$$L \cong (L \otimes L^k_{\{p\}}) \otimes (L^k_{\{p\}})^* \cong L_{D_1} \otimes L_{-D_2} \cong L_{D_1 - D_2}. \qquad \square$$

Corollary 9.62. *Every holomorphic line bundle on a connected compact Riemann surface admits a nontrivial meromorphic section.*

Proof. Let $L \to M$ be such a bundle. Theorem 9.61 shows that there is a divisor D such that $L \cong L_D$, and then Theorem 3.41 shows that there is a meromorphic section σ of L whose divisor is D. Since σ has only finitely many zeros, it is not trivial. $\square$

Recall the notations from Chapter 3: for a connected compact Riemann surface M, $\mathrm{Div}(M)$ is the group of divisors on M, $\mathrm{Cl}(M)$ (the divisor class group of M) is the group of divisors on M modulo linear equivalence, and $\mathrm{Pic}(M)$ (the Picard group of M) is the group of isomorphism classes of holomorphic line bundles on M. We also define $\mathrm{Div}^0(M) \subseteq \mathrm{Div}(M)$ to be the subgroup of divisors of degree zero, and $\mathrm{Cl}^0(M) \subseteq \mathrm{Cl}(M)$ to be the image of $\mathrm{Div}^0(M)$ in $\mathrm{Cl}(M)$. As in Chapter 6, $\mathrm{Pic}^0(M)$ (the Picard variety of M) is the subgroup of $\mathrm{Pic}(M)$ consisting of isomorphism classes of line bundles with zero Chern class; for a Riemann surface, these are exactly the line bundles of degree zero.

Corollary 9.63 (The Divisor Class Group and the Picard Group). *Suppose M is a connected compact Riemann surface. The map from* $\operatorname{Div}(M)$ *to* $\operatorname{Pic}(M)$ *that sends a divisor to the isomorphism class of its associated line bundle descends to an isomorphism from* $\operatorname{Cl}(M)$ *to* $\operatorname{Pic}(M)$*, taking* $\operatorname{Cl}^0(M)$ *to* $\operatorname{Pic}^0(M)$*.*

Proof. Theorem 3.41 showed that the natural map from $\operatorname{Div}(M)$ to $\operatorname{Pic}(M)$ descends to an injective homomorphism from $\operatorname{Cl}(M)$ to $\operatorname{Pic}(M)$, and Theorem 9.61 shows that it is surjective. Because the degree of a divisor is equal to the degree of its associated line bundle by Theorem 7.22, the image of $\operatorname{Cl}^0(M)$ is exactly $\operatorname{Pic}^0(M)$. □

Because of this result, in the algebraic geometry literature, the Picard group is sometimes *defined* to be the group $\operatorname{Cl}(M)$ of divisors modulo linear equivalence.

The following theorem is fundamental to the study of Riemann surfaces. It was first stated in 1857 by Bernhard Riemann [**Rie57**] as an inequality (without the term involving $K \otimes L^*$); the full strength of the theorem was proved in 1865 by Riemann's student Gustav Roch [**Roc65**].

Theorem 9.64 (Riemann–Roch). *Suppose M is a connected compact Riemann surface of genus g, and $L \to M$ is a holomorphic line bundle. Then*

$$\dim \mathcal{O}(M; L) = \deg L + 1 - g + \dim \mathcal{O}(M; K \otimes L^*).$$

Proof. As we observed in the proof of Theorem 9.61, for a line bundle L on a compact Riemann surface M the Euler characteristic reduces to

$$\chi(\mathcal{O}(L)) = \dim H^0(M; \mathcal{O}(L)) - \dim H^1(M; \mathcal{O}(L)).$$

Serre duality shows that $\dim H^1(M; \mathcal{O}(L)) = \dim H^0(M; \mathcal{O}(K \otimes L^*))$; thus

$$\begin{aligned}\chi(\mathcal{O}(L)) &= \dim H^0(M; \mathcal{O}(L)) - \dim H^0(M; \mathcal{O}(K \otimes L^*) \\ &= \dim \mathcal{O}(M; L) - \dim \mathcal{O}(M; K \otimes L^*),\end{aligned}$$

and the Riemann–Roch theorem is equivalent to the claim that

$$\chi(\mathcal{O}(L)) = \deg L + 1 - g \tag{9.48}$$

for every holomorphic line bundle $L \to M$.

Note that we already have a formula for $\dim \mathcal{O}(M; K)$. Because $K \cong \Lambda^{1,0} M$, the holomorphic sections of K are exactly the harmonic $(1,0)$-forms by Corollary 9.55, and then Proposition 9.49 shows that $\dim \mathcal{O}(M; K) = h^{1,0}(M) = g$.

Theorem 9.61 shows that L is isomorphic to the line bundle associated with some divisor D. We begin with the case in which $D = \sum_j n_j p_j$ is effective (meaning that all of its coefficients are nonnegative). We will prove (9.48) by induction on $\deg D = \sum_j n_j$.

The base case is $\deg D = 0$. For an effective divisor this means $D = 0$, so L is the trivial bundle. In that case, $\dim \mathscr{O}(M; L) = 1$ (because the only holomorphic sections of the trivial line bundle are the constants), and $\dim \mathscr{O}(M; K \otimes L^*) = \dim \mathscr{O}(M; K) = g$, so (9.48) holds.

Now let $k \geq 1$, and assume (9.48) holds for every line bundle L_D in which D is effective and has degree less than k. Let D be an effective divisor of degree k, and write D in the form $D = D' + p_0$, where D' has degree $k - 1$ and p_0 is some point in M. Then $L_D \cong L_{D'} \otimes L_{\{p_0\}}$ and $\deg L_D = \deg L_{D'} + 1$.

The inductive hypothesis implies $\chi(\mathscr{O}(L_{D'})) = \deg L_{D'} + 1 - g$, and then Lemma 9.60 gives

$$\chi(\mathscr{O}(L_D)) = \chi(\mathscr{O}(L_{D'})) + 1 = \deg L_D + 1 - g.$$

This completes the proof in the case that D is effective.

For a general divisor D, by separating the terms with positive and negative coefficients we can write $D = D_1 - D_2$, where D_1 and D_2 are effective; thus $L_D \cong L_{D_1} \otimes L^*_{D_2}$. It follows from Lemma 9.60 and induction that $\chi(\mathscr{O}((L_D)) = \chi(\mathscr{O}(L_{D_1})) - \deg L_{D_2}$, and therefore the result of the previous paragraph gives

$$\chi(\mathscr{O}((L_D)) = \deg L_{D_1} + 1 - g - \deg L_{D_2} = \deg L_D + 1 - g. \qquad \square$$

We will see quite a few applications of the Riemann–Roch theorem in the problems at the end of this chapter and the next; here is one to get us started.

Corollary 9.65 (Uniqueness of the Holomorphic Structure on $\mathbb{CP}^1$). *If M is a connected compact Riemann surface of genus* 0*, then M is biholomorphic to* $\mathbb{CP}^1$.

Proof. Suppose M has genus 0. Let $p \in M$ be arbitrary, and let L denote the degree-1 bundle $L_{\{p\}}$. Problem 8-15 shows that $\deg K_M = g - 2 = -2$, so $K_M \otimes L^*$ has degree -3 and therefore no nontrivial holomorphic sections. The Riemann–Roch theorem then shows that $\dim \mathscr{O}(M; L) = 2$, and Proposition 7.25 shows that M is biholomorphic to $\mathbb{CP}^1$. $\square$

The Picard and Albanese Varieties

If M is a complex manifold, recall that the Picard variety $\operatorname{Pic}^0(M) \subseteq \operatorname{Pic}(M)$ is defined as the group of isomorphism classes of line bundles on M whose sheaf-theoretic Chern classes are zero. In this section, we will prove that if M is a compact Kähler manifold, then $\operatorname{Pic}^0(M)$ has a natural structure as a complex torus. We will also introduce another complex torus canonically associated with M, called the *Albanese variety*, and prove that these tori are isomorphic when M is a connected compact Riemann surface.

Theorem 9.66. *Let M be a compact Kähler manifold. The Picard variety* $\operatorname{Pic}^0(M)$ *has a natural structure as a complex torus whose dimension is equal to $h^{0,1}(M)$.*

Proof. The long exact sequence associated with the exponential sheaf sequence (6.35) reads in part

$$H^1(M;\underline{\mathbb{Z}}) \xrightarrow{i_*} H^1(M;\mathcal{O}) \xrightarrow{\varepsilon_*} \operatorname{Pic}(M) \xrightarrow{\delta_*} H^2(M;\underline{\mathbb{Z}}).$$

The subgroup $\operatorname{Pic}^0(M)$ is the kernel of δ_* in the sequence above, and by exactness it is isomorphic to the quotient group $H^1(M;\mathcal{O})/i_*H^1(M;\underline{\mathbb{Z}})$. By the Dolbeault theorem, $H^1(M;\mathcal{O})$ is a complex vector space isomorphic to $H^{0,1}(M)$, whose dimension is $h^{0,1}(M)$. So we need only show that $i_*H^1(M;\underline{\mathbb{Z}})$ is a lattice in this space.

Consider the following commutative diagram of cohomology groups analogous to (9.43):

$$\begin{array}{ccccc} & & H^1(M;\underline{\mathbb{C}}) & \xleftarrow[\mathscr{R}]{} & H^1_{\mathrm{dR}}(M;\mathbb{C}) \\ & \nearrow^{j_*} & \downarrow{\scriptstyle k_*} & & \downarrow{\scriptstyle \pi^{0,1}} \\ H^1(M;\underline{\mathbb{Z}}) & \xrightarrow[i_*]{} & H^1(M;\mathcal{O}) & \xleftarrow[\mathscr{D}]{} & H^{0,1}(M). \end{array} \tag{9.49}$$

Since $\mathscr{D}$ is an isomorphism, it suffices to show that the image of $\pi^{0,1}\circ\mathscr{R}^{-1}\circ j_*$ is a lattice in $H^{0,1}(M)$.

Let $q = h^{0,1}(M) = \dim H^{0,1}(M)$. By the Hodge decomposition theorem, the dimension of $H^1_{\mathrm{dR}}(M;\mathbb{C})$ is $2q$. Lemma 6.27 shows that $\mathscr{R}^{-1}j_*H^1(M;\underline{\mathbb{Z}})$ is a free abelian subgroup of $H^1_{\mathrm{dR}}(M;\mathbb{C})$ of rank $2q$; and we can choose closed real 1-forms $(\eta^1,\dots,\eta^{2q})$ whose cohomology classes form a basis for this subgroup over $\mathbb{Z}$, and also a basis for $H^1_{\mathrm{dR}}(M;\mathbb{C})$ over $\mathbb{C}$. By the Hodge theorem, we may choose them to be harmonic. Then $(\pi^{0,1}\eta^1,\dots,\pi^{0,1}\eta^{2q})$ are $\bar\partial$-harmonic forms whose Dolbeault cohomology classes generate the image of $\pi^{0,1}\circ\mathscr{R}^{-1}\circ j_*$ in $H^{0,1}(M)$. It remains only to prove that these cohomology classes are linearly independent over $\mathbb{R}$.

Suppose there are real constants $a_1,\dots,a_{2q}$ such that $\sum_{j=1}^{2q} a_j\pi^{0,1}\eta^j$ represents the zero Dolbeault cohomology class. Because the harmonic representative of a cohomology class is unique, that means the form $\sum_{j=1}^{2q} a_j\pi^{0,1}\eta^j$ is identically zero, and by conjugation $\sum_{j=1}^{2q} a_j\pi^{1,0}\eta^j = 0$ as well. This implies $\sum_{j=1}^{2q} a_j\eta^j = 0$, and thus all of the a_j's are zero because the η^j's are linearly independent over $\mathbb{C}$. □

The other torus canonically associated with a compact Kähler manifold is defined as follows. Let $\Omega^1(M)$ be the complex vector space of global holomorphic 1-forms on M, and let $\Omega^1(M)^*$ be its dual space, that is, the space of complex-linear functions from $\Omega^1(M)$ to $\mathbb{C}$. Define a homomorphism $\varphi\colon H_1(M)\to\Omega^1(M)^*$ as follows: for each singular homology class $\gamma\in H_1(M)$, let $\varphi(\gamma)\in\Omega^1(M)^*$ be the linear functional

$$\varphi(\gamma)(\eta) = \int_\gamma \eta, \tag{9.50}$$

where the integral is interpreted by integrating over a smooth singular cycle representing γ. The ***Albanese variety of M*** denoted by $\mathrm{Alb}(M)$, is the following quotient group:

$$\mathrm{Alb}(M) = \Omega^1(M)^*/\varphi(H_1(M)).$$

Theorem 9.67. *Let M be a compact Kähler manifold. The Albanese variety $\mathrm{Alb}(M)$ is a complex torus whose dimension is equal to $h^{0,1}(M)$.*

Proof. Because there are no nontrivial $\bar\partial$-exact $(1,0)$-forms, $\Omega^1(M) = H^{1,0}(M)$. Let $q = \dim \Omega^1(M) = h^{1,0}(M) = h^{0,1}(M)$. By the Hodge decomposition theorem, the dimension of $H^1_{\mathrm{dR}}(M;\mathbb{C})$ is $2q$.

Because M is compact, $H_1(M)$ is finitely generated (see [**Hat02**, Corollaries A.8 and A.9]). Let $T \subseteq H_1(M)$ denote the torsion subgroup of $H_1(M)$, so the quotient group $H_1(M)/T$ is a finitely generated free abelian group. By the de Rham and universal coefficient theorems, $H^1_{\mathrm{dR}}(M;\mathbb{C})$ is isomorphic to $\mathrm{Hom}(H_1(M),\mathbb{C})$, which in turn is isomorphic to $\mathrm{Hom}(H_1(M)/T,\mathbb{C})$. Since $H^1_{\mathrm{dR}}(M;\mathbb{C})$ has dimension $2q$, it follows that $H_1(M)/T$ has rank $2q$.

Choose homology classes $(\gamma_1,\dots,\gamma_{2q})$ in $H_1(M)$ that descend to a basis for the free abelian group $H_1(M)/T$, and choose a basis $(\eta^1,\dots,\eta^q)$ for the complex vector space $\Omega^1(M)$. Let $(\varepsilon_1,\dots,\varepsilon_q)$ be the dual basis for $\Omega^1(M)^*$, defined by $\varepsilon_j(\eta^k) = \delta^k_j$. The subgroup $\varphi(H_1(M)) \subseteq \Omega^1(M)^*$ is generated by the $2q$ elements $\varphi(\gamma_1),\dots,\varphi(\gamma_{2q})$. In terms of the dual basis (ε_j), the linear functional $\varphi(\gamma_i)$ has coordinate representation $\varphi(\gamma_i) = \sum_j \Pi^j_i \varepsilon_j$, where $\Pi^j_i = \varphi(\gamma_i)(\eta^j) = \int_{\gamma_i} \eta^j$.

Let Π be the $q \times 2q$ matrix whose entries are Π^j_i; concretely, it is

$$\Pi = \begin{pmatrix} \displaystyle\int_{\gamma_1} \eta^1 & \cdots & \displaystyle\int_{\gamma_{2q}} \eta^1 \\ \vdots & \ddots & \vdots \\ \displaystyle\int_{\gamma_1} \eta^q & \cdots & \displaystyle\int_{\gamma_{2q}} \eta^q \end{pmatrix}.$$

This is called the ***period matrix*** of M with respect to the chosen bases. The columns of Π, called the ***periods*** of M with respect to these bases, represent the linear functionals $\varphi(\gamma_1),\dots,\varphi(\gamma_{2q})$ in terms of the dual basis (ε_j); thus to complete the proof, we just need to show that these columns are linearly independent over $\mathbb{R}$, so they generate a lattice in $\Omega^1(M)^*$, called the ***period lattice***.

Corollary 9.55 shows that each holomorphic 1-form η^j is harmonic. The conjugate forms $\bar{\eta}^j$ are therefore also harmonic, and thus the Hodge decomposition theorem shows that the cohomology classes of $(\eta^1, \dots, \eta^q, \bar{\eta}^1, \dots, \bar{\eta}^q)$ form a basis for $H^1_{\mathrm{dR}}(M; \mathbb{C})$ over $\mathbb{C}$. To show that the elements $\varphi(\gamma_i)$ are independent, suppose $\sum_{i=1}^{2q} a^i \varphi(\gamma_i) = 0$ for some real constants $a^1, \dots, a^{2q}$. This means $\sum_i a^i \int_{\gamma_i} \eta^j = 0$ for each j, and by conjugation $\sum_i a^i \int_{\gamma_i} \bar{\eta}^j = 0$ as well. Since $H^1_{\mathrm{dR}}(M; \mathbb{C}) \cong \mathrm{Hom}(H_1(M)/T; \mathbb{C})$, this means that $\gamma = \sum_i a^i \gamma_i$ is in the kernel of every homomorphism from $H_1(M)/T$ to $\mathbb{C}$; and since $H_1(M)/T$ is a free abelian group, this can only occur if the image of γ in $H_1(M)/T$ is zero. Since the γ_i's form a basis for $H_1(M)/T$, it follows that the coefficients a^i are all zero. Thus the images $\varphi(\gamma_i)$ are $\mathbb{R}$-linearly independent elements of $\Omega^1(M)^*$, so they span a lattice, which shows that $\mathrm{Alb}(M)$ is a complex q-torus. □

When we specialize to Riemann surfaces, these two tori are always isomorphic. The following theorem was essentially proved (in a somewhat different form) by Niels Henrik Abel and Carl Gustav Jacob Jacobi in the first half of the nineteenth century. (See also Problem 9-14.)

Theorem 9.68 (Abel–Jacobi). *Let M be a connected compact Riemann surface. The complex Lie groups $\mathrm{Pic}^0(M)$ and $\mathrm{Alb}(M)$ are holomorphically isomorphic, and both have dimension equal to the genus of M.*

Proof. Example 8.13 showed that every Riemann surface admits Kähler metrics, so fix such a metric on M.

