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Curvature estimates for four-dimensional gradient Ricci solitons

Matheus Andrade Ribeiro de Moura Horácio

Advisor: Dr. João Paulo dos Santos

Department of Mathematics,
University of Brasília

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Estimativas de curvatura para sólitons de Ricci gradiente quadridimensionais

por

Matheus Andrade Ribeiro de Moura Horácio*

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Comissão Examinadora:

Prof. Dr. João Paulo dos Santos - MAT/UnB (Orientador)

Prof. Dr. Tarcísio Castro Silva – MAT/UnB (Membro)

Prof. Dr. Ernani de Sousa Ribeiro Junior – UFC (Membro)

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What an astonishing thing a book is. It's a flat object made from a tree with flexible parts on which are imprinted lots of funny dark squiggles. But one glance at it and you're inside the mind of another person, maybe somebody dead for thousands of years. Across the millennia, an author is speaking clearly and silently inside your head, directly to you. Writing is perhaps the greatest of human inventions, binding together people who never knew each other, citizens of distant epochs. Books break the shackles of time. A book is proof that humans are capable of working magic.

Carl Sagan

We are a way for the Cosmos to know itself.

Carl Sagan

Extraordinary claims require extraordinary evidence.

Carl Sagan

Imagination will often carry us to worlds that never were. But without it we go nowhere.

Carl Sagan

For me, it is far better to grasp the Universe as it really is than to persist in delusion, however satisfying and reassuring.

Carl Sagan

If you wish to make an apple pie from scratch, you must first invent the universe.

Carl Sagan

I dedicate this work to Carl Edward Sagan.

Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding.

William Thurston

Wir müssen wissen.
Wir werden wissen.

David Hilbert

If I have seen further it is only by standing on the shoulders of giants.

Isaac Newton

A human being is part of a whole, called by us the "Universe", a part limited in time and space. He experiences himself, his thoughts and feelings, as something separated from the rest - a kind of optical delusion of his consciousness. This delusion is a kind of prison for us, restricting us to our personal desires and to affection for a few persons nearest us. Our task must be to free ourselves from this prison by widening our circles of compassion to embrace all living creatures and the whole of nature in its beauty.

Albert Einstein

Aut viam inveniam aut faciam.

Hannibal Barca

Only those who will risk going too far can possibly find out how far one can go.

T. S. Elliot

There is no royal road to geometry.

Euclid

We are absurdly accustomed to the miracle of a few written signs being able to contain immortal imagery, involutions of thought, new worlds with live people, speaking, weeping, laughing. We take it for granted so simply that in a sense, by the very act of brutish routine acceptance, we undo the work of the ages, the history of the gradual elaboration of poetical description and construction, from the treeman to Browning, from the caveman to Keats. What if we awake one day, all of us, and find ourselves utterly unable to read? I wish you to gasp not only at what you read but at the miracle of its being readable.

Vladimir Nabokov

Great men are forged in fire. It is the privilege of lesser men to light the flame.

Steven Moffat

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Abstract

In this work, we provide a study of complete gradient shrinking Ricci solitons of dimension 4. We present in detail the proofs (originally exposed in an article by Huai-Dong Cao, Ernani Ribeiro Jr, and Detang Zhou [6]) of two theorems that guarantee geometrical classifications and controls on the Ricci or Riemannian curvature, provided that pointwise estimates on the self-dual or anti-self-dual parts of the Weyl tensor or a certain control on the scalar curvature in terms of the soliton's potential function are satisfied.

Keywords: Ricci flow, Ricci solitons, curvature estimates, Weyl tensor, four-manifolds.

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Introduction

The Ricci flow, introduced in [26] by Hamilton, emerged in the 1980s as the most promising "sword" to slay the Poincaré conjecture, namely,

The Poincaré conjecture. *Every closed (id est, compact and without boundary) and simply connected three-dimensional manifold is homeomorphic to the sphere* $\mathbb{S}^3$.

Grigori Perelman extended Hamilton's initial program (which we will outline below) to prove Thurston's geometrization conjecture, which implies (as proved in [36]) the Poincaré conjecture.

Briefly and roughly speaking, Hamilton's strategy was the following: start with a closed and simply connected 3-manifold $\mathcal{M}^3$. A priori, $\mathcal{M}$ only has a topological structure, but it is a well-known fact (see Lemma **(L.2.1)**) that in dimension $n \leq 3$, any topological manifold $\mathcal{M}^n$ admits a unique differentiable structure compatible with its topology, and, *a fortiori*, admits a Riemannian metric. Let then g_0 be a Riemannian metric on $\mathcal{M}$. We will deform g_0 in such a way that we make its curvature "flow" from more curved areas to less curved areas (analogously to the way the heat equation makes temperature flow from hotter regions to colder ones), making the manifold more homogeneous over time. Ideally, one expects that such a process will eventually give us a metric g_∞ on $\mathcal{M}$ of constant Ricci curvature, or, equivalently, Einstein, and in dimension 3, *a fortiori* of constant sectional curvature (as we see in **(L.1.15)**). Since $\mathcal{M}^3$ is closed, in this case the Cartan-Hadamard Theorem would guarantee that such sectional curvature is positive, and by the Killing-Hopf Theorem, $\mathcal{M}^3$ would necessarily be homeomorphic to the sphere $\mathbb{S}^3$, as the Poincaré conjecture states.

The greatest indicators of the eventual success of this approach (and therefore of the immense power of the Ricci flow as a tool to relate topology and curvature) were the following results, proved by Hamilton in [26] and [23]:

Theorem (T.0.1) (Hamilton). *Let* $\mathcal{M}^3$ *be a closed differentiable* 3*-manifold that admits a Riemannian metric of strictly positive Ricci curvature. Then the universal cover of* $\mathcal{M}$ *is* $\mathbb{S}^3$. *In particular, if* $\mathcal{M}$ *is simply connected, then* $\mathcal{M}$ *is* $\mathbb{S}^3$.

Theorem (T.0.2) (Hamilton). *Let* $\mathcal{M}^4$ *be a closed differentiable* 4*-manifold that admits a Riemannian metric with positive curvature operator. Then the universal cover of* $\mathcal{M}$ *is* $\mathbb{S}^4$. *In particular, if* $\mathcal{M}$ *is not orientable, then* $\mathcal{M}$ *is the projective space* $\mathbb{RP}^4$.

Note that the "in particular" of Theorem **(T.0.2)** is due to the fact that the only even-dimensional space forms with positive curvature are $\mathbb{RP}^{2n}$ and $\mathbb{S}^{2n}$. Indeed, if an even-dimensional space form with

positive curvature is orientable, then it is also simply connected by Synge's Theorem and is therefore (by the Killing-Hopf Theorem) $\mathbb{S}^{2n}$. Otherwise, it is $\mathbb{RP}^{2n}$, since its orientable double cover is $\mathbb{S}^{2n}$.

In general, there are several obstructions to the effective execution of such a strategy (all of them knocked down by Grigori Perelman!). Among these, the following stand out

- the existence of possible *singularities*, regions where the curvature in some sense diverges in an a priori uncontrolled way;
- Ricci solitons, solutions that evolve only by changes of scale and diffeomorphisms, being in a certain sense stationary points of the flow. We say that a Riemannian manifold $(\mathcal{M}, g_{\mathcal{M}} = g_0)$ admits a Ricci soliton structure when there exist a field $V \in \Gamma(T\mathcal{M})$ and a constant $\lambda \in \mathbb{R}$ such that
$$\mathrm{Ric}_{g_{\mathcal{M}}} + \frac{1}{2}\mathscr{L}_V\left(g_{\mathcal{M}}\right) = \lambda\, g_{\mathcal{M}}. \tag{1}$$
Moreover, one can prove that they are also, in a certain sense (for more details, see [12]), important models of the singularities of the flow.

Both situations cause the evolution of the flow to reach an apparently insurmountable impasse. To get around such an obstacle, Perelman proposed to "cut" the manifold in order to restart the flow on the remaining pieces. However, formalizing this process naturally requires both a better comprehension of how the curvature can diverge and an understanding of the possible topologies/geometries of such solitons. The study of singularities and of Ricci solitons is then clearly motivated. In this sense,

- Perelman proved, in a sense that can be made formal (see, for example, [12]), that the injectivity radius does not decrease faster than the curvature in the regions where it diverges, which implies a certain geometric resemblance to a certain type of ancient Ricci flows. In turn, this fact allows one to conclude enough about the singularities so that the *surgery* process can be carried out;
- In [45], Perelman proved that every compact Ricci soliton is a ***gradient*** Ricci soliton, that is, that the field V in equation (1) can always be expressed as $V = \nabla f$ for some function $f \in \mathscr{C}^\infty(\mathcal{M})$.
- By the works of Ivey [27], Perelman [44], Naber [39], Ni-Wallach [42], and Cao-Chen-Zhu [5], it is known that any complete three-dimensional gradient shrinking Ricci soliton is a finite quotient of one of the following manifolds: the round sphere $\mathbb{S}^3$, the shrinking Gaussian soliton $\mathbb{R}^3$, or the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.

As mentioned in [4], one of the several remarkable particularities of the Ricci flow in dimension three is the preservation of positivity of the Ricci curvature along the flow. More specifically, one of the "magic tricks" of the Ricci flow in dimension three is the following: whenever a solution $g(t)$ of the Ricci flow develops a singularity in finite time, say $T < \infty$, the following phenomena necessarily occur:

- the curvature "blows up", more precisely,
$$\lim_{t \to T^-} \left(\sup_{x \in \mathcal{M}} \left\| \mathrm{Rm}_{g(t)} \right\| (x) \right) = \infty$$
- at any point and instant where the sectional curvature is negative and large in absolute value, one finds a much larger sectional curvature (than the absolute value of the negative sectional curvature) at the same time and instant. For details, one may consult [12].

As seen in [4] and [12], such facts imply our first curvature estimate: any three-dimensional shrinking or steady Ricci soliton not only has nonnegative sectional curvature, but even more, its curvature tensor Rm is controlled by the scalar curvature Scal, *id est*, there exists a universal constant $c > 0$ such that $\|\mathrm{Rm}\| \leq C\,\mathrm{Scal}$.

In general, information about the "divergence" of the curvature of a solution at "infinity" (more formally, of its associated singularity models, which arise as a certain limit of parabolic dilations of the flow) is crucial to the study of complete and noncompact Riemannian manifolds via the Ricci flow. Such considerations clarify the importance of studying the behavior of the curvature of gradient Ricci solitons.

However, as we increase dimensions, the situation becomes much more complex. In dimension $n \leq 3$, the curvature tensor Rm is determined by the Ricci tensor: indeed, this is an immediate consequence of Proposition **(P.3.4)** and of remark **(O.71)** (and this fact was essential for the proof of Theorem **(T.0.1)** given in [26]). In contrast, in higher dimensions, one more entity essential to the study of the control of Rm arises: the Weyl tensor. As exemplified in [41], this causes us, in dimension $n \geq 4$, to lose the preservation of positivity of the Ricci curvature by the flow.

In this sense, one of the most important recent results was the curvature estimate $\|\mathrm{Rm}\| \leq C\,\mathrm{Scal}$ for complete (and noncompact) gradient shrinking Ricci solitons with bounded scalar curvature (*id est*, $\mathrm{Scal} \leq M$ for some constant $M \in \mathbb{R}$). According to Huai-Dong Cao (see [4]), such an estimate, together with a uniqueness result obtained by Kotschwar-Wang in [28], played a crucial role in the progress of the classification of four-dimensional complete noncompact gradient shrinking Ricci solitons. In [6], the main object of study of our work, Cao-Ribeiro-Zhou extended the estimate of Munteanu and Wang, in a sense that we will make explicit shortly.

Our work is divided into three chapters. In **Chapter 1**, besides establishing our internal conventions and notation, we present in detail several fundamental concepts and results on tensors and Riemannian geometry, which, although basic, are usually not clarified in such detail in the literature and will be crucial to the following chapters.

In **Chapter 2**, we present some of the basic elements of the Ricci flow, starting with a motivation of the spirit of the flow as an analogue of the heat equation for Riemannian manifolds. Next, we study several classical examples of traditional solutions of the flow and of solitons (including the Bryant soliton and the Gaussian soliton), also clarifying their character as self-similar solutions of the flow.

Chapter 3 begins with a detailed study, based on [15], of the duality between curvature-like tensors and self-adjoint operators on $\Lambda^2(\mathcal{M})$ and of the decomposition of the bundle of curvature-like tensors, namely,

$$\mathscr{R}(T\mathcal{M}) = \mathbb{R}(g \owedge g) \oplus (g \owedge \mathcal{S}(T\mathcal{M})_0) \oplus \mathscr{W}(T\mathcal{M}).$$

As we will see, such a decomposition implies several important particularities that occur in dimension 4, the following standing out

- the fact that this is the lowest dimension in which the Weyl tensor does not vanish, and therefore it is the lowest dimension in which the curvature tensor Rm is not determined by the Ricci tensor;
- the decomposition of the bundle $\Lambda^2(\mathcal{M})$ of 2-forms on $\mathcal{M}$ into eigenbundles invariant under the Hodge star operator, which implies the following decomposition (crucial to the proofs of Theorems

(T.3.1) and **(T.3.2)**) of the curvature operator:

$$\mathrm{Rm} = \left(\begin{array}{c|c} \left(\mathcal{W}^+ + \dfrac{\mathcal{S}}{12}\right)\Big|_{\Lambda^2_+(\mathcal{M})} & \mathcal{E}|_{\Lambda^2_-(\mathcal{M})} \\ \hline \mathcal{E}|_{\Lambda^2_+(\mathcal{M})} & \left(\mathcal{W}^- + \dfrac{\mathcal{S}}{12}\right)\Big|_{\Lambda^2_-(\mathcal{M})} \end{array}\right).$$

Next, we establish some auxiliary results for our main purpose: the proofs of two results (by Huai-Dong Cao, Ernani Ribeiro Jr, and Detang Zhou, originally exposed in [6]) on complete four-dimensional gradient shrinking Ricci solitons (in Portuguese "encolhedores", also known in the literature by their English name, shrinking), namely,

- A complete gradient shrinking Ricci soliton of dimension 4 that is not Einstein and satisfies either

$$\|W^+\|^2 - \sqrt{6}\cdot\|W^+\|^3 \geq g\left((E \owedge E)^+, W^+\right);$$

  or

$$\|W^-\|^2 - \sqrt{6}\cdot\|W^-\|^3 \geq g\left((E \owedge E)^-, W^-\right),$$

  is a finite quotient of one of the following manifolds: $\mathbb{R}^4$, $\mathbb{S}^3 \times \mathbb{R}$, or $\mathbb{S}^2 \times \mathbb{R}^2$.

- If a complete and noncompact gradient shrinking Ricci soliton of dimension 4 satisfies

$$\mathrm{Scal} \leq A + \varepsilon f,$$

  where $A > 0$ and $0 \leq \varepsilon < 1$ are constants, then there exist constants c_0, c_1 and c_2 such that the following curvature estimates hold on $\mathcal{M}$:

  (i) $\|\mathrm{Ric}\| \leq c_0 + (c_1\varepsilon)f$,

  (ii) $\|\mathrm{Rm}\| \leq c_0 + (c_2\varepsilon)f^2$.

Finally, in the **Appendix**, we include the detailed proofs of several lemmas used throughout the work. Although very important to our whole exposition, the technical and index-laden nature of the computations present in such proofs justifies their placement in the appendix.

Chapter 1

Preliminary concepts and notation

The goal of this chapter is to establish in detail (based on the references [13], [14], [3], [31], [32] and [53]) the main foundations, notation, and internal conventions that we will use throughout this work. Although an effort has been made to make the whole exposition maximally self-contained, some basic concepts of Riemannian geometry and of differentiable manifolds, which we will assume to be already familiar to the reader, are necessary for a pleasant reading. We will begin by establishing the notation we will use for differentiable manifolds.

1.1 Riemannian manifolds

In this section we will recall some elementary notions of differentiable manifolds and of Riemannian geometry, based on the references [31] and [32].

Let $\mathcal{M}^n$ be a differentiable manifold of dimension n, $\psi : U \subset \mathcal{M} \to V \subset \mathbb{R}^n$ a chart around an arbitrarily fixed point $p \in \mathcal{M}$ and $\left\{ \dfrac{\partial}{\partial x^i} = \partial_i \right\}_{1 \leq i \leq n}$ the local frame induced by ψ. We will denote by $\{\mathrm{d}x^i\}_{1 \leq i \leq n}$ its associated coframe, defined by:

$$\begin{aligned} \mathrm{d}x^i : U &\to T^*U \\ p &\mapsto \mathrm{d}x^i\Big|_p : T_p\mathcal{M} \to \mathbb{R} \\ v &\mapsto \mathrm{d}x^i\Big|_p (v) = v^i, \end{aligned}$$

for each $v = \displaystyle\sum_{1 \leq i \leq n} v^i \left.\frac{\partial}{\partial x^i}\right|_p \in T_p\mathcal{M}$. Equivalently,

$$\mathrm{d}x^i\Big|_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \delta^i_j, \ \forall p \in U.$$

In the special case $\mathcal{M} = \mathbb{R}^n$, in which we have a single chart $\varphi = \mathrm{Id}$ and the (co)frames become global, we will denote them by

$$\left\{\mathrm{d}r^i\right\}_{1\leq i\leq n} \text{ and } \left\{\frac{\partial}{\partial r^i}\right\}_{1\leq i\leq n},$$

which can be naturally identified with the canonical basis of $\mathbb{R}^n$. Note that, by a very simple computation, the differential of the i-th coordinate function of the chart ψ (given by $r^i \circ \psi$, where $r^i : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the i-th coordinate of $\mathbb{R}^n$) coincides with $\mathrm{d}x^i$, *id. est,* $\mathrm{d}x^i = \mathrm{d}\left(r^i \circ \psi\right)$, which justifies our notation.

Next, we will introduce (based on the references [31] and [32]) the notions of vector bundles over differentiable manifolds and of connections and sections on such bundles. Since tensors manifest themselves as sections of appropriate vector bundles (in a sense that we will formalize shortly) and we will be interested in extending the usual covariant derivative to tensors, such an approach is clearly motivated.

Definition (D.1.1). A vector bundle of rank n over a differentiable manifold $\mathcal{M}$ is a differentiable manifold E together with a smooth surjective map $\pi : E \to \mathcal{M}$ satisfying the following conditions:

(a) For each $p \in \mathcal{M}$, the fiber

$$E_p \doteq \pi^{-1}(\{p\})$$

is a real vector space of dimension n.

(b) For each $p \in \mathcal{M}$, there exist a neighborhood $U_p \ni p$ of $p \in \mathcal{M}$ and a diffeomorphism

$$\psi_p : \pi^{-1}(U_p) \to U \times \mathbb{R}^n,$$

such that, if $\mathrm{pr}_1 : U_p \times \mathbb{R}^n \to U_p$ denotes the projection onto the first coordinate, then

$$\pi = \mathrm{pr}_1 \circ \psi_p.$$

(c) For each $q \in U$, the map

$$\psi_p|_{E_q} : E_q \to \{q\} \times \mathbb{R}^n \cong \mathbb{R}^n$$

is an isomorphism of vector spaces.

Definition (D.1.2). If $\pi : E \to \mathcal{M}$ is a vector bundle over $\mathcal{M}$, we will denote by $\Gamma(E)$ the set of sections of E over $\mathcal{M}$, that is,

$$\Gamma(E) \doteq \{s : M \to E \mid \underbrace{\pi \circ s = \mathrm{Id}_{\mathcal{M}}}_{\text{i.e, } s(p)\in E_p \ \forall p\in\mathcal{M}} \}.$$

Definition (D.1.3). A Riemannian manifold $(\mathcal{M}, g)$ is a differentiable manifold $\mathcal{M}$ (which, unless explicitly stated otherwise, we will always assume to be connected and without boundary) endowed with a section g of the bundle of positive definite symmetric bilinear forms on $\mathcal{M}$.

Remark (O.1). For each $p \in \mathcal{M}$, $g(p) : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ is an inner product on $T_p\mathcal{M}$. g is therefore a symmetric tensor field of order 2 that is positive definite in the sense that $g_p(v, v) \geq 0$ for every $v \in T_p\mathcal{M}$, with equality if and only if $v = 0$.