Let q be the genus of M, which is equal to $h^{0,1}(M)$ by Proposition 9.49. If $q = 0$, then both $\mathrm{Pic}^0(M)$ and $\mathrm{Alb}(M)$ are trivial groups. Assume from now on that $q \geq 1$.

The proof of Theorem 9.66 shows that for any real 1-forms $(\eta^1, \dots, \eta^{2q})$ representing a basis for the integral classes in $H^1_{\mathrm{dR}}(M; \mathbb{C})$, $\mathrm{Pic}^0(M)$ is isomorphic to the quotient group $H^{0,1}(M)/\Lambda$, where Λ is the lattice generated by the cohomology classes $\big([\pi^{0,1}\eta^1], \dots, [\pi^{0,1}\eta^{2q}]\big)$. On the other hand, $\mathrm{Alb}(M)$ is the quotient of $\Omega^1(M)^*$ by the lattice Γ generated by the linear functionals $\eta \mapsto \int_{\gamma_i} \eta$ for some homology classes $(\gamma_1, \dots, \gamma_{2q})$ in $H_1(M)$ that form a basis for $H_1(M)/T$ (which is equal to $H_1(M)$ itself in this case).

Since $\Omega^1(M) = H^{1,0}(M)$, it follows from the $n = 1$ case of Serre duality that $H^{0,1}(M)$ is isomorphic to $\Omega^1(M)^*$ via the map $\sigma : H^{0,1}(M) \to \Omega^1(M)^*$ that sends the cohomology class of a $\bar{\partial}$-closed $(0,1)$ form α to the linear functional $\sigma([\alpha])$ defined by

$$\sigma([\alpha])(\eta) = \int_M \alpha \wedge \eta.$$

We just need to show that $\sigma(\Lambda) = \Gamma$. To do so, we begin by introducing a more careful choice of basis for $H_1(M)$.

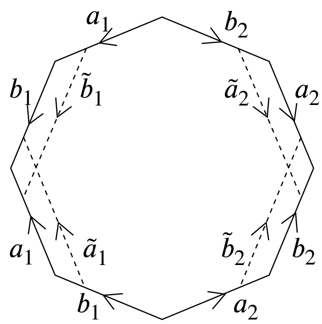

Figure 9.1. A polygon whose quotient is a surface of genus 2

Since M is a compact, connected, orientable topological 2-manifold of genus q, the classification theorem for compact surfaces [**LeeTM**, Thm. 6.15] shows that it is homeomorphic to a quotient space of a $4q$-sided polygon with edges identified in pairs according to the scheme $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_q, b_q, a_q^{-1}, b_q^{-1}$ (see Fig. 9.1 for the case $q = 2$). The images of the $4q$ sides in the quotient space (with suitable parametrizations) are $2q$ simple closed curves (thus singular cycles) whose homology classes represent a basis for the free abelian group $H_1(M)$. (See [**LeeTM**, Thm. 13.14 and Cor. 13.15].)

Any two closed curves in M that are freely homotopic represent the same element in $H_1(M)$. One way to see this is to note that a closed curve $a\colon [0,1] \to M$ descends to a continuous map $\hat{a}\colon \mathbb{S}^1 \to M$, and the homology class represented by the cycle a is equal to $\hat{a}_*(\varepsilon)$, where ε is a generator of the infinite cyclic group $H_1(\mathbb{S}^1)$ and $a_*\colon H_1(\mathbb{S}^1) \to H_1(M)$ is the induced homology homomorphism; then the claim follows from the fact that homotopic continuous maps induce the same homology homomorphism [**LeeTM**, Thm. 13.8]. By replacing the cycles a_j and b_j with freely homotopic cycles $\tilde{a}_j$ and $\tilde{b}_j$ as in Fig. 9.1, we can arrange that for each j, the images of $\tilde{a}_j$ and $\tilde{b}_j$ intersect exactly once, and otherwise the images of all the cycles are pairwise disjoint. Moreover, by the Whitney approximation theorem [**LeeSM**, Thm. 6.26], we can arrange that the cycles $\tilde{a}_j$ and $\tilde{b}_j$ are smooth cycles whose images are embedded smooth submanifolds, and by the transversality homotopy theorem [**LeeSM**, Thm. 6.36] we can arrange that the curves $\tilde{a}_j$ and $\tilde{b}_j$ meet transversely. From now on, we will drop the tildes and refer to these smooth curves as a_j and b_j. Choose smooth constant-speed parametrizations $a_j, b_j\colon [0,1] \to M$ in such a way that $p_j = a_j(0) = b_j(0) = a_j(1) = b_j(1)$ is the point where the curves meet, and the velocity vectors $\big(a_j'(0), b_j'(0)\big) = \big(a_j'(1), b_j'(1)\big)$ form an oriented basis for $T_{p_j}M$.

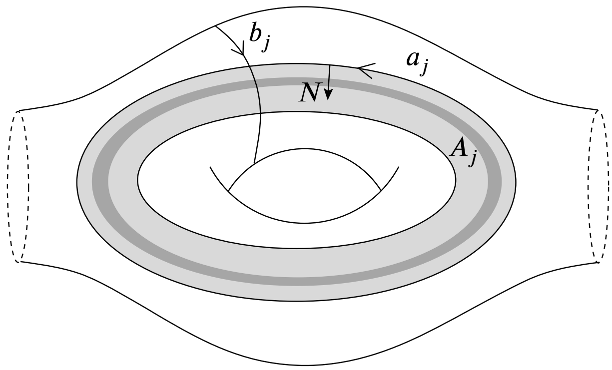

Figure 9.2. The support of α^j (the darker shaded region)

For each j, we will construct closed real 1-forms α^j and β^j on M with the following properties for all $j, k = 1, \dots, n$:

$$\int_{a_j} \alpha^k = \int_{b_j} \beta^k = 0; \tag{9.51}$$

$$\int_{a_j} \beta^k = -\int_{b_j} \alpha^k = \delta_j^k; \tag{9.52}$$

$$\int_{a_j} \eta = \int_M \alpha^j \wedge \eta \quad \text{for every closed 1-form } \eta; \tag{9.53}$$

$$\int_{b_j} \eta = \int_M \beta^j \wedge \eta \quad \text{for every closed 1-form } \eta. \tag{9.54}$$

First we choose a compact subset $A_j \subseteq M$ diffeomorphic to $\mathbb{S}^1 \times [0,1]$, with the image of a_j as one of its boundary components, such that the parametrization of a_j is consistent with the Stokes orientation of ∂A_j. This can be done by letting N be the unit normal vector field along a_j such that $(a_j'(t), N(t))$ is an oriented basis at each point $a_j(t)$, and letting A_j be the set

$$A_j = \{\exp_{a_j(t)}(sN(t)) : t \in [0,1],\ s \in [0,\varepsilon]\}$$

for some small $\varepsilon > 0$ (see Fig. 9.2). By choosing ε small enough, we can ensure that A_j is disjoint from all of the curves a_k and b_k for $k \neq j$, and such that the curve b_j enters A_j on one boundary component exactly once and leaves exactly once on the other boundary component. (It is an easy consequence of transversality that this is possible.)

Now let $f : A_j \to [0,1]$ be a smooth function such that $f \equiv 1$ in a neighborhood of the image of a_j and $f \equiv 0$ in a neighborhood of the other boundary component of A_j, and define a smooth closed 1-form α^j on M by

$$\alpha^j = \begin{cases} df, & \text{on } A_j, \\ 0, & \text{on } M \smallsetminus \operatorname{supp}(df). \end{cases}$$

This form satisfies $\int_{a_k} \alpha^j = 0$ for all k because the image of each a_k is disjoint from the support of α^j. Similarly, $\int_{b_k} \alpha^j = 0$ for $k \neq j$. For $k = j$, our choices of orientations guarantee that $b_j'(0)$ points into A_j, so there is some $t_1 > 0$ such that $b_j(t)$ lies in A_j exactly when $t \in [0, t_1]$, and $b_j(t_1)$ is on the boundary component of A_j where $f = 0$. Thus

$$\int_{b_j} \alpha^j = \int_{b_j|_{[0,t_1]}} \alpha^j = \int_{b_j|_{[0,t_1]}} df = f(b_j(t_1)) - f(b_j(0)) = -1.$$

Now suppose η is an arbitrary closed (real or complex) 1-form on M. Since $\alpha^j \wedge \eta$ is supported in A_j, by Stokes's theorem we have

$$\int_M \alpha^j \wedge \eta = \int_{A_j} \alpha^j \wedge \eta = \int_{A_j} df \wedge \eta = \int_{A_j} d(f \wedge \eta) = \int_{\partial A_j} f\eta = \int_{a_j} \eta.$$

Finally, we use the same technique to construct an annulus B_j for each b_j and a corresponding 1-form β^j satisfying $\int_{b_k} \beta^j = 0$ for all k and $\int_{a_k} \beta^j = 0$ for all $k \neq j$, and $\int_M \beta^j \wedge \eta = \int_{b_j} \eta$ for every closed 1-form η. In this case our choices of orientations ensure that the velocity vector $a_j'(0) = a_j'(1)$ is *outward-pointing* at the point where a_j and b_j meet, so there is a time $s_1 < 1$ such that $a_j(s) \in B_j$ for $s \in [s_1, 1]$, and we conclude that $\int_{a_j} \beta^j = +1$. This completes the proof of the existence of forms satisfying (9.51)–(9.54).

It follows from (9.51)–(9.52) that α^j and β^j yield integer values when integrated over smooth singular cycles, so they determine integral cohomology classes. We need to check that they generate all such cohomology classes. Since $H_0(M) \cong \mathbb{Z}$ is free abelian, the universal coefficient theorem implies that $H^1_{\text{Sing}}(M;\mathbb{Z}) \cong \operatorname{Hom}(H_1(M), \mathbb{Z})$. It follows from (9.51) and (9.52) that the de Rham cohomology classes $([\beta^1], \dots, [\beta^n], [-\alpha^1], \dots, [-\alpha^n])$ serve as the dual basis in the group $\operatorname{Hom}(H_1(M), \mathbb{Z})$ to the homology basis $([a_1], \dots, [a_n], [b_1], \dots, [b_n])$. Thus, considered as elements of $H^1_{\text{dR}}(M;\mathbb{C})$, they span the subgroup of integral cohomology classes, and as noted above, their $(0,1)$-parts span the lattice Λ.

Let $\eta \in \Omega^1(M)$ be arbitrary. Corollary 9.55 shows that η is closed. It then follows from (9.53) that when $\sigma([(\alpha^i)^{0,1}])$ is applied to η, the result is

$$\sigma([(\alpha^i)^{0,1}])(\eta) = \int_M (\alpha^i)^{0,1} \wedge \eta = \int_M \alpha^i \wedge \eta = \int_{a_i} \eta.$$

Similarly, $\sigma([(\beta^i)^{0,1}])(\eta) = \int_{b_i} \eta$. Thus the image of Λ under σ is exactly the lattice Γ. □

In the case of a connected compact Riemann surface M, the torus $\mathrm{Alb}(M)$ is usually called the ***Jacobian variety*** (or sometimes just the ***Jacobian***) of M, denoted by $\mathrm{Jac}(M)$. (Because of the preceding theorem, some authors define the Jacobian instead to be the isomorphic torus $\mathrm{Pic}^0(M)$.)

Historically, Jacobian varieties of Riemann surfaces were constructed first; the Albanese and Picard varieties came later as generalizations to higher-dimensional Kähler manifolds. In higher dimensions, the Albanese and Picard varieties may not be isomorphic, but they are dual to each other in a sense described in Problem 9-13. (The reason for the word "variety" in the names is because they are always projective algebraic varieties when M is projective; see Problem 10-12 for a proof in the Riemann surface case.)

Here is an important application of the Jacobian variety of a curve.

Theorem 9.69. *Every connected compact Riemann surface of genus* 1 *is biholomorphic to its Jacobian variety. Therefore, every holomorphic structure on* $\mathbb{S}^1 \times \mathbb{S}^1$ *is biholomorphic to a quotient* $\mathbb{C}/\Lambda$ *for some lattice* $\Lambda \subseteq \mathbb{C}$.

Proof. Let M be a connected compact Riemann surface of genus 1. Since the space $\Omega^1(M) = H^{1,0}(M)$ is 1-dimensional, we can choose a holomorphic section ζ spanning $\Omega^1(M)$. Note that $\Lambda^{1,0}M$ is equal to the canonical bundle K, which has degree 0 in this case by the result of Problem 8-15. It follows that ζ cannot have any zeros, because if it did, the degree of K would be equal to the number of zeros counted with multiplicity by Theorem 7.22.

As in the proof of Proposition 9.67, we will represent $\mathrm{Jac}(M)$ as the quotient $\mathbb{C}/\Gamma$, where $\Gamma \subseteq \mathbb{C}$ is the lattice spanned by the numbers $\left(\int_a \zeta, \int_b \zeta\right)$ for some piecewise-smooth loops a, b representing a basis for $H_1(M)$.

Choose a point $p_0 \in M$, and define a map $\Phi\colon M \to \mathrm{Jac}(M)$ as follows: for each $q \in M$, $\Phi(q)$ is the image in $\mathbb{C}/\Gamma$ of $\int_{p_0}^q \zeta$, where the integral is interpreted as the line integral $\int_\gamma \zeta$ for some piecewise smooth curve segment γ from p_0 to q. First we note that Φ is well defined. If γ_1 and γ_2 are two such curves, then the path product $\gamma_1 \cdot \gamma_2^{-1}$ is a loop and therefore represents a homology class in $H_1(M)$; thus $\int_{\gamma_1} \zeta$ and $\int_{\gamma_2} \zeta$ differ by an element of the lattice Γ and their projections onto $\mathrm{Jac}(M)$ are equal.

To see that Φ is holomorphic, let $q_0 \in M$ be arbitrary and choose a holomorphic coordinate z on a disk $U \subseteq M$ centered at q_0. Because ζ is a closed $(1,0)$-form, there is some holomorphic function h defined on U such that $\zeta|_U = dh$. Because the quotient map $\mathbb{C} \to \mathrm{Jac}(M)$ is a holomorphic covering map, we can use the standard coordinate in $\mathbb{C}$ as a local holomorphic coordinate on $\mathrm{Jac}(M)$; let $\widehat{\Phi}$ denote the coordinate representation of Φ with respect to these coordinates. For $z \in U$, the fundamental theorem of calculus gives

$$\widehat{\Phi}(z) = \widehat{\Phi}(q_0) + \int_{\gamma_z} \zeta = \widehat{\Phi}(q_0) + h(z) - h(q_0),$$

where γ_z is, say, the radial path in U from q_0 to z. This shows that Φ is holomorphic in U. Moreover, since the coordinate representation of Φ in this chart is given by a line integral of the closed form ζ, its differential there is equal to ζ; so the fact that ζ never vanishes implies that Φ is a holomorphic immersion.

Next we show that Φ is injective. Suppose p, q are distinct points in M such that $\Phi(p) = \Phi(q)$. Letting γ_1 and γ_2 be piecewise smooth paths from p_0 to p and q, respectively, this means $\int_{\gamma_1} \zeta$ differs from $\int_{\gamma_2} \zeta$ by an element of Γ, or equivalently that $\int_{\gamma_1^{-1} \cdot \gamma_2} \zeta \in \Gamma$. Since every element of Γ is the integral of ζ over some piecewise smooth closed curve in M, by setting $\gamma = \gamma_1^{-1} \cdot \gamma_2 \cdot \sigma$ for some loop σ based at q, we can arrange that $\int_\gamma \zeta = 0$ for a path γ from p to q.

The key fact is Lemma 9.70 below, which shows that under this hypothesis, there is a global meromorphic function f on M whose divisor is equal to $p - q$. It then follows from Proposition 7.25 that M is biholomorphic to $\mathbb{CP}^1$. This contradicts the assumption that M has genus 1, and completes the proof that Φ is injective. Since Φ is an injective holomorphic immersion between connected compact manifolds of the same dimension, it is an open and closed map and therefore also surjective. Thus it is a biholomorphism, and the theorem is proved. □

Here is the lemma that was used in the preceding proof.

Lemma 9.70. *Let M be a compact Riemann surface, and $p, q \in M$ be distinct points. Suppose there is a piecewise-smooth curve γ from p to q such that $\int_\gamma \zeta = 0$ for every holomorphic $(1,0)$-form ζ. Then there exists a global meromorphic function on M whose divisor is equal to $p - q$.*

Proof. We first consider the case in which the image of $\gamma\colon [0,1] \to M$ is contained in a holomorphic coordinate chart U biholomorphic to the unit disk. Let z be the coordinate function in U, and let a, b be the coordinates of the points p and q, respectively. Let $\psi \in C^\infty(M; [0,1])$ be a smooth bump function that is supported in U and equal to 1 on a smaller disk $D_r = \{z \in U : |z| < r\}$ for some $1 > r > \max(|a|, |b|)$; and define a smooth function $u\colon M \smallsetminus \{p, q\} \to \mathbb{C}$ by setting

$$u(z) = \psi(z)\frac{z-a}{z-b} + 1 - \psi(z)$$

for $z \in U \smallsetminus \{a, b\}$ and extending it to be equal to 1 outside the support of ψ. Then u does not vanish anywhere on $M \smallsetminus \{p, q\}$, because it is equal to 1 away from the support of ψ, is nonvanishing by definition in $D_r(0) \smallsetminus \{a, b\}$, and if $u(z) = 0$ for some $r \le |z| \le 1$, we would have $z = \psi(z)a + (1 - \psi(z))b$ by direct computation, meaning that z is on the line segment connecting a and b, contradicting our choice of r. Thus the $(0,1)$-form $\alpha = \bar\partial u/u$ is smooth everywhere outside of $D_r(0)$ and vanishes identically on $D_r(0) \smallsetminus \{a, b\}$, so it extends to a smooth, $\bar\partial$-closed $(0,1)$-form on all of M.