Definition (D.1.4). A connection on a bundle $\pi : E \to \mathcal{M}$ over $\mathcal{M}$ is a map $\nabla : \Gamma(T\mathcal{M}) \times \Gamma(E) \to \Gamma(E)$ satisfying the following properties

- $\nabla_X Y$ is linear over $\mathscr{C}^\infty(\mathcal{M})$ in X; that is, for any $f_1, f_2 \in \mathscr{C}^\infty(\mathcal{M})$ and $X_1, X_2 \in \Gamma(T\mathcal{M})$, we have
$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y.$$
- $\nabla_X Y$ is linear over $\mathbb{R}$ in Y: for any $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(E)$, we have
$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2.$$
- ∇ satisfies the following product rule: given any $f \in \mathscr{C}^\infty(\mathcal{M})$,
$$\nabla_X (fY) = f \nabla_X Y + X(f)\, Y.$$

Theorem (T.1.1) (Fundamental theorem of Riemannian Geometry). *Let $(\mathcal{M}, g)$ be a Riemannian manifold. Then there exists a unique connection ∇ on $\mathcal{M}$ satisfying the following conditions:*

- $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),\ \forall X, Y, Z \in \Gamma(T\mathcal{M})$;
- $\nabla_X Y - \nabla_Y X = [X, Y],\ \forall X, Y \in \Gamma(T\mathcal{M})$.

Proof: See [32]. ∎

The two conditions above imply the following equation, commonly called the *Koszul formula*:

$$\begin{aligned} 2\, g(\nabla_X Y, Z) = {} & X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ & - g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X),\ \forall X, Y, Z \in \Gamma(T\mathcal{M}). \end{aligned} \tag{1.1}$$

Definition (D.1.5). Let $\mathcal{M}$ be a Riemannian manifold and ∇ its Levi-Civita connection. We define the curvature tensor of $\mathcal{M}$ by

$$\begin{aligned} \mathrm{Rm} : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\to \Gamma(T\mathcal{M}) \\ (X, Y, Z) &\mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned} \tag{1.2}$$

It is also common to denote $\mathrm{Rm}(X,Y,Z)$ by $\mathrm{Rm}(X,Y)Z$.

Remark (O.2). The term "tensor" used in the previous definition will be defined in detail shortly.

Remark (O.3) ("Pointwise parallelism"). As seen in [32], if $p \in \mathcal{M}$ and $x \in T_p\mathcal{M}$, it is always possible to extend x to a field $X \in \Gamma(T\mathcal{M})$ such that $X(p) = x$ and $\nabla X|_{T_p\mathcal{M}} \equiv 0$ (that is, $\nabla_v X = 0$ for every $v \in T_p\mathcal{M}$).

Definition (D.1.6). If $p \in \mathcal{M}^n$ and $\varphi : U_p \to \mathbb{R}^n$ is a chart around p, the Christoffel symbols of the Levi-Civita connection ∇ of $\mathcal{M}$ with respect to φ are the real functions $\{\Gamma_{ij}^m\}_{1 \le i,j,m \le n} \subset \mathscr{C}^\infty(U_p)$ determined by
$$(\nabla_{\partial_i} \partial_j)(q) = \sum_{1 \le \ell \le n} \Gamma_{ij}^\ell(q)\, \partial_\ell|_q\,,\ \forall q \in U_p.$$

Remark (O.4). A direct computation shows that

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{1 \leq \ell \leq n} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}). \tag{1.3}$$

Remark (O.5). Let $X, Y \in \Gamma(T\mathcal{M})$, $p \in \mathcal{M}$ an arbitrarily fixed point, and suppose that $X(p) = x \in T_p\mathcal{M}, Y(p) = y \in T_p\mathcal{M}$. The fact that $(\nabla_X Y)(p)$ depends only on x and on Y in an (arbitrarily small) neighborhood of p makes harmless the abuse of notation of denoting $(\nabla_X Y)(p)$ either by $\nabla_x Y$ or by $\nabla_x y$. We will commit such an abuse several times during this work. *Mutatis mutandis*, it is clear that an entirely analogous remark holds for connections on bundles over $\mathcal{M}$.

Definition (D.1.7). A neighborhood U_p of a point $p \in \mathcal{M}$ is said to be a normal neighborhood of p when the exponential map

$$\exp_p : (\exp_p)^{-1}(U_p) \subset T_p\mathcal{M} \to U_p \subset \mathcal{M}$$

is a diffeomorphism.

Definition (D.1.8). Geodesics starting from p whose images are contained in a normal neighborhood of p are called *radial geodesics*.

Lemma (L.1.1). *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold and $p \in \mathcal{M}$ an arbitrarily fixed point. Then there exist a neighborhood $U_p \ni p$ and a local frame $\{E_i\}_{1 \leq i \leq n} \subset \Gamma(TU_p)$ satisfying*

$$g(E_i(q), E_j(q)) = \delta_{ij} \ \forall q \in U_p, \ \text{and} \ \nabla_{E_i(p)} E_j = 0,$$

for any $1 \leq i, j \leq n$.

Proof: Let U_p be a normal neighborhood of p and fix an orthonormal basis $\{b_i\}_{1 \leq i \leq n}$ of $T_p\mathcal{M}$. This basis induces an isomorphism

$$\begin{aligned} B : \mathbb{R}^n &\to T_p\mathcal{M} \\ (x^1, \cdots, x^n) &\mapsto \sum_{1 \leq i \leq n} x^i \, b_i. \end{aligned}$$

Now consider the chart $\varphi = \left(\exp_p \circ B\right)^{-1}$ around p and let $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}_{1 \leq i \leq n}$ be its associated coordinate basis. Since

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p &= \mathrm{d}(\exp_p \circ B) \left(\left.\frac{\partial}{\partial r^i}\right|_0 \right) \\ &= \underbrace{\mathrm{d}(\exp_p)_0}_{=\mathrm{Id}_{T_p\mathcal{M}}} \circ \underbrace{\mathrm{d}B_0}_{=B} \left(\left.\frac{\partial}{\partial r^i}\right|_0 \right) \\ &= B \left(\left.\frac{\partial}{\partial r^i}\right|_0 \right) \\ &= b_i, \end{aligned}$$

it follows that $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}_{1\leq i\leq n}$ is an orthonormal basis of $T_p\mathcal{M}$, so that we can therefore define $E_i(p) = \left.\frac{\partial}{\partial x^i}\right|_p$ for each $1 \leq i \leq n$. To extend such a basis to a local orthonormal frame, it suffices to recall the fact that parallel transport is an isometry and to perform parallel transport along radial geodesics. More precisely, for each $q \in U_p$, there exists a unique radial geodesic $\gamma_{p,q} : [0,1] \to \mathcal{M}$ joining p and q, and we can then define

$$E_i(q) = P_{\gamma_{p,q},0,1}(E_i(p)).$$

With respect to the chart φ, it is clear that the coordinate representation of the unique geodesic $\gamma_v : I \to \mathcal{M}$ starting from p with initial velocity $v \in T_p\mathcal{M}$ is given by $t \mapsto tv$, and therefore the geodesic equation is written as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\gamma_v(t)\right)^k + \sum_{1\leq i,j\leq n} \Gamma_{ij}^k\left(\gamma(t)\right) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_v(t)\right)^i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_v(t)\right)^j = 0, \ \forall t \in I \text{ and } \forall 1 \leq k \leq n.$$

In particular, we have

$$\sum_{1\leq i,j\leq n} \Gamma_{ij}^k(p)\, v^i v^j = 0, \ \forall v = \sum_{1\leq i\leq n} v^i b_i \in T_p\mathcal{M}, \text{ for every } 1 \leq k \leq n.$$

Setting $v = \partial_a$ for any fixed $1 \leq a \leq n$, we conclude that $\Gamma_{aa}^k(p) = 0$ for all $1 \leq a, k \leq n$. Finally, setting $v \in \{\partial_a + \partial_b, \partial_b - \partial_a\}$ for any fixed indices $1 \leq a, b \leq n$ and subtracting the resulting equations, we obtain $\Gamma_{ab}^k(p) = 0$ for any $1 \leq a, b, k \leq n$, from which it follows that $(\nabla_{E_i} E_j)(p) = 0$ for all $1 \leq i, j \leq n$.

■

Remark (O.6). It is easy to verify that in normal coordinates (coordinates of the form $\varphi = \left(\exp_p \circ B\right)^{-1}$, just as seen in the proof of the previous lemma) all the first partial derivatives of g_{ij} vanish at p. This shows that there are no geometric invariants of order < 2. As seen in [32], this fact can also be seen from the expansion of the metric in normal coordinates, given by

$$g_{ij}(t) = \delta_{ij} + \frac{1}{3} \sum_{1\leq k,\ell\leq n} \mathrm{Rm}_{ik\ell j}\, x^k\, x^\ell + O\left(|t|^3\right).$$

Remark (O.7). By the definition of geodesics and by the very construction of $\{E_i\}_{1\leq i\leq n}$, it is also clear that the geodesic frame of Lemma **(L.1.1)** satisfies

$$\nabla_v E_i = 0,$$

for every $v \in T_p\mathcal{M}$.

As we will see shortly, remark **(O.3)** and Lemma **(L.1.1)** will be very useful to us. The frame above is also called a *geodesic frame centered at p*.

Remark (O.8). The notation $a \cdot b$ will be reserved to ***always*** denote the multiplication of two real numbers; we will not use it to denote the inner product. Moreover, we will adopt the notation $\doteq$ instead of the usual $:=$ when writing an equality that defines an object.

1.2 Tensors

We will devote this section to exploring the main concepts of tensors present in this work. It is important to point out that our approach closely follows those found in [14] and [3].

Let $\mathbb{V}$ be a finite-dimensional real vector space, say of dimension n. We will denote by $\mathbb{V}^*$ its dual:

$$\mathbb{V}^* = \{f : \mathbb{V} \to \mathbb{R} \mid f \text{ is linear}\},$$

and by $\mathrm{End}(\mathbb{V})$ the space of all endomorphisms of $\mathbb{V}$ (that is, of linear transformations from $\mathbb{V}$ to $\mathbb{V}$). For integers $r, s \geq 1$, we will also denote:

$$(\mathbb{V}^*)^r = \underbrace{\mathbb{V}^* \times \mathbb{V}^* \cdots \times \mathbb{V}^*}_{r \text{ times}},$$

and

$$\mathbb{V}^s = \underbrace{\mathbb{V} \times \mathbb{V} \cdots \times \mathbb{V}}_{s \text{ times}}.$$

Definition (D.1.9). A real tensor of type (r, s) on $\mathbb{V}$ is a multilinear map $T : (\mathbb{V}^*)^r \times \mathbb{V}^s \to \mathbb{R}$. The set of real tensors of type (r, s) on $\mathbb{V}$ will be denoted by $\mathscr{T}^r_s(\mathbb{V})$.

Remark (O.9). We will say that $T \in \mathscr{T}^r_s(\mathbb{V})$ is r times contravariant and s times covariant. The number $r + s$ is called the *order* of T. It is clear that $\mathscr{T}^r_s(\mathbb{V})$ is a vector space.