Choose some Kähler metric on M. We will show that α is orthogonal to the space of $\bar\partial$-harmonic $(0,1)$-forms. Because each holomorphic $(1,0)$-form ζ is $\bar\partial$-harmonic and conjugation takes $\mathscr{H}^{1,0}(M)$ to $\mathscr{H}^{0,1}(M)$, the $\bar\partial$-harmonic $(0,1)$-forms are exactly those of the form $\bar\zeta$ for $\zeta \in \Omega^1(M)$. Thus we have to show $(\alpha, \bar\zeta) = \int_M \alpha \wedge *\zeta = 0$ for all such ζ. Since $*\zeta = -i\zeta$ by the result of Example 9.24, this is equivalent to $\int_M \alpha \wedge \zeta = 0$.

For small $\varepsilon > 0$, let $D_\varepsilon(a), D_\varepsilon(b) \subseteq U$ be the coordinate disks of radius ε centered at a and b, respectively, and let $U_\varepsilon = U \smallsetminus (D_\varepsilon(a) \cup D_\varepsilon(b))$. For any holomorphic 1-form ζ, using the fact that α is supported in U, we compute

$$\int_M \alpha \wedge \zeta = \lim_{\varepsilon \to 0} \int_{U_\varepsilon} \alpha \wedge \zeta = \lim_{\varepsilon \to 0} \int_{U_\varepsilon} \frac{\bar\partial u}{u} \wedge \zeta = \lim_{\varepsilon \to 0} \int_{U_\varepsilon} \frac{du}{u} \wedge \zeta.$$

Because ζ is closed, there is a holomorphic function h on U such that $\zeta|_U = dh$, so Stokes's theorem gives

$$\begin{aligned} \int_{U_\varepsilon} \frac{du}{u} \wedge \zeta &= \int_{U_\varepsilon} \frac{du}{u} \wedge dh = -\int_{U_\varepsilon} d\left(h\frac{du}{u}\right) \\ &= -\int_{\partial U_\varepsilon} h\frac{du}{u} = \int_{\partial D_\varepsilon(a)} h\frac{du}{u} + \int_{\partial D_\varepsilon(b)} h\frac{du}{u}. \end{aligned}$$

To compute these last integrals, note that on $\partial D_\varepsilon(a)$ and $\partial D_\varepsilon(a)$, we have $du/u = dz/(z-b) - dz/(z-a)$. We parametrize $\partial D_\varepsilon(a)$ by $z = a + \varepsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$, which yields

$$\int_{\partial D_\varepsilon(a)} h\frac{du}{u} = \int_0^{2\pi} h(a + \varepsilon e^{i\theta}) \left(\frac{\varepsilon i e^{i\theta} d\theta}{a - b + \varepsilon e^{i\theta}} - i\, d\theta \right).$$

As $\varepsilon \to 0$, this approaches $-2\pi i h(a)$. Similarly, the integral over $\partial D_\varepsilon(b)$ approaches $2\pi i h(b)$. Putting these results together, we obtain $\int_M \alpha \wedge \zeta = 2\pi i(h(b) - h(a))$. But our hypothesis is that $0 = \int_\gamma \zeta = h(b) - h(a)$ for each such ζ, so it follows that $\alpha \perp \operatorname{Ker} \Delta_{\bar\partial}$.

The Fredholm theorem then shows that there is some smooth $(0,1)$-form β such that $\alpha = \Delta_{\bar\partial}\beta$, which is equal to $\bar\partial\bar\partial^*\beta$ because $\bar\partial = 0$ on $(0,1)$-forms. Let $v = \bar\partial^*\beta$ and $f = e^{-v}u$ on $M \smallsetminus \{p, q\}$. Then $\bar\partial f = e^{-v}(\bar\partial u - u\bar\partial v) = 0$, so f is holomorphic away from p and q. Since it agrees with $(z-a)/(z-b)$ in a neighborhood of p and q, it is meromorphic on M and its divisor is equal to $p - q$.

It remains only to consider the general case in which the image of γ is not necessarily contained in a coordinate disk. In that case, we can choose numbers $0 = t_0 < t_1 < \cdots < t_k = 1$ such that the image of $\gamma_j = \gamma|_{[t_{j-1},t_j]}$ is contained in a coordinate disk. The construction above yields a function u_j for each j that has a zero at $\gamma_j(t_{j-1})$ and a pole at $\gamma_j(t_j)$; but $\alpha_j = \bar\partial u_j/u_j$ no longer satisfies $\int_M \alpha_j \wedge \zeta = 0$ for holomorphic 1-forms. Instead, the calculation above shows that $\int_M \alpha_j \wedge \zeta = 2\pi i(h(\gamma_j(t_j)) - h(\gamma_j(t_{j-1}))) = 2\pi i \int_{\gamma_j} \zeta$. Letting u denote the product of the u_j's and $\alpha = \bar\partial u/u$, we see that the intermediate poles and zeros cancel, and we have $\int_M \alpha \wedge \zeta = \sum_j \int_M \alpha_j \wedge \zeta = \sum_j \int_{\gamma_j} \zeta = \int_\gamma \zeta = 0$. The previous argument then applies to prove the existence of the desired meromorphic function f. □

It follows from Theorem 9.69 that every connected compact Riemann surface M of genus 1 can be given the structure of an abelian Lie group, once a particular point $p_0 \in M$ is chosen to define the map $M \to \mathrm{Jac}(M)$ (so that p_0 becomes the identity in the induced Lie group structure). A connected compact Riemann surface of genus 1 (sometimes endowed with a specific choice of p_0, depending on whose definition you read) is called an ***elliptic curve***. The name derives from the fact that ***elliptic integrals*** (certain indefinite integrals that appear in the computation of the arc length of an ellipse) were used by Abel and Jacobi in the early nineteenth century to show that nonsingular cubic curves in $\mathbb{CP}^2$ are biholomorphic to complex tori (Problem 10-4). (See [**Jos06**, Section 5.10] for a discussion of this.) Just for the record, it should be noted that an ellipse is not an elliptic curve: an ellipse (or rather a complex curve in $\mathbb{CP}^2$ with the same equation as an ellipse) is a nonsingular quadric, which as we have seen is always biholomorphic to $\mathbb{CP}^1$.

Problems

9-1. Suppose P is a constant-coefficient second-order scalar differential operator acting on smooth real-valued functions on $\mathbb{R}^2$. Show that P is elliptic if and only if there is a linear change of coordinates that transforms P into $\pm\Delta$ plus lower-order terms, where Δ is the Laplace–Beltrami operator with respect to the Euclidean metric.

9-2. Prove Proposition 9.28 (principal symbols of $\bar\partial$, $\bar\partial^*$, and $\Delta_{\bar\partial}$).

9-3. Let (M, g) be a Kähler manifold. Prove that in every holomorphic coordinate chart, the Laplace–Beltrami operator on scalar functions is given by the following formula:

$$\Delta u = g^{j\bar k} \frac{\partial^2 u}{\partial z^j \partial \bar z^k}.$$

9-4. Let (M, g) be a Kähler manifold, and let ω be its Kähler form, ρ its Ricci form, and S its scalar curvature. Let L_ω be the Lefschetz operator and L_ω^* its adjoint.

(a) Show that $L_\omega^* : \mathscr{E}^{1,1}(M) \to \mathscr{E}^{0,0}(M)$ has the coordinate formula

$$L_\omega^* \left(i\eta_{j\bar{k}} dz^j \wedge d\bar{z}^k \right) = g^{j\bar{k}} \eta_{j\bar{k}}.$$

(b) Use part (a) to derive the following formulas:

$$L_\omega^* \omega = \dim M,$$
$$L_\omega^* \rho = \tfrac{1}{2} S,$$
$$L_\omega^*(i\partial\bar{\partial}u) = \Delta u \quad \text{for } u \in C^\infty(M;\mathbb{C}).$$

9-5. Let (M, g) be a Kähler manifold and ω its Kähler form. Show that ω is harmonic.

9-6. A cohomology class $\gamma \in H^k_{\mathrm{dR}}(M;\mathbb{C}))$ on an n-dimensional Kähler manifold M is said to be ***primitive*** if $L_\omega^{n-k+1}\gamma = 0$, where L_ω is the Lefschetz operator. Let $P^k(M) \subseteq H^k_{\mathrm{dR}}(M;\mathbb{C})$ denote the space of primitive degree-k cohomology classes, and $P^{p,q}(M) \subseteq H^{p,q}(M)$ the space of primitive (p,q)-classes. Prove the ***Lefschetz decomposition theorem:*** *Let M be a compact n-dimensional Kähler manifold. There are direct sum decompositions*

$$H^k_{\mathrm{dR}}(M;\mathbb{C}) = \bigoplus_{0 \le r \le k/2} L_\omega^r P^{k-2r}(M),$$
$$H^{p,q}(M) = \bigoplus_{0 \le r \le (p+q)/2} L_\omega^r P^{p-r,q-r}(M).$$

[Hint: Use the hard Lefschetz theorem and induction on k.]

9-7. Let (M, g) be a Riemannian manifold.

(a) Prove that every harmonic 1-form η on M satisfies the following identity:

$$\Delta|\eta|^2 = |\nabla\eta|^2 + \langle Rc^\sharp(\eta), \eta \rangle,$$

where $Rc^\sharp : \Lambda^1 M \to \Lambda^1 M$ is the bundle homomorphism given by $Rc^\sharp(\eta)(v) = Rc(\eta^\sharp, v)$. [Hint: Do the computation in Riemannian normal coordinates.]

(b) Use the result of part (a) to prove the ***Bochner vanishing theorem***, due to Salomon Bochner [**Boc46**]: *If (M, g) is a compact Riemannian n-manifold with nonnegative Ricci curvature, then $b^1(M) \le n$. If in addition the Ricci curvature is positive somewhere, then $b^1(M) = 0$.*

9-8. Let (M, g) be a Kähler manifold.

(a) Prove that every holomorphic p-form η on M satisfies the following identity:

$$\Delta|\eta|^2 = |\nabla\eta|^2 + \langle Rc^\sharp(\eta), \eta\rangle,$$

where $Rc^\sharp \colon \Lambda^{p,0}M \to \Lambda^{p,0}M$ is the bundle endomorphism given in holomorphic coordinates by

$$Rc^\sharp(\eta)_{j_1\dots j_p} = \sum_{s=1}^{p} g^{m\bar{k}} R_{j_s\bar{k}} \eta_{j_1\dots m\dots j_p}.$$

[Hint: One way to do this is to compute in coordinates satisfying (8.8) at an arbitrary point a, and use the result of Problem 8-11.]

(b) Prove the following theorem, also due to Bochner [**Boc46**]: *If (M, g) is a compact Kähler manifold with nonnegative Ricci curvature, then every holomorphic differential form on M is parallel. If in addition the Ricci curvature is positive somewhere, then $h^{p,0}(M) = 0$ for all $p > 0$.*

9-9. Prove that every automorphism of $\mathbb{CP}^n$ is a projective transformation. [Hint: Given an automorphism $F \colon \mathbb{CP}^n \to \mathbb{CP}^n$, prove that $F^*T \cong T$, where $T \to \mathbb{CP}^n$ is the tautological bundle, and use this fact to construct a holomorphic map $\widetilde{F} \colon T \to T$, linear on fibers, such that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\widetilde{F}} & T \\ \downarrow & & \downarrow \\ \mathbb{CP}^n & \xrightarrow[F]{} & \mathbb{CP}^n. \end{array}$$

]

9-10. Prove that a holomorphic line bundle on a compact complex manifold cannot be both positive and negative.

9-11. Show that there is an orientation-reversing diffeomorphism $F \colon \mathbb{CP}^n \to \mathbb{CP}^n$ if and only if n is odd. Together with the result of Problem 3-10, this shows that the blowup of a complex n-manifold M at a point is diffeomorphic to $M \# \overline{\mathbb{CP}^n}$ when n is odd, but not necessarily when n is even. [Hint: In the even case, consider the de Rham cohomology class of $F^*(\omega^n)$, where ω is a Kähler form on $\mathbb{CP}^n$.]

9-12. Suppose (M, g) is a compact Kähler manifold with constant scalar curvature whose Kähler class is equal to a multiple of $c_1^{\mathbb{R}}(M)$. Prove that M is Kähler-Einstein. [Hint: Show that a constant multiple of the Ricci form is equal to $\omega + i\partial\bar{\partial}u$ for some scalar function u, and use the result of Problem 9-4 to conclude that Δu is constant. Conclude from this that u is constant.]

9-13. Suppose $T = V/\Lambda$ is a complex torus of dimension n. Define the ***dual torus*** to be the group $T^* = \operatorname{Hom}(T, \mathrm{U}(1))$ (the set of homomorphisms from T to the circle group, which is a group under pointwise multiplication).

(a) Let $\overline{V}^*$ denote the complex vector space of conjugate-linear functionals from V to $\mathbb{C}$. Show there is a surjective group homomorphism $\Phi\colon \overline{V}^* \to T^*$ that sends f to the homomorphism $\Phi(f)(v) = e^{2\pi i \operatorname{Im} f(v)}$, and its kernel is the lattice

$$\Lambda^* = \{f \in \overline{V}^* : \operatorname{Im} f(v) \in \mathbb{Z} \text{ for all } v \in \Lambda\}.$$

Conclude that T^* also has the structure of a complex torus of dimension n, isomorphic to $\overline{V}^*/\Lambda^*$.

(b) For a compact Kähler manifold M, show that $\operatorname{Alb}(M)$ is holomorphically isomorphic to the dual torus of $\operatorname{Pic}^0(M)$.

9-14. Let M be a connected compact Riemann surface of genus $g \geq 1$, and let $\operatorname{Div}^0(M)$ be the group of divisors of degree 0 on M. Define a map $\mu\colon \operatorname{Div}^0(M) \to \operatorname{Jac}(M) = \Omega^1(M)^*/\Lambda$ by choosing a point $p_0 \in M$ and setting

$$\mu\Big(\sum_j n_j q_j\Big) = \Big(\zeta \mapsto \sum_j n_j \int_{p_0}^{q_j} \zeta\Big) \mod \Lambda.$$

Prove ***Abel's theorem***: *A divisor $D \in \operatorname{Div}^0(M)$ is a principal divisor if and only if $\mu(D) = 0$.* [Hint: For the "only if" part, assume $D = (f)$ and construct a map $\Phi\colon \mathbb{CP}^1 \to \operatorname{Jac}(M)$ by $\Phi([t^0, t^1]) = \mu\big((t^0 f + t^1)\big)$; then use the result of Problem 4-6 to show Φ is constant. For the "if" part, adapt the proof of Lemma 9.70.] [Remark: This was Abel's half of the original proof of the Abel–Jacobi theorem (Thm. 9.68). Abel's theorem shows that the map μ defined above descends to an injective homomorphism from $\operatorname{Cl}^0(M)$ (which is isomorphic to $\operatorname{Pic}^0(M)$ by Corollary 9.63) to $\operatorname{Jac}(M)$. The other half of the theorem, surjectivity of μ, is known as the *Jacobi inversion theorem*. For a proof, see [**GH94**, p. 235].]

9-15. Let M be a connected compact Riemann surface of genus $g \geq 1$. Prove that the canonical bundle $K \to M$ has no base points. [Hint: Assuming p is a base point, show that $\mathcal{O}(K) \cong \mathcal{I}_{\{p\}}(K) \cong \mathcal{O}(K \otimes L^*_{\{p\}})$, and use Riemann–Roch to calculate $\dim \mathcal{O}(M; L_{\{p\}})$.]

9-16. Let M be a compact Kähler manifold. Define a map $A_M\colon M \to \operatorname{Alb}(M)$, called the ***Albanese map***, by choosing a point $p_0 \in M$ and setting

$$A_M(q) = \Big(\zeta \mapsto \int_{p_0}^{q} \zeta\Big) \mod \Lambda.$$

(This generalizes the map defined in the proof of Theorem 9.69.)

(a) Show that A_M is holomorphic.
(b) Show that if M is a compact Riemann surface of genus $g \geq 1$, then A_M is a holomorphic embedding.
(c) Show that if $M = \mathbb{C}^n/\Lambda_0$ is a complex torus, then A_M is a biholomorphism.
(d) Let $F : M \to N$ be a holomorphic map between compact Kähler manifolds. Show that there is a holomorphic map $\widetilde{F} : \operatorname{Alb}(M) \to \operatorname{Alb}(N)$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\quad F \quad} & N \\ {\scriptstyle A_M}\downarrow & & \downarrow{\scriptstyle A_N} \\ \operatorname{Alb}(M) & \xrightarrow{\quad \widetilde{F} \quad} & \operatorname{Alb}(N). \end{array}$$

[Hint: Begin by defining $\widehat{F} : \Omega^1(M)^* \to \Omega^1(N)^*$ by $\widehat{F}(\varphi)(\eta) = \varphi(F^*\eta)$. You will need to choose the base points for the Albanese maps carefully.]
(e) Prove the ***universal property of the Albanese variety:*** If M is a compact Kähler manifold and $F : M \to T$ is a holomorphic map to a complex torus, then F factors through the Albanese variety of M; that is, there is a holomorphic map $\psi : \operatorname{Alb}(M) \to T$ such that $F = \psi \circ A_M$.

9-17. Suppose E, F are smooth Hermitian vector bundles on a Riemannian manifold M, and $P : \Gamma(E) \to \Gamma(F)$ is a differential operator. We say P has ***injective symbol*** if the principal symbol map $\sigma_P(x,\xi) : E_x \to F_x$ is injective for every $x \in M$ and every nonzero $\xi \in T_x^*M$. Prove the following ***Fredholm theorem for operators with injective symbol***: *If M is compact and $P : \Gamma(E) \to \Gamma(F)$ has injective symbol, then* Ker P *is finite-dimensional and there is an orthogonal direct sum decomposition*

$$\Gamma(E) = \operatorname{Ker} P \oplus \operatorname{Im} P^*.$$

[Hint: Consider the principal symbol of P^*P.]