Remark (O.10). It is easy to see that $\mathscr{T}^1_0(\mathbb{V}) = \mathbb{V}^{**} \cong \mathbb{V}$ and that $\mathscr{T}^0_1(\mathbb{V}) = \mathbb{V}^*$, where a natural isomorphism (in the sense that it does not depend on any choice of bases) between $\mathbb{V}$ and $\mathbb{V}^{**}$ is given by:

$$\mathbb{V} \ni v \mapsto (f \in \mathbb{V}^* \mapsto f(v) \in \mathbb{R}) \in \mathbb{V}^{**}.$$

We are then justified in calling linear functionals *covectors*.

Example (E.1.1). The *evaluation* map $\delta : \mathbb{V}^* \times \mathbb{V} \to \mathbb{R}$ defined by $\delta(f, v) \doteq f(v)$ is a tensor of type $(1, 1)$.

Example (E.1.2). The *determinant* map $\det : (\mathbb{R}^n)^n \to \mathbb{R}$, which takes n vectors and returns the determinant of the matrix obtained by placing all these vectors as columns, is a tensor of type $(0, n)$. One can also prove that it is the unique alternating multilinear tensor of such type (in other words, a differential n-form) that evaluates to 1 on the canonical basis. This fact is of important geometric significance, and in fact serves as motivation for the very definition of det.

Definition (D.1.10). Given $f \in \mathbb{V}^*$ and $v \in \mathbb{V}$, we define the $(1, 1)$ tensor $v \otimes f = f \otimes v$ by

$$(v \otimes f)(g, w) = g(v)f(w)\,, \ \forall (g, w) \in \mathbb{V}^* \times \mathbb{V}.$$

Definition (D.1.11). We define the tensor product

$$\begin{aligned} \otimes : \mathscr{T}^r_s(\mathbb{V}) \otimes \mathscr{T}^k_\ell(\mathbb{V}) &\to \mathscr{T}^{r+k}_{s+\ell}(\mathbb{V}) \\ (T, S) &\mapsto T \otimes S, \end{aligned}$$

by

$$(T \otimes S)(\omega^1, \cdots, \omega^{r+k}, \cdots, v_1, \cdots, v_{s+\ell}) = T(\omega^1, \cdots, \omega^r, v_1, \cdots, v_s) \cdot S(\omega^{r+1}, \cdots, \omega^{r+k}, v_{s+1}, \cdots, v_{s+\ell}),$$

for all $\omega^1, \cdots, \omega^{r+k} \in \mathbb{V}^*$ and $v_1, \cdots, v_{s+\ell} \in \mathbb{V}$.

Definition (D.1.12). Let $\beta = \{\mathbf{e}_i\}_{1 \leq i \leq n}$ and $\beta^* = \{\mathbf{e}^i\}_{1 \leq i \leq n}$ be dual bases of $\mathbb{V}$ and $\mathbb{V}^*$, respectively (that is, $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$ for each $1 \leq i, j \leq n$). Given $T \in \mathscr{T}^r_s(\mathbb{V})$, the components of T in the basis β are the real numbers defined by:

$$T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} \doteq T(\mathbf{e}^{i_1}, \cdots, \mathbf{e}^{i_r}, \mathbf{e}_{j_1}, \cdots, \mathbf{e}_{j_s}),$$

for all $1 \leq i_1, \cdots i_r, j_1, \cdots j_s \leq n$.

Remark (O.11) ("*One chart to rule them all!*"). It is common in the literature to use the fact that tensors are *coordinate-free* without formally making explicit what is meant by this. An erroneous interpretation of this fact would be to say that given any two bases $\{\mathbf{e}_i\}_{1 \leq i \leq n}$ and $\{\tilde{\mathbf{e}}_i\}_{1 \leq i \leq n}$ and any $T \in \mathscr{T}^r_s(\mathbb{V})$, we have

$$T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} = T^{\widetilde{i_1} \cdots \widetilde{i_r}}{}_{\widetilde{j_1} \cdots \widetilde{j_s}},$$

for all $1 \leq i_1, \cdots, i_r, j_1, \cdots, j_s, \widetilde{i_1}, \cdots, \widetilde{i_r}, \widetilde{j_1} \cdots \widetilde{j_r} \leq n$, which is evidently absurd! In fact, what happens is that to prove that two tensors $T, Q \in \mathscr{T}^r_s(\mathbb{V})$ are equal (that is, that

$$T(\omega^1, \cdots, \omega^r, v_1, \cdots, v_s) = Q(\omega^1, \cdots, \omega^r, v_1, \cdots, v_s),$$

for every $(\omega^1, \cdots, \omega^r, v_1, \cdots, v_s) \in (\mathbb{V}^*)^r \times \mathbb{V}^s)$, it suffices to prove that

$$T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} = Q^{i_1 \cdots i_r}{}_{j_1 \cdots j_s},$$

for all $1 \leq i_1, \cdots, i_r, j_1, \cdots, j_s \leq n$, which is evidently true by the multilinearity of T and Q. Equivalently, to define a tensor $T \in \mathscr{T}^r_s(\mathbb{V})$, it suffices to declare what its action on $\{\mathbf{e}^1, \cdots, \mathbf{e}^r, \mathbf{e}_1, \cdots, \mathbf{e}_s\}$ is, that is, it suffices to specify the n^{r+s} real numbers $\{T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}\}$. When $r + s = n = 3$, for example, we can think of T as a "cube" of dimensions n of $3^3 = 27$ numbers. Therefore, in a certain sense tensors are generalizations of the usual "two-dimensional" matrices (equivalently, linear transformations) that we already know.

A fortiori, when we have two tensors (a priori not necessarily equal) defined on a Riemannian manifold, the fact that tensors can (by definition) be evaluated pointwise guarantees that, to prove the equality between them, it suffices to arbitrarily fix a point $p \in \mathcal{M}^n$ and a convenient local frame $\{E_i\}_{1 \leq i \leq n}$ around p and to prove that the equalities between components are satisfied. Again, this does *not* mean that the components are independent of the fixed frame: we merely use the fact of multilinearity to guarantee that the definition of a tensor by its components is well defined (in contrast with not necessarily multilinear maps, which are not determined by their values on bases). Briefly: two tensorial geometric quantities that coincide in *one* given coordinate system (resp. one given frame) coincide in *any* coordinate system (resp. any frame).

Finally, as we see in more detail in remark (O.18), these considerations can be "fiberized". More precisely, given any atlas of a differentiable manifold $\mathcal{M}^n$, say

$$\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n\}_{\alpha \in I},$$

any section of the bundle $T \in \Gamma\left(\mathscr{T}_b^a\left(T\mathcal{M}\right)\right)$ (commonly called simply a tensor of type (a,b) on $\mathcal{M}$) is determined by the knowledge of all its n^{a+b} components $\left\{T^{i_1\cdots i_a}{}_{j_1\cdots j_b}\right\} \subset \mathscr{C}^\infty\left(\mathcal{M}\right)$, given explicitly by

$$\left\{T\left(\mathbf{e}^{(\alpha,i_1)},\ldots,\mathbf{e}^{(\alpha,i_a)},\mathbf{e}_{(\alpha,j_1)},\ldots,\mathbf{e}_{(\alpha,j_b)}\right)\right\}_{\alpha\in I} \subset \mathscr{C}^\infty\left(\mathcal{M}\right),$$

where

$$\mathbf{e}_{(\alpha,j_k)}(p) = \mathrm{d}\left(\varphi_\alpha^{-1}\right)_{\varphi_\alpha(p)}\left(\left.\frac{\partial}{\partial r^{j_k}}\right|_{\varphi(p)}\right),$$

and $\left\{\mathbf{e}^{(\alpha,i_\ell)}\right\}_{1\le\ell\le n}$ is the dual basis $\left\{\mathbf{e}^{(\alpha,i_\ell)}\right\}_{1\le\ell\le n}$ associated with the basis $\left\{\mathbf{e}_{(\alpha,j_k)}\right\}_{1\le k\le n}$, determined by

$$\mathbf{e}^{(\alpha,i_\ell)}\left(\mathbf{e}_{(\alpha,j_k)}\right) = \delta^{i_\ell}_{j_k}.$$

The converse direction (that is, when certain real functions indexed over an atlas of $\mathcal{M}$ are in fact tensor components) is a little more complicated, so we will not elaborate further on it, but it is worth observing that such a problem appears when trying, for example, to "globalize" the method of moving frames (for more details, see [18] or [53]).

Proposition (P.1.1). The map

$$\begin{aligned}\psi : \mathrm{End}(\mathbb{V}) &\to \mathscr{T}_1^1(\mathbb{V})\\ T &\mapsto \psi(T) : \mathbb{V}^*\times\mathbb{V}\to\mathbb{R}\\ &\qquad (f,v)\mapsto f(T(v)).\end{aligned}$$

is an isomorphism.

Proof: See [14]. ∎

Definition (D.1.13). Let $\mathbb{V}$ and $\mathbb{W}$ be finite-dimensional real vector spaces. A multilinear map

$$T : (\mathbb{V}^*)^r \times \mathbb{V}^s \to \mathbb{W}$$

is called a $\mathbb{W}$-tensor of type (r,s) on $\mathbb{V}$. We will denote the vector space of all such maps by $\mathrm{Hom}_s^r(\mathbb{V},\mathbb{W})$.

Remark (O.12). *A fortiori,*

$$\mathrm{Hom}_s^r(\mathbb{V},\mathbb{R}) = \mathscr{T}_s^r(\mathbb{V}).$$

Proposition (P.1.2). $\mathrm{Hom}_s^r(\mathbb{V},\mathbb{V})$ is isomorphic to $\mathscr{T}_s^{r+1}(\mathbb{V})$.

Proof: Define $\psi : \text{Hom}_s^r(\mathbb{V}, \mathbb{V}) \to \mathscr{T}_s^{r+1}(\mathbb{V})$ by:

$$[\psi(T)]\left(\omega^1, \cdots, \omega^{r+1}, v_1, \cdots, v_s\right) = \omega^{r+1}\left(T\left(\omega^1, \cdots, \omega^r, v_1, \cdots, v_s\right)\right),$$

for each $\left(\omega^1, \cdots, \omega^{r+1}, v_1, \cdots, v_s\right) \in (\mathbb{V}^*)^{r+1} \times \mathbb{V}^s$. It is simple to verify that ψ is an isomorphism.