9-18. Let M be a Riemannian manifold and $E_0, E_1, \dots$ a sequence of Hermitian vector bundles on M. A sequence of first-order linear differential operators

$$0 \to \Gamma(E_0) \xrightarrow{P_0} \Gamma(E_1) \xrightarrow{P_1} \Gamma(E_2) \xrightarrow{P_2} \cdots \tag{9.55}$$

is called an ***elliptic complex*** if for each j, $P_{j+1} \circ P_j = 0$ and the principal symbol sequence

$$\cdots \to E_j\big|_x \xrightarrow{\sigma_{P_j}(x,\xi)} E_{j+1}\big|_x \xrightarrow{\sigma_{P_{j+1}}(x,\xi)} E_{j+2}\big|_x \to \cdots$$

is exact for every $x \in M$ and every nonzero $\xi \in T_x^* M$. Given such a complex, define

$$\mathscr{H}^j(E_*) = \left\{\sigma \in \Gamma(E_j) : P_j\sigma = 0 \text{ and } P_{j-1}^*\sigma = 0\right\};$$
$$H^j(E_*) = \frac{\operatorname{Ker}\left(P_j : \Gamma(E_j) \to \Gamma(E_{j+1})\right)}{\operatorname{Im}\left(P_{j-1} : \Gamma(E_{j-1}) \to \Gamma(E_j)\right)}.$$

Prove the following ***Hodge theorem for elliptic complexes***: *If M is compact and* (9.55) *is an elliptic complex on M, then for every j, $\mathscr{H}^j(E_*)$ is finite-dimensional, and we have an orthogonal direct sum decomposition*

$$\Gamma(E_j) = \mathscr{H}^j(E_*) \oplus \operatorname{Im} P_{j-1} \oplus \operatorname{Im} P_{j+1}^*,$$

and an isomorphism

$$H^j(E_*) \cong \mathscr{H}^j(E_*).$$

[Hint: First show that $P_j \oplus P_{j-1}^* : \Gamma(E_j) \to \Gamma(E_{j+1} \oplus E_{j-1})$ has injective symbol.]

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Chapter 10

The Kodaira Embedding Theorem

In this chapter, we present the proof of the Kodaira embedding theorem, which completely characterizes those compact complex manifolds that can be holomorphically embedded in projective spaces. The theorem comes in two versions: The first, more technical version states that a compact complex manifold is projective if and only if it carries a positive line bundle, meaning one whose first real Chern class is represented by a positive $(1,1)$-form. The second, more geometric version states that a compact complex manifold is projective if and only if it admits a Kähler metric whose Kähler class is *integral*, meaning it lies in the image of integral cohomology under the coefficient homomorphism.

The proof of this theorem uses most of the tools we have developed in this book: blowups, Hartogs's theorem, holomorphic line bundles, Chern connections, sheaf cohomology, and Hodge theory. It was first published in 1954 by Kunihiko Kodaira [**Kod54**].

Preliminaries

The heart of Kodaira's embedding theorem is the statement that every positive holomorphic line bundle over a compact complex manifold is ample, meaning that some positive tensor power of it is very ample: its global holomorphic sections separate points and directions. It then follows from Corollary 3.44 that a compact complex manifold carrying a positive line bundle is projective.

Let us consider an arbitrary holomorphic line bundle $L \to M$, and think about how one might go about proving it is very ample. We need to show that $\mathcal{O}(M;L)$ separates points and separates directions.

The first step is to transform the desired conclusions into statements about sections of sheaves. Thus let p and q be a pair of distinct points in M, and let $(L_p)_p \oplus (L_q)_q$ be the "double skyscraper sheaf" whose stalks at p and q are the fibers L_p and L_q, respectively, and all other stalks are zero. There is a sheaf morphism $e\colon \mathscr{O}(L) \to (L_p)_p \oplus (L_q)_q$ called the ***evaluation map***, defined as follows: for $\sigma \in \mathscr{O}(U; L)$ over some open set $U \subseteq M$, we set $e_U(\sigma) = (\sigma(p), \sigma(q)) \in L_p \oplus L_q$ if p and q both lie in U; if only one of them does, then $e_U(\sigma) = \sigma(p)$ or $\sigma(q)$ as appropriate; and otherwise $e_U(\sigma) = 0$. Then the statement that $\mathscr{O}(M; L)$ separates points is equivalent to the global section map $e_M\colon \mathscr{O}(M; L) \to L_p \oplus L_q$ being surjective for all p and q.

The evaluation map fits into a short exact sheaf sequence:

$$0 \to \mathscr{I}_{\{p,q\}}(L) \hookrightarrow \mathscr{O}(L) \xrightarrow{e} (L_p)_p \oplus (L_q)_q \to 0, \tag{10.1}$$

where $\mathscr{I}_{\{p,q\}}(L)$ is the sheaf of holomorphic sections of L that vanish at p and q. It is easy to check that this sheaf sequence is exact, so the obstruction to the global section map e_M being surjective lies in the cohomology group $H^1(M; \mathscr{I}_{\{p,q\}}(L))$.

Similarly, the question of whether $\mathscr{O}(M; L)$ separates directions can also be reframed as a question about sheaves. Let p be an arbitrary point in M, and choose a local holomorphic frame s for L in a neighborhood of p. Let $\mathscr{I}_{\{p\}}(L)$ be the sheaf of holomorphic sections of L that vanish at p, and let $\mathscr{I}^2_{\{p\}}(L)$ be the subsheaf of $\mathscr{I}_{\{p\}}(L)$ consisting of sections that vanish to second order at p (see Example 5.2 for the definition). Then we have an exact sheaf sequence

$$0 \to \mathscr{I}^2_{\{p\}}(L) \hookrightarrow \mathscr{I}_{\{p\}}(L) \xrightarrow{\delta} (\Lambda^{1,0}_p M)_p \to 0, \tag{10.2}$$

where the sheaf on the right is the skyscraper sheaf whose stalk at p is $\Lambda^{1,0}_p M$ with all other stalks zero, and $\delta(fs) = df_p$. Surjectivity at $(\Lambda^{1,0}_p M)_p$ can be proved by choosing holomorphic coordinates $(z^1, \dots, z^n)$ on a neighborhood U of p and centered at p, and noting that for any $\eta = \sum_j c_j dz^j|_p \in \Lambda^{1,0}_p M$, the section $\sigma(z) = \left(\sum_j c_j z^j\right)s(z)$ lies in $\mathscr{I}_{\{p\}}(U; L)$ and satisfies $\delta(\sigma) = \eta$. Exactness at $\mathscr{I}_{\{p\}}(L)$ follows from the fact that, thanks to Taylor's theorem, a holomorphic function f satisfying $f(p) = 0$ and $df_p = 0$ can be written in local holomorphic coordinates centered at p in the form $f(z) = \sum_{j,k} z^j z^k g_{jk}(z)$ for some holomorphic functions g_{jk}.

The next exercise shows that proving $\mathscr{O}(M; L)$ separates directions is equivalent to proving that the global section map $\delta_M\colon \mathscr{I}_{\{p\}}(M; L) \to \Lambda^{1,0}_p M$ is surjective for every $p \in M$. Once again, the obstruction lies in the cohomology group $H^1(M; \mathscr{I}^2_{\{p\}}(L))$.

► **Exercise 10.1.** With notation as above, show that $\delta_M : \mathscr{I}_{\{p\}}(M;L) \to \Lambda^{1,0}_p M$ is surjective if and only if for every nonzero $v \in T'_p M$, there exists a global section $\sigma \in \mathscr{I}_{\{p\}}(M;L)$ that can be expressed in a neighborhood of p as $\sigma = fs$, where $f(p) = 0$ and $vf \neq 0$.

This setup is promising, but unfortunately we have no tools for evaluating cohomology of sheaves like $\mathscr{I}_{\{p,q\}}(L)$ or $\mathscr{I}^2_{\{p\}}(L)$ (except in the case $\dim M = 1$, as we will see below). On the other hand, the sheaf of sections of L that vanish on a *hypersurface* is isomorphic to the sheaf of sections of a holomorphic line bundle by the result of Proposition 5.16, and we will prove a theorem below (the Kodaira vanishing theorem) that gives conditions under which the cohomology groups of such a sheaf are zero. So the plan is to transfer the problem to the blowup of M at the points p and q (for separating points) or at p alone (for separating directions), where the selected points are replaced by hypersurfaces, and try to find an appropriate line bundle to which we can apply the Kodaira vanishing theorem. (When $\dim M = 1$, individual points *are* hypersurfaces, so blowing up is not necessary. As Problem 10-3 shows, the proof is simpler in that case.)

The Kodaira Vanishing Theorem

The main tool for proving that the crucial cohomology groups are zero is a vanishing theorem for certain cohomology groups of sheaves of sections of line bundles. We begin by extending the Hodge-theoretic results of Chapter 9 to bundle-valued forms on a Kähler manifold.

Suppose M is a complex manifold and $E \to M$ is a Hermitian vector bundle. Let ∇ be a connection on E, and let $D : \mathscr{E}^k(M;E) \to \mathscr{E}^{k+1}(M;E)$ be the exterior covariant derivative determined by ∇ (Prop. 7.11). On (p,q)-forms, we can decompose this operator as $D = D' + D''$, where D' maps $\mathscr{E}^{p,q}(M;E)$ to $\mathscr{E}^{p+1,q}(M;E)$ and D'' maps $\mathscr{E}^{p,q}(M;E)$ to $\mathscr{E}^{p,q+1}(M;E)$.

We define two Laplace-type operators $\Delta', \Delta'' : \mathscr{E}^{p,q}(M;E) \to \mathscr{E}^{p,q}(M;E)$:

$$\Delta'\alpha = D'D'^*\alpha + D'^*D'\alpha, \qquad \Delta''\alpha = D''D''^*\alpha + D''^*D''\alpha. \tag{10.3}$$

Unlike the case of the Dolbeault Laplacian and its conjugate acting on scalar-valued forms, these two operators are generally not equal on Kähler manifolds; but in the case of a holomorphic bundle with its Chern connection, there is an important formula for their difference. The first step in relating these operators is to prove a bundle-valued analogue of the Kähler identities.

Proposition 10.2 (Kähler Identities for Bundle-Valued Forms). *Suppose M is a Kähler manifold with Kähler form ω; $E \to M$ is a smooth Hermitian vector bundle; ∇ is a metric connection on E; D', D'' are the exterior covariant derivative operators defined with respect to ∇; and $L_\omega : \mathscr{E}^{p,q}(M;E) \to \mathscr{E}^{p+1,q+1}(M;E)$ is the* *bundle homomorphism* $L_\omega \eta = \omega \wedge \eta$. *The following identities hold:*

(a) $[D''^*, L_\omega] = iD'$.

(b) $[D'^*, L_\omega] = -iD''$.

(c) $[L_\omega^*, D''] = -iD'^*$.

(d) $[L_\omega^*, D'] = iD''^*$.

Proof. We begin with (a). Suppose (s_j) is a local frame for E on an open set $U \subseteq M$, and $\alpha = \alpha^j \otimes s_j$ is an E-valued $(p, q-1)$-form. If θ_j^k denotes the matrix of connection 1-forms with respect to this frame, then (7.11) gives the following local expression for $D\alpha$:

$$D\alpha = d\alpha^j \otimes s_j + (-1)^{p+q-1}\left(\alpha^j \wedge \theta_j^k\right) \otimes s_k,$$

and therefore

$$D'\alpha = \partial\alpha^j \otimes s_j + (-1)^{p+q-1}\left(\alpha^j \wedge (\theta')_j^k\right) \otimes s_k,$$
$$D''\alpha = \bar\partial\alpha^j \otimes s_j + (-1)^{p+q-1}\left(\alpha^j \wedge (\theta'')_j^k\right) \otimes s_k,$$

where θ' and θ'' represent the $(1,0)$- and $(0,1)$-parts of θ, respectively. Let us write the second terms on the right-hand sides as $\theta'\alpha$ and $\theta''\alpha$, so

$$D'\alpha = \partial\alpha^j \otimes s_j + \theta'\alpha,$$
$$D''\alpha = \bar\partial\alpha^j \otimes s_j + \theta''\alpha,$$

Now suppose α is as above and $\beta = \beta^k \otimes s_k$ is an E-valued (p,q)-form, and both are compactly supported in U. Then by the way we have defined the inner product of E-valued forms and the fact that the frame (s_j) is orthonormal, we have

$$\begin{aligned}(D''\alpha, \beta) &= \int_M \langle \bar\partial\alpha^j \otimes s_j,\ \beta^k \otimes s_k\rangle dV_g + (\theta''\alpha,\ \beta)\\ &= \int_M \sum_j \langle \bar\partial\alpha^j,\ \beta^j\rangle dV_g + (\theta''\alpha,\ \beta)\\ &= \int_M \sum_j \langle \alpha^j,\ \bar\partial^*\beta^j\rangle dV_g + (\alpha,\ \theta''^*\beta),\end{aligned}$$

which shows that

$$D''^*\beta = \bar\partial^*\beta^k \otimes s_k + \theta''^*\beta$$

for such forms. Since every form agrees on a neighborhood of each point with a form compactly supported in U, this expression holds everywhere in U.

Since $L_\omega\beta = (L_\omega\beta^k) \otimes s_k$, we have

$$\begin{aligned} D''^* L_\omega\beta &= D''^*\big((L_\omega\beta^k) \otimes s_k\big) \\ &= (\bar\partial^* L_\omega\beta^k) \otimes s_k + \theta''^* L_\omega\beta, \end{aligned}$$

and

$$L_\omega D''^*\beta = L_\omega(\bar\partial^*\beta^k) \otimes s_k + L_\omega\theta''^*\beta.$$

Therefore, using the corresponding Kähler identity for scalar-valued forms (Prop. 9.38(a)), we find

$$\begin{aligned} [D''^*, L_\omega]\beta &= ([\bar\partial^*, L_\omega]\beta^k) \otimes s_k + [\theta''^*, L_\omega]\beta \\ &= (i\partial\beta^k) \otimes s_k + [\theta''^*, L_\omega]\beta \\ &= iD'\beta - i\theta'\beta + [\theta''^*, L_\omega]\beta. \end{aligned} \tag{10.4}$$

This formula holds with respect to every local orthonormal frame. On the other hand, since both sides of (a) are defined independently of any choice of frame, for each $x_0 \in M$ we can use any convenient frame to verify the identity at x_0. Lemma 7.3 shows that we can choose an orthonormal frame with the property that $\theta_k^j = 0$ at x_0. Because θ' and θ'' do not involve differentiation, it follows that θ' and θ''^* both vanish at x_0, and then (10.4) shows that (a) holds at x_0. This proves (a).

The proof of (b) is exactly the same, with single-primed and double-primed expressions interchanged; and then (c) and (d) follow by taking adjoints. □

Besides the Kähler identities, the other key ingredient in proving that $\Delta_\partial = \Delta_{\bar\partial}$ on scalar-valued forms (Lemma 9.42) was the fact that $\partial\bar\partial = -\bar\partial\partial$, which follows immediately from $d^2 = 0$. But in the present context, the role of d is played by the exterior covariant derivative D, whose square is not zero. Instead, we have the following lemma.

Lemma 10.3. *Suppose $E \to M$ is a Hermitian holomorphic vector bundle and ∇ is its Chern connection. Let $D = D' + D''$ be the exterior covariant derivative associated with ∇, and let $\Theta \in \mathscr{E}^{1,1}(M; E)$ be its curvature. Then for every E-valued differential form α,*

$$D'D''\alpha + D''D'\alpha = \Theta \wedge \alpha.$$

Proof. For $\alpha \in \mathscr{E}^{p,q}(M; E)$, Proposition 7.11(iii) shows that

$$\Theta \wedge \alpha = D^2\alpha = (D')^2\alpha + \big(D'D''\alpha + D''D'\alpha\big) + (D'')^2\alpha.$$

Because Θ is of type $(1, 1)$ (Prop. 7.18), the first and last terms on the right-hand side are zero, and the result follows. □

Here is the fundamental relation between the two bundle-valued Laplace operators Δ' and Δ'', due to Yasuo Akizuki and Shigeo Nakano [**AN54**].

Theorem 10.4 (Akizuki–Nakano Identity). *Suppose M is a Kähler manifold and $E \to M$ is a Hermitian holomorphic vector bundle. Let Δ' and Δ'' be the Laplace operators defined by* (10.3) *with respect to the Chern connection on E, and let $\Omega : \mathscr{E}^{p,q}(M;E) \to \mathscr{E}^{p+1,q+1}(M;E)$ be the operator $\Omega(\alpha) = i\Theta \wedge \alpha$, where Θ is the curvature of the Chern connection. Then*

$$\Delta'' = \Delta' + [\Omega, L_\omega^*].$$

Proof. Using Proposition 10.2, we compute

$$\begin{aligned}\Delta'' &= D''D''^* + D''^*D'' \\ &= -iD''[L_\omega^*, D'] - i[L_\omega^*, D']D'' \\ &= -i(D''L_\omega^*D' - D''D'L_\omega^* + L_\omega^*D'D'' - D'L_\omega^*D''),\end{aligned}$$

and

$$\begin{aligned}\Delta' &= D'D'^* + D'^*D' \\ &= i(D'[L_\omega^*, D''] + [L_\omega^*, D'']D') \\ &= i(D'L_\omega^*D'' - D'D''L_\omega^* + L_\omega^*D''D' - D''L_\omega^*D').\end{aligned}$$

Therefore, Lemma 10.3 gives

$$\begin{aligned}\Delta'' - \Delta' &= i(D''D' + D'D'')L_\omega^* - iL_\omega^*(D''D' + D'D'') \\ &= [\Omega, L_\omega^*].\end{aligned}$$

□

This does not directly yield results analogous to the Hodge decomposition for scalar-valued forms; but in special cases when something can be said about the curvature, it can lead to very powerful results. We will see one such application below—the Kodaira–Nakano–Akizuki vanishing theorem (Thm. 10.6).