■

Due to Proposition **(P.1.2)**, we will not make (as is also not done in the literature) any distinction between real tensors and vector-valued tensors.

As we will see next, the existence of an inner product g on $\mathbb{V}$ allows us to make even more identifications.

Definition (D.1.14). We define the flat map $\flat : \mathbb{V} \to \mathbb{V}^*$ by

$$\flat(v) = g(v, \bullet).$$

Proposition (P.1.3). The flat map is an isomorphism.

Proof: See [14].

■

Remark (O.13). Since the map $\flat$ (defined in **(D.1.14)**) is an isomorphism on each fiber of $T\mathcal{M}$, we can then define ∇ by equation **(1.1)**. This is a common technique in proofs in Riemannian geometry: we determine the image of a field via flat, which in turn determines the field itself.

Remark (O.14). It is *not* true in general that $\flat(\mathbf{e}_i) = \mathbf{e}^i$. In fact, it is easy to see that this is equivalent to the basis $\{\mathbf{e}_i\}_{1\leq i\leq n}$ being orthonormal.

Remark (O.15). By Proposition **(P.1.3)**, it is clear that every $v \in \mathbb{V}$ determines and is determined by $\flat(v)$. Now, since $\flat(v)$ is determined by $\{\flat(v)(\mathbf{e}_k)\}_{1\leq k\leq n} = \{g(v, e_k)\}_{1\leq k\leq n}$, it is clear that v is determined by the values $\{g(v, e_k)\}_{1\leq k\leq n}$. It is then reasonable to expect that an explicit relation between the components of v and such values can be obtained. Indeed, denoting $g_{ij} \doteq g(\mathbf{e}_i, \mathbf{e}_j)$, $v = \sum_{1\leq i\leq n} v^i\, \mathbf{e}_i$ and by g^{ij} the coefficients of the inverse matrix of $\tilde{g} = (g_{ij})$ (which is well defined, since $\tilde{g}v = 0 \iff v = 0$ by the nondegeneracy of the metric), observe that the equality

$$g(v, \mathbf{e}_k) = g\left(\sum_{1\leq i\leq n} v^i\, \mathbf{e}_i, \mathbf{e}_k\right) = \sum_{1\leq i\leq n} v^i\, g_{ki}, \ \forall k \in \{1, \ldots, n\},$$

holds, and therefore,

$$\sum_{1\leq k\leq n} g^{kt}\, g(v, \mathbf{e}_k) = \sum_{1\leq i,k\leq n} v^i g_{ki} g^{kt} = \sum_{1\leq i\leq n} v^i\, \delta_{it} = v^t, \ \forall t \in \{1, \cdots, n\}.$$

Due to Proposition **(P.1.3)**, the inverse map of the flat map is then well defined:

Definition (D.1.15). Given $f \in \mathbb{V}^*$, there exists a unique $f^\#$ such that

$$f(v) = g\left(f^\#, v\right),$$

for every $v \in \mathbb{V}$. The sharp map $\sharp : \mathbb{V}^* \to \mathbb{V}$ is then defined by:

$$\begin{aligned} \sharp : \mathbb{V}^* &\to \mathbb{V} \\ f &\mapsto f^\#. \end{aligned}$$

Proposition (P.1.4). Let $\{\mathbf{e}_i\}_{1\leq i\leq n}$ be a basis of $\mathbb{V}$. If $v = \sum_{1\leq i\leq n} v^i \mathbf{e}_i \in \mathbb{V}$ and $f = \sum_{1\leq i\leq n} f_i \mathbf{e}^i \in \mathbb{V}^*$, then writing:

$$\flat(v) = v_\flat = \sum_{1\leq i\leq n} (v_\flat)_i \mathbf{e}^i \text{ , as well as } \sharp(f) = f^\# = \sum_{1\leq i\leq n} (f^\#)^i \mathbf{e}_i,$$

we have that

$$(v_\flat)_i = \sum_{1\leq j\leq n} g_{ij} v^j, \text{ as well as } (f^\#)^i = \sum_{1\leq j\leq n} g^{ij} f_j.$$

Proof: See [14].

■

Corollary (C.1.1). *For each $1 \leq i \leq n$, we have that $(\mathbf{e}_i)_\flat = \sum_{1\leq j\leq n} g_{ij}\mathbf{e}^j$, and $(\mathbf{e}^i)^\# = \sum_{1\leq j\leq n} g^{ij}\mathbf{e}_j$.*

To motivate the several different maps that can be used to identify tensors of different types, we will state some propositions below, whose proofs can be found in [14].

Proposition (P.1.5). The map:

$$\begin{aligned} \sharp_1 : \mathscr{T}_0^2(\mathbb{V}) &\to \mathscr{T}_1^1(\mathbb{V}) \\ T &\mapsto T^{\sharp_1} : \mathbb{V}^* \times \mathbb{V} \to \mathbb{R}, \end{aligned}$$

where

$$T^{\sharp_1}(f, v) = T\left(f^\#, v\right), \ \forall (f^\#, v) \in \mathbb{V}^* \times \mathbb{V},$$

is an isomorphism.

Proposition (P.1.6). The map

$$\begin{aligned} \flat_1 : \mathscr{T}_0^2(\mathbb{V}) &\to \mathscr{T}_1^1(\mathbb{V}) \\ T &\mapsto T_{\flat_1} : \mathbb{V}^* \times \mathbb{V} \to \mathbb{R}, \end{aligned}$$

where

$$[T_{\flat_1}](f, v) = T(f, v_\flat), \ \forall (f, v) \in \mathbb{V}^* \times \mathbb{V},$$

is an isomorphism.

Remark (O.16). It is common not to make explicit the isomorphisms used when identifying tensors of different types. For example, a persistent abuse of notation in the literature is to denote the components of $T^{\sharp_1}$ simply by T^i_j instead of $(T^{\sharp_1})^i_j$. Such an abuse is reasonable, because although T is not of type $(1,1)$, the isomorphism $T \mapsto T^{\sharp_1}$ is obvious enough not to be made explicit repeatedly. In general, we have several possible identifications, one of which we see in Proposition **(P.1.7)**. It is these isomorphisms that allow us to "lower" or "raise" indices at will. Formally, if $T \in \mathscr{T}^k_\ell(\mathbb{V})$ is a tensor of type (k,ℓ), we say that the tensor $T^{\sharp a} \in \mathscr{T}^{k+1}_{\ell-1}(\mathbb{V})$ determined by

$$(T^{\sharp a})(\omega^1, \cdots, \omega^k, \bullet, v_1, \cdots, v_{\ell-1}) = T(\omega^1, \cdots, \omega^k, v_1, \cdots, v_{a-1}, \sharp(\bullet), \cdots, v_{\ell-1})$$

is the tensor obtained from T by raising its a-th index. Analogously, we say that the tensor $T_{\flat r} \in \mathscr{T}^{k-1}_{\ell+1}(\mathbb{V})$ determined by

$$T_{\flat r}(\omega^1, \cdots, \omega^{r-1}, \bullet, \cdots, \omega^k, v_2, \cdots, v_{\ell+1}) = T(\omega^1, \cdots, \omega^{r-1}, \flat(\bullet), \cdots, \omega^k, v_2, \cdots, v_{\ell+1})$$

is the tensor obtained from T by lowering its r-th index.

Remark (O.17). In general, the identification we use to view tensors of type $(0,k)$ as tensors of type $(1,k-1)$ (or, equivalently, of type $(1,k)$ as of type $(0,k+1)$) is made by raising the last index of T, *id est*, if T is a $(0,k)$ tensor, we also view it as the tensor $T \in \mathscr{T}^1_{k-1}(\mathbb{V})$ determined by

$$\mathbb{V} \ni T(y_1, \cdots, y_{k-1}) = \sharp[T(y_1, \cdots, y_{k-1}, \bullet)],$$

for all $y_1, \cdots, y_{k-1} \in \mathbb{V}$.

Proposition (P.1.7). Let $a, b, r, s \geq 0$ be integers such that $r + s = a + b$. Then $\mathscr{T}^a_b(\mathbb{V}) = \mathscr{T}^r_s(\mathbb{V})$.

Proof: There are several possible isomorphisms. We can, for example, lower all the possible contravariant indices and identify both spaces with $\mathscr{T}^0_{r+s}(\mathbb{V})$, that is, consider

$$\mathscr{T}^r_s(\mathbb{V}) \xrightarrow{\varphi} \mathscr{T}^0_{r+s}(\mathbb{V}) \xrightarrow{\mathrm{Id}} \mathscr{T}^0_{a+b}(\mathbb{V}) \xrightarrow{\psi} \mathscr{T}^a_b(\mathbb{V}),$$

where

$$[\varphi(T)](v_1, \ldots, v_{r+s}) = T\left((v_1)_\flat, (v_2)_\flat, \ldots, (v_r)_\flat, v_{r+1}, \ldots, v_{r+s}\right),$$

and

$$[\psi(T)]\left(\omega^1, \ldots, \omega^a, v_1, \ldots, v_b\right) = T\left(\left(\omega^1\right)^\sharp, \left(\omega^2\right)^\sharp, \ldots, \left(\omega^a\right)^\sharp, v_1, \ldots, v_b\right).$$

■

Remark (O.18). It is obvious that all the constructions recalled in this section "fiberize" naturally. If $(\mathcal{M}^n, g)$ is a Riemannian manifold, we can, for example, consider the (a,b)-tensor bundle

$$\mathscr{T}^a_b(T\mathcal{M}) = \bigcup_{p \in \mathcal{M}} \mathscr{T}^a_b(T_p\mathcal{M}) = \{(p,T) \mid p \in \mathcal{M} \text{ and } T \in \mathscr{T}^a_b(T_p\mathcal{M})\},$$

which is a differentiable manifold of dimension $n + n^{a+b}$. Its topology and atlas are induced analogously to those of the usual tangent bundle. That is, if

$$\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n\}_{\alpha \in I},$$

is a maximal atlas of $\mathcal{M}$ and

$$\begin{aligned} \pi : \mathscr{T}^a_b(T\mathcal{M}) &\to \mathcal{M} \\ (p, T) &\mapsto p, \end{aligned}$$

denotes the natural projection, we define an atlas of $\mathscr{T}^a_b(T\mathcal{M})$ by

$$\tilde{\mathcal{A}} = \left\{ \psi_\alpha : \mathscr{T}^a_b(U_\alpha) = \bigcup_{p \in U_\alpha} \mathscr{T}^a_b(T_p\mathcal{M}) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^{n^{a+b}} \subset \mathbb{R}^{n+n^{a+b}} \right\}_{\alpha \in I},$$

where ψ_α is defined by

$$\begin{aligned} &\psi_\alpha \left(q, \sum_{\substack{1 \le i_1, \ldots, i_b \le n \\ 1 \le j_1, \ldots, j_a \le n}} T^{j_1 \ldots j_a}_{i_1 \ldots i_b}(q) \cdot \left.\frac{\partial}{\partial x^{j_1}}\right|_q \otimes \left.\frac{\partial}{\partial x^{j_2}}\right|_q \otimes \ldots \otimes \left.\frac{\partial}{\partial x^{j_a}}\right|_q \otimes \left.\mathrm{d}x^{i_1}\right|_q \otimes \left.\mathrm{d}x^{i_2}\right|_q \otimes \ldots \otimes \left.\mathrm{d}x^{i_b}\right|_q \right) \\ &= \left(\varphi_\alpha(q), \sum_{\substack{1 \le i_1, \ldots, i_b \le n \\ 1 \le j_1, \ldots, j_a \le n}} T^{j_1 \ldots j_a}_{i_1 \ldots i_b}(q) \cdot \left.\frac{\partial}{\partial r^{j_1}}\right|_q \otimes \left.\frac{\partial}{\partial r^{j_2}}\right|_q \otimes \ldots \otimes \left.\frac{\partial}{\partial r^{j_a}}\right|_q \otimes \left.\mathrm{d}r^{i_1}\right|_q \otimes \left.\mathrm{d}r^{i_2}\right|_q \otimes \ldots \otimes \left.\mathrm{d}r^{i_b}\right|_q \right), \end{aligned}$$

and we declare that a subset $O \subset \mathscr{T}^a_b(U_\alpha)$ is open in $\mathscr{T}^a_b(U_\alpha)$ if, and only if, $\varphi_\alpha(O)$ is open in $\varphi_\alpha(U_\alpha) \times \mathbb{R}^{n^{a+b}}$. We will say that a section $T \in \Gamma(\mathscr{T}^a_b(T\mathcal{M}))$ is a tensor field on $\mathcal{M}$. For simplicity, we will also commit the abuse of notation of writing that $T \in \mathscr{T}^a_b(T\mathcal{M})$ is a tensor of type (a, b) on $\mathcal{M}$ (which corresponds to the abuse of language of regarding tensor fields as tensors).

Example (E.1.3). Let T, S be tensors of type $(0, 4)$ and $(0, 2)$ on $\mathcal{M}$, respectively. Then, considering local coordinates around an arbitrarily fixed $p \in \mathcal{M}$, the components (and consequently T and S themselves) of T and S have, among several other possible ones, the following different "faces":

$$\underbrace{T^{ijk\ell}}_{=T\left((\mathrm{d}x^i)^\sharp, (\mathrm{d}x^j)^\sharp, (\mathrm{d}x^k)^\sharp, (\mathrm{d}x^\ell)^\sharp\right)}, \qquad \underbrace{T^{ij}{}_k{}^\ell}_{=T\left((\mathrm{d}x^i)^\sharp, (\mathrm{d}x^j)^\sharp, \partial_k, (\mathrm{d}x^\ell)^\sharp\right)}, T^{ij}{}_{k\ell}, T_{ijk\ell}, \ldots, \text{ as well as } S^{ij}, S^i{}_j, S_{ij}, S_i{}^j,$$

where i, j, k, ℓ range over $\{1, \ldots, n\}$.

Example (E.1.4). Consider the Riemannian curvature tensor defined in **(1.2)**. Let us fix local coordinates on a neighborhood $U_p \ni p \in \mathcal{M}^n$, denoting its associated frame by $\{\partial_i\}_{1 \le i \le n}$. For each $q \in U_p$, there exist n real numbers, which we will denote by $\{A^\ell_{ijk}\}_{1 \le \ell \le n}$ (and which in turn determine a tensor A of type $(3, 1)$), determined by the relation

$$\mathrm{Rm}_{\partial_i, \partial_j}(\partial_k) = \sum_{1 \le \ell \le n} A^\ell_{ijk} \partial_\ell, \text{ at } q.$$

Taking the inner product with ∂_s on both sides of the equation above, we obtain

$$\langle \mathrm{Rm}_{\partial_i,\partial_j}(\partial_k), \partial_s \rangle = \mathrm{Rm}_{ijks} = \sum_{1\leq \ell \leq n} A^\ell_{ijk} g_{\ell s},$$

and therefore

$$\sum_{1\leq t\leq n} g^{st} \cdot \mathrm{Rm}_{ijks} = \sum_{1\leq \ell,t\leq n} A^\ell_{ijk} \cdot g_{\ell s} \cdot g^{st} = A^t_{ijk}.$$

Recalling Corollary (C.1.1) and committing the same (harmless) abuse of notation as in the previous example, we see that

$$\mathrm{Rm}^t_{ijk} = \mathrm{Rm}\left(\left(\mathrm{d}x^t\right)^\sharp, \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\right) = A^t_{ijk},$$

for every $1 \leq t \leq n$. Hence, A is determined by Rm, and we are now completely justified in writing

$$\mathrm{Rm}_{\partial_i,\partial_j}(\partial_k) = \sum_{1\leq \ell \leq n} \mathrm{Rm}^\ell_{ijk} \partial_\ell.$$

Example (E.1.5). The way we defined it, a priori $\mathrm{Rm} = \mathrm{Rm}^1_3$ is a tensor of type $(1,3)$. But recalling Propositions **(P.1.7)** and **(P.1.2)**, it is reasonable to denote the following tensors

$$\begin{aligned} \mathrm{Rm}^{0,4} : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\to \mathscr{C}^\infty(\mathcal{M}) \\ (X,Y,Z,W) &\mapsto [\flat(W)](\mathrm{Rm}(X,Y,Z)), \end{aligned}$$

and

$$\begin{aligned} \widetilde{\mathrm{Rm}}^1_3 : \Gamma(T^*\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\to \mathscr{C}^\infty(\mathcal{M}) \\ (\flat(W),X,Y,Z) &\mapsto [\flat(W)](\mathrm{Rm}(X,Y,Z)), \end{aligned}$$

besides

$$\begin{aligned} \mathrm{Rm}^{0,2} : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\to \mathrm{End}\left(\Gamma(T\mathcal{M})\right) \\ (X,Y) &\mapsto \mathrm{Rm}^{0,2}(X,Y) : \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M}) \\ Z &\mapsto \mathrm{Rm}(X,Y)Z, \end{aligned}$$

simply by Rm as well.

Example (E.1.6). If $\mathcal{M}$ is a Riemannian manifold, every vector field $X \in \Gamma(T\mathcal{M})$ can be seen as a 1-form $X_\flat = g(X, \bullet) \in \Gamma(T^*\mathcal{M})$.

Definition (D.1.16). Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a smooth map and suppose that $T \in \mathscr{T}^0_k(T\mathcal{N})$. The *pull-back* of T is the tensor $\varphi^*T \in \mathscr{T}^0_k(T\mathcal{M})$ determined by

$$\left(\varphi^*T\right)_p (v_1,\ldots,v_k) = T_{\varphi(p)} \left(\mathrm{d}\varphi_p(v_1),\ldots,\mathrm{d}\varphi_p(v_k)\right),$$

for all $p \in \mathcal{M}$ and $v_1,\ldots,v_k \in T_p\mathcal{M}$.

1.2 The trace of tensors

The *naive* trace of an endomorphism $T : \mathbb{V} \to \mathbb{V}$, which is the trace of its matrix representation $A = (a_{ij})_{1\leq i,j\leq n}$ in any basis, given by

$$\operatorname{tr}(T) = \operatorname{tr}(A) = \sum_{1\leq i\leq n} a_{ii}, \tag{1.4}$$

makes no sense at all for tensors of type $(0, 2)$, because if $h \in \mathscr{T}^0_2(T\mathcal{M})$, the expression

$$\sum_{1\leq i\leq n} h_{ii}$$

is *not* invariant under change of coordinates. The goal of this subsection is to extend the definition of the trace of endomorphisms to tensors in a well-defined way.

Remark (O.19). Alternatively, with $\{\mathbf{e}_i\}_{1\leq i\leq n}$ being any basis of $\mathbb{V}$, equation (1.4) can be equivalently written as:

$$\operatorname{tr}(T) = \sum_{1\leq i,j\leq n} g^{ij}\langle T(\mathbf{e}_i), \mathbf{e}_j\rangle = \sum_{1\leq i\leq n} T^i{}_i = \sum_{1\leq i\leq n} T_i{}^i.$$

Lemma (L.1.2). There exists a unique linear map $\operatorname{tr}^1_1 : \mathscr{T}^1_1(\mathbb{V}) \to \mathbb{R}$ such that:

$$\operatorname{tr}^1_1(v \otimes f) = f(v),\ \forall (f, v) \in \mathbb{V}^* \times \mathbb{V}.$$

Proof: See [14].

■

Definition (D.1.17). The contraction in the a-th contravariant entry and b-th covariant entry is the map $\operatorname{tr}^a_b : \mathscr{T}^r_s(\mathbb{V}) \to \mathscr{T}^{r-1}_{s-1}(\mathbb{V})$ given by:

$$[\operatorname{tr}^a_b(T)](f^1, \ldots, f^{r-1}, v_1, \ldots, v_{s-1}) \doteq \operatorname{tr}^1_1(T(f^1, \ldots, f^{a-1}, \bullet, f^a, \ldots, f^{r-1}, v_1, \ldots, v_{b-1}, \bullet, v_b, \ldots, v_s)).$$

We also define:

$$\begin{aligned}\operatorname{tr}_{1,2} : \mathscr{T}^0_2(\mathbb{V}) &\to \mathbb{R}\\ T &\mapsto \operatorname{tr}^1_1(T^{\sharp_1}),\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}^{1,2} : \mathscr{T}^2_0(\mathbb{V}) &\to \mathbb{R}\\ T &\mapsto \operatorname{tr}^1_1(T_{\flat_1}),\end{aligned}$$