The Kodaira–Nakano–Akizuki Vanishing Theorem

Recall that a holomorphic line bundle L is said to be *positive* if its first real Chern class is represented by a positive $(1,1)$-form, that is, a real $(1,1)$-form ω such that $\omega(X, JX) > 0$ for all nonzero X. For example, the hyperplane bundle on $\mathbb{CP}^n$ is positive because it has a fiber metric whose Chern form is a positive multiple of the Kähler form of the Fubini–Study metric (see Example 8.14).

Proposition 10.5 (Basic Properties of Positive Line Bundles). *Suppose M is a compact complex manifold and $L \to M$ is a positive holomorphic line bundle.*

(a) *There exist a Kähler metric on M and a Hermitian fiber metric on L such that $\frac{i}{2\pi}\Theta_L$ is equal to the Kähler form, where Θ_L is the curvature of the Chern connection on L.*

(b) *If M' is another complex manifold and $f : M' \to M$ is a holomorphic immersion, then $f^*L \to M'$ is also positive.*

Proof. To prove (a), let ω be a closed, positive $(1,1)$-form representing $c_1^{\mathbb{R}}(L)$. Then ω is a Kähler form, so we can endow M with the Kähler metric $g = \omega(\cdot, J\cdot)$. By Corollary 9.58, there is a Hermitian fiber metric on L whose Chern connection has a curvature form Θ_L that satisfies $\frac{i}{2\pi}\Theta_L = \omega$.

For (b), suppose $f : M' \to M$ is a holomorphic immersion. Give L the Hermitian fiber metric described in part (a), and give f^*L the pullback metric. Then Proposition 7.21 shows that the curvature of the Chern connection on f^*L is given by $\Theta_{f^*L} = f^*\Theta_L$. Because F is a holomorphic immersion, it follows that for all $x \in M'$ and all nonzero $X \in T_xM'$,

$$\begin{aligned}\frac{i}{2\pi}\Theta_{f^*L}\big|_x(X, JX) &= \frac{i}{2\pi} f^*\Theta_L\big|_x(X, JX)\\ &= \frac{i}{2\pi}\Theta_L\big|_{f(x)}(Df(x)(X), JDf(x)(X)) > 0,\end{aligned}$$

which shows that f^*L is positive. □

The following theorem expresses one of the deepest properties of positive line bundles. It was proved in 1954 by Yasuo Akizuki and Shigeo Nakano [**AN54**], generalizing an earlier result by Kunihiko Kodaira (Cor. 10.7 below).

Theorem 10.6 (Kodaira–Nakano–Akizuki Vanishing Theorem). *Suppose M is a compact n-dimensional complex manifold and $L \to M$ is a positive line bundle. Then $H^{p,q}(M; L) = 0$ for $p + q > n$.*

Proof. Give M the Kähler metric and L the Hermitian fiber metric described in Proposition 10.5(a), and endow L with the corresponding Chern connection. By the Hodge–Dolbeault theorem for bundle-valued forms (Thm. 9.35), every cohomology class in $H^{p,q}(M; L)$ has a $\bar\partial_L$-harmonic representative. Let $\alpha \in \mathscr{H}^{p,q}(M; L)$ be such a form. Because we are using the Chern connection on L, the exterior covariant derivative operator is $D = D' + D''$ where $D'' = \bar\partial_L$, and the Dolbeault Laplacian $\Delta_{\bar\partial_L} = \bar\partial_L\bar\partial_L^* + \bar\partial_L^*\bar\partial_L$ is equal to Δ''.

The Akizuki–Nakano identity (Thm. 10.4) gives

$$0 = \Delta_{\bar\partial_L}\alpha = \Delta''\alpha = \Delta'\alpha + [\Omega, L_\omega^*]\alpha. \tag{10.5}$$

On the other hand, our choice of Kähler metric gives

$$\Omega\alpha = i\Theta_L \wedge \alpha = 2\pi L_\omega \alpha.$$

Therefore, (10.5) combined with Lemma 9.39(a) yields

$$-\Delta'\alpha = 2\pi[L_\omega, L_\omega^*]\alpha = 2\pi(p + q - n)\alpha.$$

(Here we are using the fact that L_ω and L_ω^* act only on the differential form part of a bundle-valued form, so the commutation relation of Lemma 9.39(a) still holds for bundle-valued forms.) Taking the global inner product with α, we find

$$\begin{aligned}2\pi(p+q-n)\|\alpha\|^2 &= -(\Delta'\alpha,\alpha) = -(D'D'^*\alpha,\alpha) - (D'^*D'\alpha,\alpha)\\ &= -\|D'^*\alpha\|^2 - \|D'\alpha\|^2 \le 0.\end{aligned}$$

When $p+q-n>0$, this implies $\alpha=0$. □

The most useful case of the preceding theorem is the one below; it is the original vanishing theorem proved by Kunihiko Kodaira in 1953 [**Kod53**] and later generalized as above by Akizuki and Nakano.

Corollary 10.7 (Kodaira Vanishing Theorem). *If M is a compact complex manifold, $L \to M$ is a positive line bundle, and $K \to M$ is the canonical bundle, then $H^q(M;\mathcal{O}(K\otimes L)) = 0$ for all $q>0$.*

Proof. Note that $K = \Lambda^{n,0}M$ (where $n = \dim M$), so $\mathcal{O}(K\otimes L) = \Omega^n(L)$. By the Dolbeault theorem, $H^q(M;\Omega^n(L)) \cong H^{n,q}(M;L)$, and the result follows from Theorem 10.6. □

In practical applications of the Kodaira vanishing theorem, it is usually more useful to express the conclusion in terms of the cohomology of the sheaf of sections of a specific line bundle L, instead of $K\otimes L$. The next corollary shows how.

Corollary 10.8. *Suppose M is a compact complex manifold and $L \to M$ is a complex line bundle such that $K^*\otimes L$ is positive. Then $H^q(M;\mathcal{O}(L)) = 0$ for all $q>0$.*

Proof. This follows from the previous corollary applied to $L' = K^*\otimes L$, noting that $K\otimes L' \cong L$. □

Line Bundles on Blowups

As mentioned at the beginning of this chapter, the strategy for proving the Kodaira embedding theorem is to transfer the problem to a blowup, thus converting sheaves of sections vanishing at a point to sheaves of sections vanishing on a hypersurface. The next preliminary result we need is a general fact about lifting a positive line bundle to a blowup. This is the most technically complicated part of the proof of the embedding theorem.

Proposition 10.9. *Suppose M is a compact complex manifold of dimension $n\ge 2$ and $L\to M$ is a positive line bundle. For any $p\in M$, let $\pi_p\colon M_p\to M$ denote the blowup of M at p, let $S_p = \pi_p^{-1}(p)$ be the exceptional hypersurface, and let $L_{S_p}\to M_p$ be the line bundle associated with S_p. There exists an integer $k_0>0$ such that for every integer $k\ge k_0$ and every point $p\in M$, the following line bundle on M_p is positive:*

$$\pi_p^* L^k \otimes L_{S_p}^*.$$

Proof. Give M the Kähler metric and L the Hermitian fiber metric described in Proposition 10.5(a), and endow L with the corresponding Chern connection, so the Chern form $\frac{i}{2\pi}\Theta_L$ is equal to the Kähler form ω. Fix $p \in M$ and let $\pi_p : M_p \to M$ be the blowup. By Proposition 7.21, the Chern form of $\pi_p^* L$ with respect to the pullback metric is equal to

$$\frac{i}{2\pi}\pi_p^*\Theta_L = \pi_p^*\omega.$$

Since π_p restricts to a biholomorphism from $M_p \smallsetminus S_p$ to $M \smallsetminus \{p\}$, and biholomorphisms pull positive $(1,1)$-forms back to positive $(1,1)$-forms, this shows that the Chern form of $\pi_p^* L$ is positive on $M_p \smallsetminus S_p$, and thus so is the Chern form of $\pi_p^* L^k$ for all $k > 0$. However, at a point $x \in S_p$, because $D\pi_p(x)(Y) = 0$ exactly when $Y \in T_x S_p$, this form is positive when applied to vectors in $T_x M_p \smallsetminus T_x S_p$, but zero when restricted to $T_x S_p$. We will use the bundle $L_{S_p}^*$ to correct this, at the cost of introducing some negativity in $M_p \smallsetminus S_p$, which we can eliminate by raising L to a high enough power.

Let $(v^1, \dots, v^n)$ be a holomorphic coordinate chart on an open set $W \subseteq M$ centered at p, whose image contains a closed ball $\overline{B}_\varepsilon(0) \subseteq \mathbb{C}^n$. For each $s \in (0, \varepsilon]$, let W_s denote the subset of W where $|v| < s$. Let $\widetilde{W} = \pi_p^{-1}(W_\varepsilon) \subseteq M_p$, identified with the subset $T(\varepsilon)$ of the tautological bundle T as in (3.13), so we can write

$$\widetilde{W} = \{([w], v) : [w] \in \mathbb{CP}^{n-1},\ v \in [w],\ |v| < \varepsilon\}.$$

With this identification, $S_p = \{([w], v) : v = 0\}$, and the restriction of the blow-down map $\pi : \widetilde{W} \to W$ is given by $\pi([w], v) = (v^1, \dots, v^n)$. There is a holomorphic retraction $r : \widetilde{W} \to S_p$ given by $r([w], v) = ([w], 0)$.

Cover $\widetilde{W}$ with open sets U_α, $\alpha = 1, \dots, n$, where $U_\alpha = \{([w], v) : w^\alpha \neq 0\}$, and let $U_0 = M_p \smallsetminus S_p$. We construct a system of local defining functions for S_p as follows. On each set U_α with $\alpha \neq 0$, we take $f_\alpha = v^\alpha$, which vanishes simply on $U_\alpha \cap S_p$; and on U_0, we take $f_0 \equiv 1$. Theorem 3.39 shows that $\{U_0, U_1, \dots, U_n\}$ is a trivializing cover for L_{S_p}. The transition function on $U_\alpha \cap U_\beta$ for nonzero α, β is the holomorphic extension of v^α / v^β to $U_\alpha \cap U_\beta$, which is equal to w^α / w^β on all of $U_\alpha \cap U_\beta$. On $U_\alpha \cap U_0$, on the other hand, the transition function is $f_\alpha / f_0 = v^\alpha$. Thus for this trivializing cover, L_{S_p} has transition functions

$$\begin{aligned} \tau_{\alpha 0} &= v^\alpha \quad \text{on } U_\alpha \cap U_0, \\ \tau_{\alpha\beta} &= \frac{w^\alpha}{w^\beta} \quad \text{on } U_\alpha \cap U_\beta \text{ for } \alpha, \beta \in \{1, \dots, n\}. \end{aligned}$$

On the other hand, S_p is biholomorphic to $\mathbb{CP}^{n-1}$, and Proposition 3.32 shows that the tautological bundle H^* over S_p is trivial over $\{U_1 \cap S_p, \dots, U_n \cap S_p\}$ with the restrictions of the same transition functions. Since the retraction $r : \widetilde{W} \to S_p$ satisfies $r^*(w^\alpha / w^\beta) = w^\alpha / w^\beta$ for each α and β, this shows that the transition functions for L_{S_p} over $\widetilde{W}$ are the pullbacks by r of those for H^*, so $L_{S_p}\big|_{\widetilde{W}} \cong r^*H^*$. Taking duals shows that $L^*_{S_p}\big|_{\widetilde{W}} \cong r^*H$.

Because $H \to \mathbb{CP}^{n-1}$ is a positive line bundle, it has a Hermitian fiber metric whose Chern form $\frac{i}{2\pi}\Theta_H$ is a positive $(1,1)$-form, so on $\widetilde{W}$ we can give $L^*_{S_p}\big|_{\widetilde{W}}$ the pullback metric $\langle\cdot,\cdot\rangle_{r^*H}$, which has Chern form

$$\frac{i}{2\pi}\Theta_{r^*H} = \frac{i}{2\pi}r^*\Theta_H.$$

This form is nonnegative everywhere, and it is positive when restricted to vectors tangent to S_p because r restricts to the identity on S_p.

By Theorem 3.39, there is a global section $\sigma \in \mathscr{O}(M_p, L_{S_p})$ that vanishes only on S_p. Thus the restriction of σ to $M_p \smallsetminus S_p$ is a local holomorphic frame for L_{S_p} there, and we can define a fiber metric $\langle\cdot,\cdot\rangle_1$ on $L_{S_p}\big|_{M_p\smallsetminus S_p}$ by setting $\langle\sigma,\sigma\rangle_1 \equiv 1$. Because $|\sigma|_1^2$ is constant, the curvature of the Chern connection for this metric is identically zero. Give $L^*_{S_p}\big|_{M_p\smallsetminus S_p}$ the dual metric, also denoted by $\langle\cdot,\cdot\rangle_1$; it too has zero curvature.

We now create a global fiber metric on $L^*_{S_p}$ by blending these two metrics together. Fix once and for all a smooth cutoff function $\varphi\colon [0,\infty) \to [0,1]$ supported in $[0, 3\varepsilon/4)$ and identically 1 on $[0,\varepsilon/2]$, and define a smooth function $\rho\colon M_p \to [0,1]$ by $\rho([w],v) = \varphi(|v|)$ for $([w],v) \in \widetilde{W}$, extended by zero to the rest of M_p. Define a Hermitian fiber metric $\langle\cdot,\cdot\rangle_p$ on $L^*_{S_p}$ by

$$\langle\cdot,\cdot\rangle_p = \rho\langle\cdot,\cdot\rangle_{r^*H} + (1-\rho)\langle\cdot,\cdot\rangle_1.$$

The Chern form $\frac{i}{2\pi}\Theta_p$ associated with $\langle\cdot,\cdot\rangle_p$ is nonnegative on $\pi_p^{-1}(W_{\varepsilon/2})$ (where $\rho \equiv 1$), is strictly positive when restricted to TS_p, and is zero on $M_p \smallsetminus \pi_p^{-1}(W_{3\varepsilon/4})$; but it may have positive and/or negative values on $\pi_p^{-1}(W_{3\varepsilon/4} \smallsetminus W_{\varepsilon/2})$ where ρ is not constant.

Let $\sigma_p\colon M \smallsetminus \{p\} \to M_p \smallsetminus S_p$ be the inverse of the biholomorphism $\pi_p|_{M_p\smallsetminus S_p}$, and let $\Psi_p = \sigma_p^*\Theta_p$, which is a smooth $(1,1)$-form on $M \smallsetminus \{p\}$. It is nonnegative everywhere on $M \smallsetminus \{p\}$ except perhaps in the compact annulus $K = \overline{W}_{3\varepsilon/4} \smallsetminus W_{\varepsilon/2}$. The set UTK of unit tangent vectors to K is compact [**LeeRM**, Prop. 2.9], so the expression $\frac{i}{2\pi}\Psi_p(X,JX)$ is bounded when applied to unit vectors there. Because $\frac{i}{2\pi}\Theta_L(X,JX) = \omega(X,JX) = g(X,X) = 1$ for every unit vector X, it follows that the form

$$\frac{i}{2\pi}\left(k\Theta_L + \Psi_p\right) \tag{10.6}$$

is positive on $M \smallsetminus \{p\}$ for k sufficiently large.

The tensor product metric on $\pi_p^* L^k \otimes L_{S_p}^*$ has its Chern form equal to

$$\frac{i}{2\pi}\left(k\pi_p^*\Theta_L + \Theta_p\right). \tag{10.7}$$

For k sufficiently large, this form is positive on $M_p \smallsetminus S_p$ because it is the pullback of the positive form (10.6); at points of S_p, both terms are nonnegative, while the second term is positive when applied to vectors tangent to S_p, and the first term is positive when applied to everything else.

To see that we can choose k independently of p, we start by deriving an explicit formula for the form Ψ_p defined above. Example 7.26 shows that the tautological bundle H^* over $S_p \approx \mathbb{CP}^{n-1}$ has a trivializing cover $\{U_1 \cap S_p, \dots, U_n \cap S_p\}$, and on $U_\alpha \cap S_p$ we have a local frame s_α whose norm is

$$|s_\alpha|^2_{H^*} = \frac{|w|^2}{|w^\alpha|^2}.$$

Therefore, the dual frame s_α^* for H has norm

$$|s_\alpha^*|^2_H = \frac{|w^\alpha|^2}{|w|^2}. \tag{10.8}$$

On each set U_α, the local frame s_α^* for H pulls back under r to a local frame for $L_{S_p}^*\big|_{U_\alpha}$, which for simplicity we also denote by s_α^*. The norm of the pullback frame with respect to the pullback metric is given by the same formula (10.8) on U_α because $r^*\left(|w^\alpha|^2/|w|^2\right) = |w^\alpha|^2/|w|^2$. However, on $U_\alpha \smallsetminus S_p$, note that $|w^\alpha|^2/|w|^2 = |v^\alpha|^2/|v|^2$, so we can also express the norm there as

$$|s_\alpha^*|^2_{r^*H} = \frac{|v^\alpha|^2}{|v|^2}.$$

On $U_\alpha \cap U_0 = U_\alpha \smallsetminus S_p$, on the other hand, Theorem 3.39 shows that the global section σ of L_{S_p} can be expressed as $\sigma = f_\alpha s_\alpha = v^\alpha s_\alpha$ (no implicit summation), and therefore the dual frame σ^* for $L_{S_p}^*$ over $U_\alpha \smallsetminus S_p$ satisfies $\sigma^* = (1/v^\alpha)s_\alpha^*$. Since $|\sigma^*|_1 = 1$ by design, we have

$$|s_\alpha^*|_1^2 = |v^\alpha|^2.$$

Thus our blended metric has the local formula

$$|s_\alpha^*|_p^2 = \rho\frac{|v^\alpha|^2}{|v|^2} + (1-\rho)|v^\alpha|^2 = \frac{|v^\alpha|^2\left(\rho + (1-\rho)|v|^2\right)}{|v|^2}$$

on $U_\alpha \smallsetminus S_p$, and the curvature of its Chern connection is

$$
\begin{aligned}
\Theta_p\big|_{U_\alpha \smallsetminus S_p} &= \bar\partial\partial \log \frac{|v^\alpha|^2\big(\rho + (1-\rho)|v|^2\big)}{|v|^2} \\
&= \bar\partial\partial \left(\log |v^\alpha|^2 + \log \frac{\big(\rho + (1-\rho)|v|^2\big)}{|v|^2} \right) \\
&= \bar\partial\partial \log \frac{\big(\rho + (1-\rho)|v|^2\big)}{|v|^2},
\end{aligned} \tag{10.9}
$$

where we have used the fact that $\bar\partial\partial \log |v^\alpha|^2 = 0$, which can be seen by writing $\log |v^\alpha|^2 = \log v^\alpha + \log \bar v^\alpha$ in a neighborhood of each point where a branch of the complex logarithm is defined. The notable thing about this formula is that it is independent of α, so it actually holds on all of $\widetilde W \smallsetminus S_p$. Since the restriction of σ_p to $W_\varepsilon \smallsetminus \{p\}$ is given by $\sigma_p(v) = ([v], v)$, it follows that the pullback $\Psi_p = \sigma_p^* \Theta_p$ is given by the same formula (10.9) on $W_\varepsilon \smallsetminus \{p\}$.