Given $T \in \mathscr{T}^r_s(\mathbb{V})$ and integers $1 \leq a \leq r, 1 \leq b \leq s$, a direct computation shows that the components of the tensor $\mathrm{tr}^a_b(T) \in \mathscr{T}^{r-1}_{s-1}(\mathbb{V})$ relative to a previously fixed basis are given by:

$$(\mathrm{tr}^a_b(T))^{i_1\dots i_{r-1}}{}_{j_1\dots j_{s-1}} = \sum_{1\leq k\leq n} T^{i_1\dots i_{a-1}k i_a\dots i_{r-1}}{}_{j_1\dots j_{b-1}k j_b\dots j_{s-1}},$$

where the dummy index k appears in the a-th upper entry and in the b-th lower entry. When $\mathbb{V}$ is endowed with an inner product (which from now on we will assume is always the case) and either r or s is ≥ 2, we also define the *metric traces* $\mathrm{tr}_{a,b}(T) \in \mathscr{T}^r_{s-2}(\mathbb{V})$ and $\mathrm{tr}^{a,b}(T) \in \mathscr{T}^{r-2}_s(\mathbb{V})$ by raising or lowering an index and applying $\mathrm{tr}_{1,2}$ or $\mathrm{tr}^{1,2}$. More precisely, we define

$$\mathrm{tr}_{a,b}(T)(f^1,\dots,f^r,x_1,\dots,x_{s-2}) \doteq \mathrm{tr}_{1,2}(T(f^1,\dots,f^r,x_1,\dots,\bullet,\dots,\bullet,\dots,x_{s-2})),$$

and

$$\mathrm{tr}^{a,b}(T)(f^1,\dots,f^{r-2},x_1,\dots,x_s) \doteq \mathrm{tr}^{1,2}(T(f^1,\dots,\bullet,\dots,\bullet,\dots,f^{r-2},x_1,\dots,x_s)),$$

with the $\bullet$ in the a-th and b-th entries, respectively. In coordinates, we have

$$(\mathrm{tr}_{a,b}(T))^{i_1\dots i_r}{}_{j_1\dots j_{s-2}} = \sum_{1\leq k,\ell\leq n} g^{k\ell} T^{i_1\dots i_r}{}_{j_1\dots k\dots\ell\dots j_{s-2}}, \text{ and } (\mathrm{tr}^{a,b}(T))^{i_1\dots i_{r-2}}{}_{j_1\dots j_s} = \sum_{1\leq k,\ell\leq n} g_{k\ell} T^{i_1\dots k\dots\ell\dots i_r}{}_{j_1\dots j_s},$$

where $(g_{ij})_{1\leq i,j\leq n}$ are the components of g with respect to such a fixed basis and $(g^{ij})_{1\leq i,j\leq n}$ denotes its inverse.

Example (E.1.7). A useful intuition for this operation is illustrated by the following situation: given $\alpha \in \mathscr{T}^0_2(\mathbb{V})$, the fact that g is nondegenerate guarantees the existence of $A \in \mathrm{End}(\mathbb{V})$ such that $\alpha(x,y) = g(A(x),y)$ for all $x,y \in \mathbb{V}$. Denoting by $\{\mathbf{e}_i\}_{1\leq i\leq n}$ such a basis and setting $x = \mathbf{e}_i, y = \mathbf{e}_k$, we obtain $\alpha_{jk} = g(A(\mathbf{e}_j),\mathbf{e}_k) = \sum_{1\leq i\leq n} g(A^i{}_j\mathbf{e}_i,\mathbf{e}_k) = \sum_{1\leq i\leq n} A^i{}_j g_{ik}$, so that $A^i{}_j = \sum_{1\leq k\leq n} g^{ik}\alpha_{jk}$ and $\mathrm{tr}_{1,2}(\alpha) = \mathrm{tr}(A)$.

Definition (D.1.18). The Ricci curvature of a Riemannian manifold $(\mathcal{M}^n, g)$ is defined as $\mathrm{Ric} = \mathrm{tr}_{1,4}(\mathrm{Rm})$.

Remark (O.20). Explicitly, having fixed $p \in \mathcal{M}$ and with $\{\mathbf{e}_i\}_{1\leq i\leq n}$ a basis of $T_p\mathcal{M}$, we have

$$\begin{aligned}
\mathrm{Ric}(v,w) &= \mathrm{tr}\left(T_p\mathcal{M} \ni u \mapsto \mathrm{Rm}(x,u)w\right) \\
&= \sum_{1\leq i\leq n} g\left(\mathrm{Rm}\left(\mathbf{e}_i,v\right)w,\mathbf{e}_i\right) \\
&= \sum_{1\leq i\leq n} g\left(\mathrm{Rm}\left(v,\mathbf{e}_i\right)\mathbf{e}_i,w\right) \\
&= \sum_{1\leq i\leq n} g\left(\mathrm{Rm}\left(\mathbf{e}_i,w\right)v,\mathbf{e}_i\right),
\end{aligned}$$

for all $v,w \in T_p\mathcal{M}$, and therefore Ric is a symmetric bilinear form. Equivalently, we could have adopted the definition of Ric as a tensor of type $(1,1)$, determined by

$$\mathrm{Ric}(v) = \sum_{1\leq i\leq n} \mathrm{Rm}\left(v,\mathbf{e}_i\right)\mathbf{e}_i, \ \forall v \in T_p\mathcal{M}.$$

Definition (D.1.19). An *Einstein manifold* is a Riemannian manifold $(\mathcal{M}^n, g)$ that satisfies $\mathrm{Ric}_g = \lambda\, g$ for some real function $\lambda \in \mathscr{C}^\infty(\mathcal{M})$.

Remark (O.21). As we will see in Lemma **(L.1.15)**, in dimension $n \geq 3$ there is no loss of generality in replacing "for some real function $\lambda \in \mathscr{C}^\infty(\mathcal{M})$" in Definition **(D.1.19)** by "for some constant $\lambda \in \mathbb{R}$." In this case, Ric is constant (and equal to λ) on the unit tangent bundle of $\mathcal{M}$. By the polarization formula, namely,

$$\mathrm{Ric}(v,w) = \frac{\mathrm{Ric}(v+w,v+w) - \mathrm{Ric}(v,v) - \mathrm{Ric}(w,w)}{2},$$

we see that $\left(\mathcal{M}^{n\geq 3}, g\right)$ is an Einstein manifold if and only if the quadratic form induced by Ric is constant on the unit tangent bundle $\mathcal{UM}$.

Remark (O.22). Given $T \in \mathrm{End}(\mathbb{V})$, the tensor $\mathscr{T}^0_2(\mathbb{V}) \ni \tilde{T}(\bullet,\star) \doteq \langle T(\bullet), \star\rangle$ satisfies $\mathrm{tr}_{1,2}(\tilde{T}) = \mathrm{tr}(T)$. Conversely, given any $(0,2)$ tensor $\tilde{T}$, there exists a unique such T satisfying $\tilde{T}(\bullet,\star) \doteq \langle T(\bullet), \star\rangle$. *Mutatis mutandis*, the same holds for tensors of type $(2,0)$. We then see that although the metric was necessary to define the trace of tensors of any types, the fact that we brought everything back to the case of the trace of endomorphisms (whose traces are independent of any choice of bases or inner product) made the trace in fact independent of the metric.

Definition (D.1.20). Let $T \in \mathscr{T}^0_k(\mathbb{V})$ be a k-tensor. We can view T as a $(1,k-1)$ tensor $\widetilde{T}$, determined by

$$\widetilde{T}(y_1,\ldots,y_{k-1}) = \sharp(T(y_1,\ldots,y_{k-1},\bullet)),$$

The g-trace of T is the $(k-2)$-tensor defined by

$$(\mathrm{tr}_g(T))(y_1,\ldots,y_{k-2}) = \mathrm{tr}(\underbrace{\widetilde{T}(y_1,\ldots,y_{k-2},\bullet)}_{\in \mathrm{End}(\mathbb{V})}). \tag{1.5}$$

Note that its components are given by

$$\begin{aligned}(\mathrm{tr}_g(T))(\mathbf{e}_1,\ldots,\mathbf{e}_{k-2}) &= \sum_{1\leq i,j\leq n} g^{ij}\langle \widetilde{T}_{i_1\ldots i_{k-2}i}, \mathbf{e}_j\rangle \\ &= \sum_{1\leq i,j\leq n} g^{ij} T_{i_1\ldots i_{k-2}ij}.\end{aligned}$$

Example (E.1.8). The Ricci tensor of a Riemannian manifold $\mathcal{M}$, denoted by Ric, can be seen as an endomorphism:

$$\begin{aligned}\mathrm{Ric} : \Gamma(T\mathcal{M}) &\to \Gamma(T\mathcal{M}) \\ X &\mapsto \mathrm{Ric}(X) = \sharp(\mathrm{Ric}(X,\bullet)),\end{aligned}$$

that is,

$$g(\mathrm{Ric}(X),Y) = \mathrm{Ric}(X,Y)\ \forall X,Y \in \Gamma(T\mathcal{M}).$$

Proposition (P.1.8). *Let $(\mathbb{V},g)$ be a real vector space and $B : \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ a symmetric bilinear form on $\mathbb{V}$. Let us denote by*

$$\mathbb{S}_r \doteq \{v \in \mathbb{V} \mid \|v\| = r\}$$

the sphere of radius $r > 0$ centered at the origin of $\mathbb{V}$ and by $\mathrm{d}\mathbb{V}$ the volume form on $\mathbb{V}$ induced by g, so that

$$\mathrm{vol}(\mathbb{S}_r) = \int_{\mathbb{S}_r} \mathrm{d}\mathbb{V}.$$

Then

$$\frac{1}{\mathrm{vol}(\mathbb{S}_r)} \int_{\mathbb{S}_r} B(x,x)\ \mathrm{d}\mathbb{V} = \frac{r^2}{\dim(\mathbb{V})} \cdot \mathrm{tr}_g(B).$$

Proof: By the spectral theorem, there exists a basis of $\mathbb{V}$, say $\{\mathbf{e}_i\}_{1\leq i\leq n}$, that is g-orthonormal and B-orthogonal. Let us call $\lambda_k \doteq B(\mathbf{e}_k, \mathbf{e}_k)$ for each $1 \leq k \leq n$. Given any $x = \sum_{1\leq k\leq n} x_k e_k \in \mathbb{V}$, the square of the norm of x is then given by $g(x,x = \sum_{1\leq k\leq n} x_k^2$, and B is determined by $B(x,x) = \sum_{1\leq k\leq n} \lambda_k x_k^2$. Therefore,

$$\int_{\mathbb{S}_r} B(x,x)\ \mathrm{d}\mathbb{V} = \sum_{1\leq k\leq n} \lambda_k \int_{\mathbb{S}_r} x_k^2\ \mathrm{d}\mathbb{V}. \tag{1.6}$$