Now let $q = (q^1, \dots, q^n)$ be any other point in the open set $W_{\varepsilon/4}$. We can carry out the same construction of a fiber metric on $L^*_{S_q}$ using coordinates $(\tilde v^1, \dots, \tilde v^n)$, where $\tilde v^j = v^j - q^j$. Defining $\sigma_q \colon M \smallsetminus \{q\} \to M_q$ and $\Psi_q = \sigma_q^* \Theta_q$ as above, we find that that construction results in

$$
\begin{aligned}
\Psi_q &= \bar\partial\partial \log \frac{\big(\varphi(|\tilde v|) + (1 - \varphi(|\tilde v|))|\tilde v|^2\big)}{|\tilde v|^2} \\
&= \bar\partial\partial \log \frac{\big(\varphi(|v-q|) + (1 - \varphi(|v-q|))|v-q|^2\big)}{|v-q|^2}
\end{aligned}
$$

on $\overline W_\varepsilon \smallsetminus \{q\}$. For any such q, the form Ψ_q is nonnegative except possibly in the compact annulus $K' = \overline W_\varepsilon \smallsetminus W_{\varepsilon/4}$, and the expression $\frac{i}{2\pi}\Psi_q\big|_x(X, JX)$ depends continuously on $q \in W_{\varepsilon/4}$ and $(x, X) \in UTK'$. Since $\overline W_{\varepsilon/8} \times UTK'$ is compact, there exists an integer k_p such that $\frac{i}{2\pi}\left(k\Theta_L + \Psi_q\right)$ is positive for all $k \ge k_p$ and all q in the neighborhood $W_{\varepsilon/8}$ of p, which implies as above that $\pi_q^* L^k \otimes L^*_{S_q}$ is positive for all such k and q. Since M is compact, we can cover it with finitely many such neighborhoods and let k_0 be the largest such integer, thus completing the proof. □

For proving that $\mathcal{O}(M; L^k)$ separates points, we will also need an adaptation of the previous result for blowups at two points.

Proposition 10.10. *With M and L as in the hypothesis of Proposition* 10.9*, let k_0 be the integer guaranteed by that proposition. For a pair of distinct points $p, q \in M$, let $\pi_{pq} \colon M_{pq} \to M$ be the blowup of M at p and q, and let $S_p = \pi_{pq}^{-1}(\{p\})$ and $S_q = \pi_{pq}^{-1}(\{q\})$. For any $k \ge k_0$ and any distinct $p, q \in M$, the following line* *bundle on M_{pq} is positive:*

$$\pi_{pq}^* L^{2k} \otimes L_{S_p}^* \otimes L_{S_q}^*.$$

Proof. Let $p, q \in M$ be an arbitrary pair of distinct points, and $k \geq k_0$. If we let $\widetilde{\pi}_p : \widetilde{M}_p \to M$ and $\widetilde{\pi}_q : \widetilde{M}_q \to M$ be the blowups of M at p and q, respectively, we have a commutative diagram of blowdown maps:

$$\begin{array}{ccccc} & & M_{pq} & & \\ & \swarrow^{\pi_q} & \Big\downarrow{\scriptstyle \pi_{pq}} & \searrow^{\pi_p} & \\ \widetilde{M}_p & & & & \widetilde{M}_q \\ & \searrow_{\widetilde{\pi}_p} & & \swarrow_{\widetilde{\pi}_q} & \\ & & M. & & \end{array}$$

With $\widetilde{S}_p = \widetilde{\pi}_p^{-1}(\{p\})$ and $\widetilde{S}_q = \widetilde{\pi}_q^{-1}(\{q\})$, Proposition 10.9 shows that both of the bundles $\widetilde{\pi}_p^* L^k \otimes L_{\widetilde{S}_p}^* \to \widetilde{M}_p$ and $\widetilde{\pi}_q^* L^k \otimes L_{\widetilde{S}_q}^* \to \widetilde{M}_q$ are positive. Because $\pi_q : M_{pq} \to \widetilde{M}_p$ is a biholomorphism from a neighborhood of S_p to a neighborhood of $\widetilde{S}_p$, it follows from Problem 3-12 that $\pi_q^* L_{\widetilde{S}_p} \cong L_{S_p}$. Thus the bundle $\pi_{pq}^* L^k \otimes L_{S_p}^* \cong \pi_q^* \big(\widetilde{\pi}_p^* L^k \otimes L_{\widetilde{S}_p}^* \big)$ has a Chern form that is positive on $M_{pq} \smallsetminus S_q$, where π_q is a biholomorphism onto its image; at points of S_q, it is merely nonnegative. Similarly, $\pi_{pq}^* L^k \otimes L_{S_q}^*$ has a Chern form that is positive on $M_{pq} \smallsetminus S_p$ and nonnegative at points of S_p. Therefore, the tensor product of these two bundles, namely

$$\big(\pi_{pq}^* L^k \otimes L_{S_p}^*\big) \otimes \big(\pi_{pq}^* L^k \otimes L_{S_q}^*\big) \cong \pi_{pq}^* L^{2k} \otimes L_{S_p}^* \otimes L_{S_q}^*,$$

has a Chern form that is the sum of the two, which is positive everywhere. □

Because the Kodaira vanishing theorem involves the canonical bundle, it will also be important to have explicit information about the canonical bundle of a blowup.

Proposition 10.11. *Suppose M is a complex n-manifold and $\pi : \widetilde{M} \to M$ is the blowup of M at finitely many points $p_1, \dots, p_m \in M$. Let K denote the canonical bundle of M, $\widetilde{K}$ the canonical bundle of $\widetilde{M}$, and for each i, L_{S_i} the line bundle associated with the exceptional hypersurface $S_i = \pi^{-1}(\{p_i\})$. Then*

$$\widetilde{K} \cong \pi^* K \otimes L_{S_1}^{n-1} \otimes \cdots \otimes L_{S_m}^{n-1}. \tag{10.10}$$

Proof. For simplicity, we will prove the proposition for the blowup at one point. The generalization of the proof to multiple points is more complicated notationally but not conceptually, so it is left to the reader.

Thus let $\widetilde{M}$ be the blowup of M at $p \in M$, and let L_S be the line bundle associated with the exceptional hypersurface $S = \pi^{-1}(\{p\})$. We will prove that

$$\widetilde{K} \cong \pi^* K \otimes L_S^{n-1} \tag{10.11}$$

by showing that the bundles on both sides of the equality have a trivializing cover with the same transition functions.

We begin by constructing a trivializing cover of M for K. Choose a holomorphic coordinate chart $(W_0, (v^1, \dots, v^n))$ centered at p, whose image is a ball $B_\varepsilon(0) \subseteq \mathbb{C}^n$; and let $\{W_c\}_{c \in C}$ be a cover of $M \smallsetminus \{p\}$ by holomorphic coordinate domains, with coordinates $\left(\zeta^1_{(c)}, \dots, \zeta^n_{(c)}\right)$ on W_c. In each such open set there is a nonvanishing holomorphic section of K, namely $s_c = d\zeta^1_{(c)} \wedge \cdots \wedge d\zeta^n_{(c)}$ in W_c and $s_0 = dv^1 \wedge \cdots \wedge dv^n$ in W_0. The transition functions for this cover are given by

$$\tau^K_{cd} = \det\left(\frac{\partial \zeta^j_{(d)}}{\partial \zeta^k_{(c)}}\right) \quad \text{on } W_c \cap W_d,$$

$$\tau^K_{0d} = \det\left(\frac{\partial \zeta^j_{(d)}}{\partial v^k}\right) \quad \text{on } W_0 \cap W_d.$$

Next we construct a trivializing cover of $\widetilde{M}$ for $\widetilde{K}$. The map π restricts to a biholomorphism from $\widetilde{M} \smallsetminus S$ to $M \smallsetminus \{p\}$, so we can define $\widetilde{W}_c = \pi^{-1}(W_c)$ for $c \in C$, and on each such set there is a holomorphic local frame $\tilde{s}_c = \pi^* s_c$ for $\widetilde{K}$. We let $\widetilde{W}_0 = \pi^{-1}(W_0) \subseteq \widetilde{M}$, identified with $T(\varepsilon)$ as in the proof of Proposition 10.9. As before, we cover $\widetilde{W}_0$ with open sets $U_\alpha = \{([w], v) : w^\alpha \neq 0\}$. On each such set there are holomorphic coordinates $\left(z^1_{(\alpha)}, \dots, z^n_{(\alpha)}\right)$ defined by

$$z^j_{(\alpha)} = \begin{cases} \dfrac{w^j}{w^\alpha}, & j \neq \alpha, \\ v^\alpha, & j = \alpha, \end{cases}$$

and the inverse of the coordinate map is given by

$$([w], v) = \left(\left(\left[z^1_{(\alpha)}, \dots, 1 \dots, z^n_{(\alpha)}\right]\right),\ z^\alpha_{(\alpha)}\left(z^1_{(\alpha)}, \dots, 1, \dots \dots, z^n_{(\alpha)}\right)\right),$$

with the 1's in the αth positions. (The motivation for these coordinates comes from the description of T as the tautological bundle over $\mathbb{CP}^{n-1}$: the αth coordinate is a fiber coordinate for T, and the remaining coordinates are affine coordinates for $\mathbb{CP}^{n-1}$.)

Over each U_α, there is a holomorphic local frame for $\widetilde{K}$ given by

$$\tilde{s}_\alpha = dz^1_{(\alpha)} \wedge \cdots \wedge dz^n_{(\alpha)}.$$

To compute the transition functions for these frames, it is easiest to express them in terms of the functions $(v^1, \dots, v^n)$ on $\widetilde{W}_0$. On the complement of S in U_α, we have $w^j/w^\alpha = v^j/v^\alpha$, so

$$\begin{aligned}
\tilde{s}_\alpha &= d\left(\frac{v^1}{v^\alpha}\right) \wedge \cdots \wedge dv^\alpha \wedge \cdots \wedge d\left(\frac{v^n}{v^\alpha}\right) \\
&= \left(\frac{1}{v^\alpha}\right)^{n-1} dv^1 \wedge \cdots \wedge dv^\alpha \wedge \cdots \wedge dv^n \\
&= \left(\frac{1}{v^\alpha}\right)^{n-1} \pi^* s_0,
\end{aligned}$$

where the second equality follows from the fact that the remaining terms involving derivatives of v^α cancel because $dv^\alpha \wedge dv^\alpha = 0$. Therefore $\tilde{s}_\beta = (v^\alpha/v^\beta)^{n-1}\tilde{s}_\alpha = (w^\alpha/w^\beta)^{n-1}\tilde{s}_\alpha$ on $(U_\alpha \cap U_\beta) \smallsetminus S$, and by continuity $\tilde{s}_\beta = (w^\alpha/w^\beta)^{n-1}\tilde{s}_\alpha$ on all of $U_\alpha \cap U_\beta$. It follows that the collection $\{U_1, \dots, U_n, \widetilde{W}_c : c \in C\}$ is a trivializing cover of $\widetilde{M}$ for $\widetilde{K}$, with transition functions

$$\begin{aligned}
\tau^{\widetilde{K}}_{cd} &= \pi^* \tau^K_{cd} && \text{on } \widetilde{W}_c \cap \widetilde{W}_d, \\
\tau^{\widetilde{K}}_{\alpha d} &= (v^\alpha)^{n-1} \pi^* \tau^K_{0d} && \text{on } U_\alpha \cap \widetilde{W}_d, \\
\tau^{\widetilde{K}}_{\alpha\beta} &= \left(\frac{w^\alpha}{w^\beta}\right)^{n-1} && \text{on } U_\alpha \cap U_\beta.
\end{aligned}$$

This same cover is a trivalizing cover for π^*K. Since K is trivial over W_0, it follows that π^*K is trivial over each U_α, and it has transition functions

$$\begin{aligned}
\tau^{\pi^*K}_{cd} &= \pi^* \tau^K_{cd} && \text{on } \widetilde{W}_c \cap \widetilde{W}_d, \\
\tau^{\pi^*K}_{0d} &= \pi^* \tau^K_{0d} && \text{on } U_\alpha \cap \widetilde{W}_d, \\
\tau^{\pi^*K}_{\alpha\beta} &= 1 && \text{on } U_\alpha \cap U_\beta.
\end{aligned}$$

Finally, the proof of Proposition 10.9 shows that there are trivializations for L_S over the same open cover, with transition functions

$$\begin{aligned}
\tau^{L_S}_{cd} &= 1 && \text{on } \widetilde{W}_c \cap \widetilde{W}_d, \\
\tau^{L_S}_{\alpha d} &= v^\alpha && \text{on } U_\alpha \cap \widetilde{W}_d, \\
\tau^{L_S}_{\alpha\beta} &= \frac{w^\alpha}{w^\beta} && \text{on } U_\alpha \cap U_\beta.
\end{aligned}$$

Putting these results together, we see that the transition functions for $\widetilde{K}$ satisfy

$$\tau^{\widetilde{K}}_{\bullet\bullet} = \left(\tau^{L_S}_{\bullet\bullet}\right)^{n-1} \tau^{\pi^*K}_{\bullet\bullet}$$

for each pair of indices, which proves the result. □

Proof of the Embedding Theorem

Here is the first version of the embedding theorem.

Theorem 10.12 (Kodaira Embedding Theorem, Line Bundle Version). *Suppose M is a compact complex manifold. A holomorphic line bundle $L \to M$ is ample if and only if it is positive. Thus M is projective if and only if it admits a positive holomorphic line bundle.*

Proof. For the 1-dimensional case, see Problem 10-3. We assume from now on that M has dimension $n \geq 2$.

Let $L \to M$ be a holomorphic line bundle. Suppose first that L is ample, meaning that some positive tensor power L^k is very ample. Thus by Theorem 3.43, its associated map $F : M \to \mathbb{CP}^N$ is an embedding for some N. Proposition 3.45 shows that $L^k \cong F^*H$, so L^k is positive by Proposition 10.5.

Given a Hermitian fiber metric $\langle \cdot, \cdot \rangle_{L^k}$ on L^k whose Chern form is positive, there is a unique Hermitian fiber metric $\langle \cdot, \cdot \rangle_L$ on L such that $\langle \cdot, \cdot \rangle_{L^k}$ is the tensor product metric of that on L, namely

$$|v|_L = \big(|v \otimes \cdots \otimes v|_{L^k}\big)^{1/k}.$$

The Chern form of L with respect to this metric is

$$\frac{i}{2\pi}\Theta_L = \frac{i}{2\pi}\frac{1}{k}\Theta_{L^k},$$

which is also positive. This proves that L is a positive line bundle.

Conversely, suppose L is positive. Using Proposition 10.5(a), give M a Kähler metric and L a Hermitian fiber metric whose Chern form $\frac{i}{2\pi}\Theta_L$ is equal to the Kähler form ω. Choose a fiber metric on the anticanonical bundle K^* and let Θ_{K^*} be the curvature of its associated Chern connection. Because $\frac{i}{2\pi}\Theta_L$ is positive, there is some $k_1 > 0$ such that for every $k' \geq k_1$, the $(1,1)$-form

$$\frac{i}{2\pi}(\Theta_{K^*} + k'\Theta_L)$$

is positive on M, and thus $K^* \otimes L^{k'}$ is a positive line bundle.

Suppose k is any positive integer such that $k \geq 2nk_0 + k_1$, where k_0 is the integer guaranteed by Proposition 10.9 and k_1 is defined in the preceding paragraph.

We will show first that $\mathcal{O}(M; L^k)$ separates points. As explained at the beginning of this chapter, this is equivalent to showing that the evaluation map

$$e_M : \mathcal{O}(M; L^k) \to L^k_p \oplus L^k_q$$

is surjective for each pair of distinct points $p, q \in M$. Let $p, q \in M$ be arbitrary distinct points and consider the following short exact sheaf sequence on M:

$$0 \to \mathcal{I}_{\{p,q\}}(L^k) \hookrightarrow \mathcal{O}(L^k) \xrightarrow{e} (L^k_p)_p \oplus (L^k_q)_q \to 0. \tag{10.12}$$

By Proposition 5.22, the sheaf on the right is isomorphic to the quotient sheaf $\mathcal{O}(L^k)/\mathcal{I}_{\{p,q\}}(L^k)$, so our problem is equivalent to showing the surjectivity of the global section map Π_M in the following sequence:

$$0 \to \mathcal{I}_{\{p,q\}}(M; L^k) \hookrightarrow \mathcal{O}(M; L^k) \xrightarrow{\Pi_M} \Gamma\big(\mathcal{O}(L^k)/\mathcal{I}_{\{p,q\}}(L^k)\big), \tag{10.13}$$

where the right-hand group is the group of global sections of the quotient sheaf and Π_M is the global section map associated with the canonical sheaf homomorphism Π given by Proposition 5.22. (Bear in mind that this group of global sections is obtained from the sheafification functor, and is generally not the same as the quotient group $\mathcal{O}(M; L^k)/\mathcal{I}_{\{p,q\}}(M; L^k)$.)

Let $\pi : M_{pq} \to M$ be the blowup of M at p and q. Let $S_p = \pi^{-1}(p)$, $S_q = \pi^{-1}(q)$, $S = S_p \cup S_q$, and $\widetilde{L} = \pi^* L$. There is an analogous short exact sheaf sequence on M_{pq}:

$$0 \to \mathcal{I}_S(\widetilde{L}^k) \hookrightarrow \mathcal{O}(\widetilde{L}^k) \xrightarrow{\widetilde{\Pi}} \mathcal{O}(\widetilde{L}^k)/\mathcal{I}_S(\widetilde{L}^k)) \to 0. \tag{10.14}$$

We can combine the global section sequence of this sheaf sequence with (10.13) to obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{I}_S(M_{pq}; \widetilde{L}^k) & \hookrightarrow & \mathcal{O}(M_{pq}; \widetilde{L}^k) & \xrightarrow{\widetilde{\Pi}_{M_{pq}}} & \Gamma\big(\mathcal{O}(\widetilde{L}^k)/\mathcal{I}_S(\widetilde{L}^k)\big) \\ & \pi_0^* \Big\uparrow & & \pi^* \Big\uparrow & & \pi_1^* \Big\uparrow \\ 0 \longrightarrow & \mathcal{I}_{\{p,q\}}(M; L^k) & \hookrightarrow & \mathcal{O}(M; L^k) & \xrightarrow{\Pi_M} & \Gamma\big(\mathcal{O}(L^k)/\mathcal{I}_{\{p,q\}}(L^k)\big), \end{array} \tag{10.15}$$

where π^* is the pullback operator on sections (see Prop. 3.10), π_0^* is the restriction of π^*, and π_1^* is obtained from π^* by passing to the quotient: specifically, given a global section σ of the quotient sheaf $\mathcal{O}(L^k)/\mathcal{I}_{\{p,q\}}(L^k)$, for each $\tilde{x} \in M_{pq}$, we choose a representative $s \in \mathcal{O}(U; L^k)/\mathcal{I}_{\{p,q\}}(U; L^k)$ for the germ $\sigma(\pi(x))$, and define $(\pi_1^*\sigma)(\tilde{x})$ to be the germ of $\pi^* s$ in $\mathcal{O}(\pi^{-1}(U); \widetilde{L}^k)/\mathcal{I}_S(\pi^{-1}(U); \widetilde{L}^k)$. Because $\pi^* s$ vanishes on S whenever s vanishes at p and q, this is well defined.

We will show that π^* is surjective and π_1^* is injective. For π_*, suppose $\widetilde{\sigma} \in \mathcal{O}(M_{pq}; \widetilde{L}^k)$ is arbitrary. Because π restricts to a biholomorphism from $M_{pq} \smallsetminus S$ to $M \smallsetminus \{p, q\}$, we can define a holomorphic section $\sigma \in \mathcal{O}(M \smallsetminus \{p, q\}; L^k)$ by pulling back $\widetilde{\sigma}$ by the inverse of $\pi|_{M_{pq} \smallsetminus S}$. Choosing holomorphic coordinates for M and a holomorphic local frame s for L in a neighborhood U of p, we can write the restriction of σ to $U \smallsetminus \{p\}$ as $\sigma = f s^k$ for some holomorphic function $f : U \smallsetminus \{p\} \to \mathbb{C}$. Hartogs's extension theorem shows that this function extends holomorphically to all of U. A similar argument works in a neighborhood of q, so we obtain a globally defined section $\sigma \in \mathcal{O}(M; L^k)$ satisfying $\pi^*\sigma = \widetilde{\sigma}$ on $M_{pq} \smallsetminus S$, and by continuity on all of M_{pq}. Thus π^* is surjective.

To see that π_1^* is injective, suppose σ is a global section of the quotient sheaf $\mathcal{O}(L^k)/\mathcal{I}_{\{p,q\}}(L^k)$ satisfying $\pi_1^*\sigma = 0$. Given $x \in M$ and a representative section $s \in \mathcal{O}(U; L^k)$ for the germ $\sigma(x)$, let $\tilde{x}$ be any point in $\pi^{-1}(x)$. The fact that $(\pi_1^*\sigma)(\tilde{x}) = 0$ means that the pullback π^*s lies in $\mathcal{I}_S(\widetilde{U}; \widetilde{L}^k)$ on some neighborhood $\widetilde{U}$ of $\tilde{x}$. Since $\pi^*s(\tilde{x}) = (\tilde{x}, s(x))$, this implies that $s(x) = 0$ when x is equal to p or q, so s represents zero in the quotient space $\mathcal{O}(U; L^k)/\mathcal{I}_{\{p,q\}}(U; L^k)$ and thus $\sigma(x) = 0$.

Now comes the heart of the proof. We will use the Kodaira vanishing theorem to show that the map $\widetilde{\Pi}_{M_{pq}}$ in (10.15) is surjective. From the long exact cohomology sequence associated with the sheaf sequence (10.14), we see that this will be the case provided $H^1(M_{pq}; \mathcal{I}_S(\widetilde{L}^k)) = 0$. By Proposition 5.16, the sheaf $\mathcal{I}_S(\widetilde{L}^k)$ is isomorphic to $\mathcal{O}(\widetilde{L}^k \otimes L_S^*)$, which in turn is isomorphic to $\mathcal{O}(\widetilde{L}^k \otimes L_{S_p}^* \otimes L_{S_q}^*)$ by Problem 3-11. Corollary 10.8 to the Kodaira vanishing theorem shows that $H^1(M_{pq}; \mathcal{O}(\widetilde{L}^k \otimes L_{S_p}^* \otimes L_{S_q}^*)) = 0$ provided that $\widetilde{K}^* \otimes \widetilde{L}^k \otimes L_{S_p}^* \otimes L_{S_q}^*$ is a positive line bundle, where $\widetilde{K}$ is the canonical bundle of M_{pq}. But Proposition 10.11 shows that

$$\widetilde{K} \cong \pi^*K \otimes L_{S_p}^{n-1} \otimes L_{S_q}^{n-1},$$

from which it follows that

$$\widetilde{K}^* \otimes \widetilde{L}^k \otimes L_{S_p}^* \otimes L_{S_q}^* \cong \pi^*(K^* \otimes L^k) \otimes (L_{S_p}^*)^n \otimes (L_{S_q}^*)^n. \tag{10.16}$$

Because $k \geq 2nk_0 + k_1$, we can write $k = 2nk_0 + k'$ with $k' \geq k_1$, and rewrite the tensor product on the right-hand side of (10.16) as

$$\pi^*\left(K^* \otimes L^{k'}\right) \otimes \left(\pi^*L^{2k_0} \otimes L_{S_p}^* \otimes L_{S_q}^*\right)^n.$$

The first bundle in this tensor product is the pullback of a positive bundle, which is therefore nonnegative; the second is positive by Proposition 10.10. Thus the tensor product bundle is positive, which completes the proof that $\widetilde{\Pi}_{M_{pq}}$ is surjective.

Using this result, we can show that the map Π_M in (10.14) is also surjective. Let $\sigma \in \Gamma\big(\mathcal{O}(L^k)/\mathcal{I}_{\{p,q\}}(L^k)\big)$ be arbitrary. The surjectivity of $\widetilde{\Pi}_{M_{pq}}$ shows that there exists $\tilde{\tau} \in \mathcal{O}(M_{pq}; \widetilde{L}^k)$ such that $\widetilde{\Pi}_{M_{pq}}(\tilde{\tau}) = \pi_1^*(\sigma)$. Since π^* is surjective, there exists $\tau \in \mathcal{O}(M; L^k)$ such that $\pi^*(\tau) = \tilde{\tau}$, and then commutativity of (10.15) implies

$$\pi_1^*\Pi_M(\tau) = \widetilde{\Pi}_{M_{pq}}\pi^*(\tau) = \widetilde{\Pi}_{M_{pq}}(\tilde{\tau}) = \pi_1^*\sigma,$$

and the fact that π_1^* is injective means that $\Pi_M(\tau) = \sigma$. This completes the proof that $\mathcal{O}(M; L^k)$ separates points.

Next we address the question of separating directions. Let $p \in M$ be arbitrary and let s be a holomorphic local frame for L on a neighborhood of p. Using the result of Exercise 10.1, we need to show that the map $\delta_M : \mathcal{I}_{\{p\}}(M; L^k) \to \Lambda_p^{1,0}M$

given by $\delta_M(fs^k) = df_p$ is surjective. There is a short exact sheaf sequence

$$0 \to \mathscr{I}^2_{\{p\}}(L^k) \hookrightarrow \mathscr{I}_{\{p\}}(L^k) \xrightarrow{\delta} (\Lambda^{1,0}_p M)_p \to 0, \tag{10.17}$$

and the sheaf on the right is isomorphic to the quotient sheaf $\mathscr{I}_{\{p\}}(L^k)/\mathscr{I}^2_{\{p\}}(L^k)$, so we need to show the surjectivity of Π_M in the following sequence:

$$0 \to \mathscr{I}^2_{\{p\}}(M; L^k) \hookrightarrow \mathscr{I}_{\{p\}}(M; L^k) \xrightarrow{\Pi_M} \Gamma\big(\mathscr{I}_{\{p\}}(L^k)/\mathscr{I}^2_{\{p\}}(L^k)\big). \tag{10.18}$$

Let $\pi : M_p \to M$ be the blowup of M at p, $S = \pi^{-1}(p)$, and $\widetilde{L} = \pi^* L$. We have a short exact sheaf sequence on M_p:

$$0 \to \mathscr{I}^2_S(\widetilde{L}^k) \hookrightarrow \mathscr{I}_S(\widetilde{L}^k) \xrightarrow{\widetilde{\Pi}} \mathscr{I}_S(\widetilde{L}^k)/\mathscr{I}^2_S(\widetilde{L}^k)) \to 0,$$

and a commutative diagram of global section maps:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{I}^2_S(M_p; \widetilde{L}^k) & \hookrightarrow & \mathscr{I}_S(M_p; \widetilde{L}^k) & \xrightarrow{\widetilde{\Pi}_{M_p}} & \Gamma\big(\mathscr{I}_S(\widetilde{L}^k)/\mathscr{I}^2_S(\widetilde{L}^k)\big) \\
& \pi_0^* \big\uparrow & & \pi^* \big\uparrow & & \pi_1^* \big\uparrow \\
0 \longrightarrow & \mathscr{I}^2_{\{p\}}(M; L^k) & \hookrightarrow & \mathscr{I}_{\{p\}}(M; L^k) & \xrightarrow{\Pi_M} & \Gamma\big(\mathscr{I}_{\{p\}}(L^k)/\mathscr{I}^2_{\{p\}}(L^k)\big).
\end{array} \tag{10.19}$$

As before, π^* is surjective. We can also show that π_1^* is injective, but it takes a little more work this time. Suppose $\sigma \in \Gamma\big(\mathscr{I}_{\{p\}}(L^k)/\mathscr{I}^2_{\{p\}}(L^k)\big)$ with $\pi_1^*\sigma = 0$. Since the stalks of $\mathscr{I}_{\{p\}}(L^k)/\mathscr{I}^2_{\{p\}}(L^k)$ are zero at all points of M other than p, we need only show $\sigma(p) = 0$. Choose holomorphic coordinates $(v^1, \dots, v^n)$ on a domain W_0 centered at p with image $B_\varepsilon(0) \subseteq \mathbb{C}^n$, and identify $\widetilde{W} = \pi^{-1}(W_0) \subseteq M_p$ with $\{([w], v) \in \mathbb{CP}^{n-1} \times B_\varepsilon(0) : v \in [w]\}$ as in the proof of Proposition 10.9. After shrinking W_0 if necessary, we can choose a section $s_0 \in \mathscr{I}_{\{p\}}(W_0; L^k)$ whose image in the quotient space $\mathscr{I}_{\{p\}}(W_0; L^k)/\mathscr{I}^2_{\{p\}}(W_0; L^k)$ is a representative of the germ $\sigma(p)$. Because s_0 vanishes at p, we can write its coordinate representation in W_0 as

$$s_0(v) = \sum_j v^j f_j(v) s(v) \otimes \cdots \otimes s(v),$$

for some holomorphic functions $f_1, \dots, f_n$, where s is our chosen local frame for L. The hypothesis implies (after shrinking W_0 further if necessary) that $\pi^* s_0 \in \mathscr{I}^2_S(\widetilde{W}; \widetilde{L}^k)$. To show that $s_0 \in \mathscr{I}^2_{\{p\}}(W_0; L^k)$ and therefore $\sigma(p) = 0$, we need to show that $f_j(p) = 0$ for each j. Let $j \in \{1, \dots, n\}$ be arbitrary, and define a function $\lambda_j : D_\varepsilon(0) \to \widetilde{W}$ by

$$\lambda_j(z) = ([0, \dots, 1, \dots, 0], (0, \dots, z, \dots, 0)),$$

with the nonzero entries in the jth positions. Then $\lambda_j(z) \in S$ if and only if $z = 0$, and

$$(\pi_1^* s_0)(\lambda_j(z)) = \big([0, \dots, 1, \dots, 0],\ z f_j(0, \dots, z, \dots, 0) s(0, \dots, z, \dots, 0)^k\big).$$

The assumption that $\pi_1^* s_0 \in \mathscr{I}_S^2(\widetilde{W}; \widetilde{L}^k)$ implies that $f_j(0) = 0$. Since this is true for each j, it follows that $s_0 \in \mathscr{I}_{\{p\}}^2(W_0; L^k)$, and thus $\sigma(p) = 0$.

The map $\widetilde{\Pi}_{M_p}$ in (10.19) is surjective provided $H^1(M_p; \mathscr{I}_S^2(\widetilde{L}^k)) = 0$. Proposition 5.16 shows that $\mathscr{I}_S^2(\widetilde{L}^k) \cong \mathscr{O}(\widetilde{L}^k \otimes L_S^* \otimes L_S^*)$, and Corollary 10.8 shows that $H^1(M_p; \mathscr{O}(\widetilde{L}^k \otimes L_S^* \otimes L_S^*)) = 0$ provided $\widetilde{K}^* \otimes \widetilde{L}^k \otimes L_S^* \otimes L_S^*$ is positive. By Proposition 10.11, this bundle is isomorphic to

$$\pi^*(K^* \otimes L^k) \otimes (L_S^*)^{n+1}.$$

Because $k \geq 2nk_0 + k_1 \geq (n+1)k_0 + k_1$, we can write $k = (n+1)k_0 + k''$ with $k'' \geq k_1$, and rewrite the tensor product above as

$$\pi^*(K^* \otimes L^{k''}) \otimes (\pi^* L^{k_0} \times L_S^*)^{n+1}.$$

As above, this is a positive line bundle. Thus $\widetilde{\Pi}_{M_p}$ is surjective, and it follows as before that Π_M is also surjective. This completes the proof that $\mathscr{O}(M; L^k)$ separates directions, and therefore L is ample. □

There is another way to express the Kodaira embedding theorem that puts the emphasis on the Kähler geometry of M. We say that a Kähler metric on M is a ***Hodge metric*** if its Kähler class is *integral*; that is, its image in $H^2(M; \underline{\mathbb{R}})$ lies in the image of the coefficient homomorphism $H^2(M; \underline{\mathbb{Z}}) \to H^2(M; \underline{\mathbb{R}})$.

Theorem 10.13 (Kodaira Embedding Theorem, Geometric Version). *A compact complex manifold is projective if and only if it admits a Hodge metric.*

Proof. Suppose first that M is projective, so there is a holomorphic embedding $M \hookrightarrow \mathbb{CP}^N$ for some N. Let g_{FS} be the Fubini-Study metric on $\mathbb{CP}^N$. Recall from Example 8.14 that the Kähler form of g_{FS} is equal to $\frac{i}{2}\Theta_H$, where Θ_H is the curvature of the Chern connection on the hyperplane bundle H. Thus the metric $\tilde{g} = \frac{1}{\pi} g_{\mathrm{FS}}$ has Kähler form $\frac{i}{2\pi}\Theta_H$, which is the Chern form of the hyperplane bundle and therefore is integral by Theorem 7.14. The pullback of this metric to M is a Hodge metric on M. (This is the reason some authors define the Fubini–Study metric with the additional factor of $1/\pi$.)

Conversely, suppose M admits a Hodge metric g, and let ω be its Kähler form. By the Lefschetz theorem on $(1,1)$-classes, there is a line bundle $L \to M$ whose first real Chern class is represented by ω. Since ω is a positive $(1,1)$-form, this implies L is a positive line bundle, so it follows from Theorem 10.12 that M is projective. □

Applications of the Embedding Theorem

The Kodaira embedding theorem has numerous applications in complex differential geometry and algebraic geometry. Here we consider a few of the most important ones; some others are described in the problems at the end of the chapter.

Theorem 10.14 (Blowups of Projective Manifolds Are Projective). *Suppose M is a projective complex manifold and $\widetilde{M}$ is the blowup of M at finitely many points. Then $\widetilde{M}$ is also projective.*

Proof. Since a blowup at multiple points can be viewed as the result of a sequence of blowups one point at a time, it suffices to prove this when $\pi_p : M_p \to M$ is a blowup at one point. Since M is projective, it has a positive complex line bundle $L \to M$. Proposition 10.9 shows that for some $k > 0$, the bundle $\pi_p^* L^k \otimes L_{S_p}^* \to M_p$ is positive (where $S_p \subseteq M_p$ is the exceptional hypersurface), so M_p is also projective. □

Theorem 10.15 (Covering Manifolds Are Projective). *Suppose $\widetilde{M}$ and M are compact complex manifolds and $\pi : \widetilde{M} \to M$ is a holomorphic covering map. Then $\widetilde{M}$ is projective if and only if M is projective.*

Proof. First suppose M is projective. Then it admits a positive line bundle L, and $\pi^* L$ is positive line bundle on $\widetilde{M}$ by Proposition 10.5.

Conversely, suppose $\widetilde{M}$ is projective, and let $\tilde{g}$ be a Hodge metric on $\widetilde{M}$ with Kähler form $\widetilde{\omega}$. Because $\widetilde{M}$ and M are compact, the covering π has only finitely many sheets. Define a positive $(1,1)$-form on M as follows. Given $p \in M$, let U be a connected evenly covered neighborhood of p, and let $s_1, \dots s_m : U \to \pi^{-1}(U)$ be the distinct local sections of π, where m is the number of sheets of π. Define $\omega|_U = \sum_{i=1}^m s_i^* \widetilde{\omega}$. Given any other such neighborhood U' with sections $s_1', \dots, s_m'$, let U_0 be the connected component of $U \cap U'$ containing p. Because two local sections on a connected open set that agree at a point must agree everywhere [**LeeTM**, Thm. 11.12], the restrictions of the first set of sections must agree with those of the second set after reordering, so ω is well defined. It satisfies $\pi^* \omega = m\widetilde{\omega}$.

To see that ω is integral, by Lemma 6.27 we just need to show that its integral over every smooth 2-cycle is an integer. Let $c \in \operatorname{Sing}_2^\infty(M)$ be such a cycle. By [**LeeTM**, Prop. 13.19], we can replace c by a homologous smooth singular cycle (still denoted by c) with the property that the image of every singular simplex in c is contained in an evenly covered open set. Writing $c = \sum_j n_j \sigma_j$, we define a smooth singular chain $\pi^\# c \in \operatorname{Sing}_\infty^2 \left(\widetilde{M}\right)$ by

$$\pi^\# c = \sum_j n_j \left(s_1 \circ \sigma_j + \cdots + s_m \circ \sigma_j\right),$$

where for each σ_j, we let $s_1, \dots, s_m$ denote the local sections on a neighborhood of the image of σ_j in some order. Because $\pi^\#(\partial c) = \partial(\pi^\# c)$, it follows that $\pi^\# c$ is a cycle in $\widetilde{M}$. We compute

$$\begin{aligned}\int_c \omega = \sum_j n_j \int_{\sigma_j} \omega = \sum_j n_j \int_{\sigma_j} \left(s_1^* \widetilde{\omega} + \cdots + s_m^* \widetilde{\omega} \right) \\ = \sum_j n_j \Big(\int_{s_1 \circ \sigma_j} \widetilde{\omega} + \cdots + \int_{s_m \circ \sigma_j} \widetilde{\omega} \Big) \\ = \int_{\pi^\# c} \widetilde{\omega}.\end{aligned}$$

This is an integer by our assumption on $\widetilde{\omega}$. □

Theorem 10.16. *If M is a compact Kähler manifold with $h^{2,0}(M) = 0$, then M is projective.*

Proof. Suppose M satisfies the hypothesis. By Hodge symmetry, $h^{0,2}(M)$ is also zero, and thus $H^2_{\mathrm{dR}}(M;\mathbb{C}) \cong H^{1,1}(M)$ by the Hodge decomposition theorem. Let ω be a Kähler form on M. Lemma 6.27 shows that the integral classes span $H^2_{\mathrm{dR}}(M;\mathbb{R})$ over $\mathbb{R}$, so rational linear combinations of integral classes are dense in $H^2_{\mathrm{dR}}(M;\mathbb{R})$. Therefore by perturbing ω slightly, we can find a closed real $(1,1)$-form $\widetilde{\omega}$ that is a rational linear combination of integral forms and is still positive. Then multiplying $\widetilde{\omega}$ by a suitable positive integer, we obtain a positive integral $(1,1)$-form, which thus determines a Hodge metric on M. □

For our final application, we address the question of which complex tori are projective. Problem 10-3 shows that every 1-dimensional complex torus is projective, but in higher dimensions that is no longer the case.

Suppose V is an n-dimensional complex vector space and $\Lambda \subseteq V$ is a lattice. Let $J : V \to V$ be the associated complex structure map. A ***Riemann form for Λ*** is an antisymmetric real bilinear form Ω on V that satisfies

(i) $\Omega(Jv, Jw) = \Omega(v, w)$ for all $v, w \in V$;
(ii) $\Omega(v, Jv) > 0$ for all nonzero $v \in V$;
(iii) $\Omega(v, w) \in \mathbb{Z}$ for all $v, w \in \Lambda$.

Theorem 10.17 (Characterization of Projective Tori). *Suppose V is an n-dimensional complex vector space and $\Lambda \subseteq V$ is a lattice. The torus $M = V/\Lambda$ is projective if and only if there is a Riemann form for Λ.*

Proof. Suppose first that Ω is a Riemann form for Λ. Under the canonical identification of each tangent space of V with the underlying real vector space of V, we may consider Ω to be a constant-coefficient 2-form on V. Property (i) implies it is of type $(1,1)$ by the result of Problem 4-1, and (ii) shows it is positive. Because it has constant coefficients, it is closed; and it is also invariant under translations by Λ, so it descends to a Kähler form on M, which we denote by ω.

We need to show that ω defines an integral cohomology class; by Lemma 6.27, it suffices to show that $\int_c \omega \in \mathbb{Z}$ for every smooth 2-cycle c in M. Let $(v_1, \dots, v_{2n})$ be a basis for the free abelian group Λ, which is also a basis for V over $\mathbb{R}$. Let $(x^1, \dots, x^{2n})$ denote the corresponding real coordinates on V, so v_i has coordinates $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the ith place. For each pair of indices $i < j$, the real subspace of V spanned by (v_i, v_j) projects to an embedded 2-torus $T_{ij} \subseteq M$. It follows from the Künneth theorem for homology [**Hat02**, Thm. 3B.6] and induction that $H_2(M)$ is a free abelian group of rank $\binom{2n}{2}$ generated by the images of $H_2(T_{ij})$ under inclusion, which we can represent as smooth cycles by triangulating each such torus. Thus we need to show that $\int_{T_{ij}} \omega \in \mathbb{Z}$ for each $i < j$. We can parametrize T_{ij} by the map $\varphi_{ij} : [0,1]^2 \to M$ defined by $\varphi_{ij} = \pi \circ \Phi_{ij}$, where $\pi : V \to M$ is the quotient map and $\Phi_{ij} : [0,1]^2 \to V$ is given by $\Phi_{ij}(s,t) = (0, \dots, s, \dots, t, \dots, 0)$, with s and t in positions i and j. Writing $\Omega = \sum_{kl} \Omega_{kl} dx^k \wedge dx^l$ in these coordinates, we see that $\varphi_{ij}^* \omega = \Phi_{ij}^* \Omega = \Omega_{ij} ds \wedge dt$, and therefore

$$\int_{T_{ij}} \omega = \int_{[0,1]^2} \varphi_{ij}^* \omega = \int_{[0,1]^2} \Omega_{ij} ds \wedge dt = \Omega_{ij} = \Omega(v_i, v_j),$$

which is an integer. Thus ω defines an integral cohomology class, showing that M is projective by the Kodaira embedding theorem.

Conversely, suppose M is projective. Let g be a Hodge metric on M, and let ω be its Kähler form. Since Λ is a discrete subgroup of the Lie group V, the quotient space $M = V/\Lambda$ is a complex Lie group, which acts transitively on itself by left multiplication. Define a new 2-form Ω on M by

$$\Omega_p(X,Y) = \int_M (\gamma^* \omega)_p(X,Y) dV(\gamma),$$

where dV is a left-invariant volume form on M. Then it is straightforward to show that Ω is invariant under left multiplication on M and thus lifts to a constant-coefficient 2-form on V, which is a Riemann form for Λ. □

Problem 10-5 gives an example of how to apply this theorem to distinguish projective and nonprojective tori, and Problem 10-12 applies it to prove that Jacobian varieties of compact Riemann surfaces are always projective. A projective complex torus is called an ***abelian variety***. These projective varieties are among the most intensely studied objects in algebraic geometry.

Problems

10-1. Suppose M is a compact complex n-manifold and $L \to M$ is a negative line bundle. Prove that $H^q(M; \Omega^p(L)) = 0$ for $p + q < n$.

10-2. Prove that a negative line bundle on a compact complex manifold has no nontrivial holomorphic sections.

10-3. Show that if M is a connected compact Riemann surface of genus g and $L \to M$ is a holomorphic line bundle whose degree is greater than $2g$, then L is very ample. Use this to prove that every connected compact Riemann surface is projective. [Hint: Assuming $p, q \in M$ are distinct points such that every holomorphic section of L vanishing at p also vanishes at q, show that $\mathcal{O}(M; L \otimes L^*_{\{p\}}) \cong \mathcal{O}(M; L \otimes L^*_{\{p\}} \otimes L^*_{\{q\}})$, and use Riemann–Roch to derive a contradiction.]

10-4. Use the Riemann–Roch theorem to prove that every connected compact Riemann surface of genus 1 is biholomorphic to a nonsingular cubic curve in $\mathbb{CP}^2$.

10-5. Let $a \in \mathbb{R}$ and let $\Lambda \subseteq \mathbb{C}^2$ be the lattice spanned by the following four vectors:

$$v_1 = (1, 0), \quad v_2 = (i, 0), \quad v_3 = (0, 1), \quad v_4 = (a, i).$$

Show that that the torus $\mathbb{C}^2/\Lambda$ is projective if and only if a is rational.

10-6. If $F : M \to N$ is a smooth map between compact, connected, oriented, smooth n-manifolds, the ***degree of F*** is the unique integer k such that $\int_M F^*\omega = k \int_N \omega$ for every $\omega \in \mathscr{E}^n(N)$, and the preimage of every regular value of F contains exactly k points counted with signs (+ if F is locally orientation-preserving near the point, − if not) [**LeeSM**, Thm. 17.35]. Suppose M is a connected compact Riemann surface of positive genus.

(a) Show that the degree of a holomorphic map $F : M \to \mathbb{CP}^1$ is equal to the degree of the line bundle $F^*H \to M$, where $H \to \mathbb{CP}^1$ is the hyperplane bundle.

(b) Show that there is a holomorphic map $F : M \to \mathbb{CP}^1$ of degree 2 if and only if there exists a holomorphic line bundle $L \to M$ of degree 2 with $\dim \mathcal{O}(M; L) = 2$. [Hint: Given such a bundle, you will have to show that it has no base points. Assuming p is a base point of L, show that $\mathcal{O}(M; L) \cong \mathcal{O}(M; L \otimes L^*_{\{p\}})$ and use Proposition 7.25.]

(c) Show that when M has genus 1, it always admits such a map of degree 2.

10-7. A connected compact Riemann surface M of genus $g > 1$ is called a ***hyperelliptic curve*** if there is a holomorphic map $M \to \mathbb{CP}^1$ of degree 2, or equivalently by the result of Problem 10-6, if there is a holomorphic line bundle $L \to M$ of degree 2 with $\dim \mathcal{O}(M; L) = 2$. (The name reflects the fact that this generalizes a property shared by all elliptic curves, i.e., Riemann surfaces of genus 1, as shown in Problem 10-6.)

(a) Show that if M has genus 2, then it is hyperelliptic.

(b) Because the canonical bundle $K \to M$ has no base points (Problem 9-15) and $\dim \mathcal{O}(M; K) = g$, there is a holomorphic map $M \to \mathbb{CP}^{g-1}$ associated with K, called the ***canonical map***. Any variety in $\mathbb{CP}^{g-1}$ that is the image of a canonical map is called a ***canonical curve***. Show that the canonical map is a holomorphic embedding if and only if M is not hyperelliptic.

(c) Show that every nonsingular quartic curve in $\mathbb{CP}^2$ is a canonical curve, and thus not hyperelliptic.

10-8. Suppose (M, g) is a compact Kähler manifold whose Ricci curvature is either positive-definite or negative-definite. Prove that M is projective.

10-9. Let $H \to \mathbb{CP}^n$ be the hyperplane bundle. Prove the following:

$$\dim H^q(\mathbb{CP}^n; \mathcal{O}(H^d)) = \begin{cases} \dbinom{n+d}{n}, & q = 0 \text{ and } d \geq 0, \\ \dbinom{-d-1}{n}, & q = n \text{ and } d \leq -n-1, \\ 0 & \text{otherwise.} \end{cases}$$

10-10. Suppose (M, g) is a connected compact n-dimensional Kähler manifold. For any $p \in M$, let $\operatorname{Hol}(p)$ denote the holonomy group at p (see Problem 8-19). Show that M is a Calabi–Yau manifold and g is Ricci-flat if and only if $\operatorname{Hol}(p) \subseteq \operatorname{SU}(n)$ for some $p \in M$ and an appropriate choice of basis for T_pM.

10-11. Show that if (M, g) is a connected Ricci-flat Calabi–Yau manifold of dimension $n \geq 3$ whose holonomy group is equal to all of $\operatorname{SU}(n)$ for some $p \in M$, then $h^{2,0}(M) = 0$ and therefore M is projective. [Hint: First use Problem 9-8 to show that if η is a harmonic $(2,0)$-form, then η_p is invariant under the holonomy group $\operatorname{Hol}(p)$. Then show that $\operatorname{Hol}(p) \cong \operatorname{SU}(n)$ acts transitively on the set of 2-dimensional complex-linear subspaces of T'_pM, but there is a 2-dimensional subspace V such that $\eta_p|_{V\times V} \equiv 0$.] [Remark: Because of this result, some authors define Calabi–Yau manifolds as Kähler manifolds with holonomy equal to $\operatorname{SU}(n)$.]

10-12. This problem outlines a proof that Jacobian varieties are projective. Suppose M is a connected compact Riemann surface of genus $g \geq 1$, and $\operatorname{Jac}(M) = \Omega^1(M)^*/\Lambda$ is its Jacobian, where Λ is the lattice $\varphi(H_1(M))$ defined by (9.50). Let $(a_1, \dots, a_g, b_1, \dots, b_g)$ be the cycles representing a basis for $H_1(M)$ defined in the proof of Theorem 9.68, and let $(\alpha^1, \dots, \alpha^g, \beta^1, \dots, \beta^g)$ be closed 1-forms satisfying (9.51)–(9.54).

(a) For any closed 1-forms θ and $\tilde{\theta}$ on M, show that

$$\int_M \theta \wedge \tilde{\theta} = \sum_{j=1}^{g} \Big(\int_{a_j} \theta\Big)\Big(\int_{b_j} \tilde{\theta}\Big) - \Big(\int_{b_j} \theta\Big)\Big(\int_{a_j} \tilde{\theta}\Big).$$

[Hint: First show both sides of the equation are unchanged if θ is replaced by a cohomologous form. Then write θ (mod exact forms) as a linear combination of the α^j's and β^j's and use (9.51)–(9.54).]

(b) Choose a basis $(\eta^1, \dots, \eta^g)$ for $\Omega^1(M)$ over $\mathbb{C}$, and let $(\varepsilon_1, \dots, \varepsilon_g)$ be the dual basis. Write the period matrix of M as $\Pi = (A\ B)$, where

$$A_j^k = \int_{a_j} \eta^k, \quad B_j^k = \int_{b_j} \eta^k.$$

By applying part (a) to $\eta^k \wedge \eta^l$ and $\eta^k \wedge *\overline{\eta^l}$, prove that the periods satisfy the following ***Riemann bilinear relations***:

(i) For all $k, l = 1, \dots, g$,

$$\sum_{j=1}^{g} \left(A_j^k B_j^l - B_j^k A_j^l \right) = 0.$$

(ii) The following Hermitian matrix is positive definite:

$$Q^{kl} = i \sum_{j=1}^{g} \left(A_j^k \overline{B_j^l} - B_j^k \overline{A_j^l} \right).$$

(In matrix notation, these can be written succinctly as $AB^T - BA^T = 0$ and $Q = i(A\overline{B}^T - B\overline{A}^T) >> 0$.)

(c) Show that the matrix A is nonsingular, and conclude that by choosing a new basis $\tilde{\eta}^m = \sum_k (A^{-1})_k^m \eta^k$ for $\Omega^1(M)$, we can put the period matrix into the form $(I\ Z)$ for some $g \times g$ matrix Z that is symmetric and has positive definite imaginary part. [Hint: Assuming A is singular, show that there is a nonzero column matrix w such that $w^T Q \overline{w} = 0$.]

(d) Define a real-linear isomorphism $\Phi : \mathbb{R}^{2g} \to \mathbb{C}^g$ by

$$\Phi(x^1, \dots, x^g, y^1, \dots, y^g) = (z^1, \dots, z^g), \text{ where}$$

$$z^\alpha = \sum_{j=1}^{g} x^j A_j^\alpha + \sum_{j=1}^{g} y^j B_j^\alpha,$$

and let $\Psi = \Phi^{-1} : \mathbb{C}^g \to \mathbb{R}^{2g}$. With $(A\ B) = (I\ Z)$ as above, show that the differential of Ψ satisfies

$$D\Psi\left(\frac{\partial}{\partial z^\alpha}\right) = \sum_{\alpha=1}^{g} C_\alpha^j \frac{\partial}{\partial x^j} + D_\alpha^j \frac{\partial}{\partial y^j},$$

$$D\Psi\left(\frac{\partial}{\partial \bar{z}^\alpha}\right) = \sum_{\alpha=1}^{g} \overline{C_\alpha^j} \frac{\partial}{\partial x^j} + \overline{D_\alpha^j} \frac{\partial}{\partial y^j},$$

where the $g \times g$ matrices C and D are given by $C = -\overline{Z}(Z - \overline{Z})^{-1}$, $D = (Z - \overline{Z})^{-1}$. Use these formulas to show that $C^T D - D^T C = 0$ and $-i(C^T \overline{D} - D^T \overline{C})$ is positive definite.

(e) Let Ω_0 be the 2-form $\sum_{j=1}^{g} dx^j \wedge dy^j$ on $\mathbb{R}^{2g}$, and let $\Omega = \Psi^* \Omega_0$. Show that Ω is a Riemann form for Λ, and therefore $\mathrm{Jac}(M)$ is projective.

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