We now claim that given any $1 \leq k \neq \ell \leq n$, we have

$$\int_{\mathbb{S}_r} x_k^2\ \mathrm{d}\mathbb{V} = \int_{\mathbb{S}_r} x_\ell^2\ \mathrm{d}\mathbb{V}.$$

Indeed, the orthogonal reflection φ about the line $\mathrm{span}(\mathbf{e}_k + \mathbf{e}_\ell)$ is an isometry that preserves $\mathrm{d}\mathbb{V}$ and satisfies $x_k^2 \circ \varphi = x_\ell^2$, and therefore the claim follows from the change of variables theorem. Denoting then

$$I_k \doteq \int_{\mathbb{S}_r} x_k^2\ \mathrm{d}\mathbb{V},$$

we have that

$$I_k = \frac{1}{n} \sum_{1\leq \ell\leq n} I_\ell = \frac{1}{n} \int_{\mathbb{S}_r} \left(\sum_{1\leq k\leq n} x_k^2 \right)\ \mathrm{d}\mathbb{V}.$$

Now, since

$$\sum_{1\leq k\leq n} x_k^2 = g(x,x) = r^2\ \forall x \in \mathbb{S}_r,$$

we see that

$$I_k = \frac{1}{n} \int_{\mathbb{S}_r} r^2\ \mathrm{d}\mathbb{V} = \frac{r^2}{n} \cdot \mathrm{vol}(\mathbb{S}_r).$$

Finally, recalling equation (1.6), we conclude that

$$\int_{\mathbb{S}_r} B(x,x)\ \mathrm{d}\mathbb{V} = \frac{r^2}{n} \cdot \mathrm{vol}(\mathbb{S}_r) \cdot \sum_{1\leq k\leq n} \lambda_k = \frac{r^2}{n} \cdot \mathrm{vol}(\mathbb{S}_r) \cdot \mathrm{tr}_g(B),$$

as desired.

■

Remark (O.23). In some computations in this work, we will adopt the Einstein convention when dealing with sums. It consists, essentially, of making the summation symbols disappear, agreeing on when certain indices are being summed or not, and deducing in the obvious way how far the implicit sum goes. More explicitly, all summation symbols are omitted, with the common understanding that if

the same index appears in a monomial expression once below and once above, we are in fact summing the expression over that index.

Remark (O.24). We will use the following notation commonly seen in the literature: if A and B are arbitrary tensors, the notation $A * B$ will denote a tensor that can be expressed as a finite linear combination (whose coefficients depend neither on A nor on B) of contractions or metric contractions of the tensor product $A \otimes B$. For example,

$$\begin{aligned}
&\pi^2 \cdot (\mathrm{tr}_{1,2}\, A) \otimes B + 271 \cdot \mathrm{tr}_{1,3}(A \otimes B) - 666 \cdot A \otimes (\mathrm{tr}_{1,2}\, B) = A * B,\\
&13 P_{ij}^{ki\ell} Q_{\ell qr}^{j} - P_{qi}^{\ell rs} Q_{isr}^{k} = (P * Q)_{kqr},\\
&23 P_{ij}^{ijk} Q_{k\ell m}^{\ell} = (P * Q)_{m},
\end{aligned}$$

where in the last equalities above we used the Einstein convention and P and Q denote tensors of type $(3,2)$ and $(1,3)$, respectively. A very useful property of this notation, which is a direct consequence of the tensoriality of $A * B$ and of the Cauchy-Schwarz inequality, is that

$$\|A * B\| \leq C \|A\| \|B\|, \tag{1.7}$$

where C is a dimensional constant that depends neither on A nor on B. Indeed, note that the definition of tr with respect to any indices always reduces to tr_1^1, and the Cauchy-Schwarz inequality gives us

$$|\,\mathrm{tr}_1^1(\omega \otimes X)| = |\langle \omega^\sharp, X\rangle| \leq \|\omega^\sharp\| \cdot \|X\| = \|\omega\| \cdot \|X\|,$$

from which the general inequality (1.7) follows by multilinearity.

1.2 The extension of the metric to tensors

Definition (D.1.21). Let $P, Q \in \mathscr{T}_\ell^k(T\mathcal{M}^n)$. The inner product of P and Q is the real function

$$\mathscr{C}^\infty(\mathcal{M}) \ni g(P,Q) = \sum_{\substack{1\leq i_1,\ldots,i_k\leq n\\ 1\leq j_1,\ldots,j_\ell\leq n\\ 1\leq r_1,\ldots,r_k\leq n\\ 1\leq s_1,\ldots,s_\ell\leq n}} g_{i_1 r_1} g_{i_2 r_2} \cdots g_{i_k r_k} g^{j_1 s_1} g^{j_2 s_2} \cdots g^{j_\ell s_\ell} P_{j_1\ldots j_\ell}^{i_1\ldots i_k} Q_{s_1\ldots s_\ell}^{r_1\ldots r_k},$$

where the local frame taken above is not necessarily orthonormal.

Remark (O.25). The proof that such a product is well defined (that is, that the computation of $g(P,Q)$ defined above does not depend on the chosen chart) can be found in the appendix.

Remark (O.26). The previous definition may seem obscure at first sight, but in fact it has a natural motivation. First, we can define the inner product of 1-forms by

$$\langle \omega, \eta\rangle = \langle \omega^\sharp, \eta^\sharp\rangle.$$

Then, we can define the inner product on each fiber $\mathscr{T}_b^a(T_p\mathcal{M})$ by declaring that the inner product of the following decomposable tensors

$$P \doteq \mathrm{d}x^{j_1}\Big|_p \otimes \mathrm{d}x^{j_2}\Big|_p \otimes \ldots \otimes \mathrm{d}x^{j_a}\Big|_p \otimes \frac{\partial}{\partial x^{i_1}}\Big|_p \otimes \frac{\partial}{\partial x^{i_2}}\Big|_p \otimes \ldots \otimes \frac{\partial}{\partial x^{i_b}}\Big|_p,$$

with

$$Q \doteq \mathrm{d}x^{\ell_1}\Big|_p \otimes \mathrm{d}x^{\ell_2}\Big|_p \otimes \ldots \otimes \mathrm{d}x^{\ell_a}\Big|_p \otimes \frac{\partial}{\partial x^{s_1}}\Big|_p \otimes \frac{\partial}{\partial x^{s_2}}\Big|_p \otimes \ldots \otimes \frac{\partial}{\partial x^{s_b}}\Big|_p,$$

is given by

$$\langle P, Q\rangle \doteq \left\langle \mathrm{d}x^{j_1}\Big|_p, \mathrm{d}x^{\ell_1}\Big|_p \right\rangle \ldots \left\langle \mathrm{d}x^{j_a}\Big|_p, \mathrm{d}x^{\ell_a}\Big|_p \right\rangle \left\langle \frac{\partial}{\partial x^{i_1}}\Big|_p, \frac{\partial}{\partial x^{s_1}}\Big|_p \right\rangle \ldots \left\langle \frac{\partial}{\partial x^{i_b}}\Big|_p, \frac{\partial}{\partial x^{s_b}}\Big|_p \right\rangle,$$

and then extending by multilinearity to the whole fiber $\mathscr{T}_b^a(T_p\mathcal{M})$ (and consequently, recalling remark **(O.18)**, to the whole bundle $\mathscr{T}_b^a(T\mathcal{M})$). It is easy to see that the definition resulting from this process is equivalent to Definition **(D.1.21)**, since

$$\left\langle \partial^{i_k}, \partial^{m_k} \right\rangle = \left\langle \left(\partial^{i_k}\right)^\sharp, (\partial^{m_k})^\sharp \right\rangle.$$

And as seen in Corollary **(C.1.1)**, we have $\left(\partial^{i_k}\right)^\sharp = \sum_{1\leq \ell \leq n} g^{i_k \ell}\partial_\ell$. Hence,

$$\left\langle \partial^{i_k}, \partial^{m_k} \right\rangle = g^{i_k m_k},$$

and therefore we recover Definition **(D.1.21)**.

Remark (O.27). When $k = 0$, we see that:

$$g(P, Q) = \sum_{1\leq j_1,\ldots,j_\ell \leq n} P_{j_1\ldots j_\ell} \cdot \underbrace{Q^{j_1\ldots j_\ell}}_{=Q((\mathbf{e}^{j_1})^\sharp,\ldots,(\mathbf{e}^{j_\ell})^\sharp)},$$

Moreover, we see that if $\{\mathbf{e}_i\}_{1\leq i\leq n}$ is any local orthonormal frame, we have

$$g(P, Q) = \sum_{\substack{1\leq i_1,\ldots,i_k\leq n \\ 1\leq j_1,\ldots,j_\ell\leq n}} P^{i_1\ldots i_k}_{j_1\ldots j_\ell} Q^{i_1\ldots i_k}_{j_1\ldots j_\ell}.$$

Remark (O.28). The verification that for each $p \in \mathcal{M}$, $g_p(\bullet, \star) : \mathscr{T}_\ell^k(T_p\mathcal{M}) \times \mathscr{T}_\ell^k(T_p\mathcal{M}) \to \mathbb{R}$ is a positive definite symmetric bilinear form on $\mathscr{T}_\ell^k(T_p\mathcal{M})$ is direct, which justifies our naming it an inner product.

Definition (D.1.22). The norm of a tensor $T \in \mathscr{T}_\ell^k(T\mathcal{M})$ is the real function $\|T\| : \mathcal{M} \to \mathbb{R}$ given by

$$\|T\| = \sqrt{g(T, T)}.$$

Remark (O.29). A fortiori, if $T \in \mathscr{T}^0_\ell(T\mathcal{M})$ and $\{\mathbf{e}_i\}_{1\leq i\leq n}$ is a geodesic frame on a neighborhood U_p of $p \in \mathcal{M}$, we have

$$\|T\|^2(q) = \sum_{1\leq i_1,\ldots,i_k\leq n} (T_{i_1\ldots i_k}(q))^2, \text{ for every } q \in U_p.$$

1.2 The covariant derivative of tensors

Let $(\mathcal{M}, g)$ be a Riemannian manifold and let us denote by

$$\nabla : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$$

its Levi-Civita connection. It is clear that ∇ is not a tensor, because if $X, Y \in \Gamma(T\mathcal{M})$ and $p \in \mathcal{M}$, $(\nabla_X Y)(p)$ does *not* depend only on $X(p)$ and $Y(p)$, but rather on $X(p)$ and on Y in an (arbitrarily small) neighborhood of p. The following definition provides a partial "fix" for this problem.

Definition (D.1.23). If $Y \in \Gamma(T\mathcal{M})$, we define the total covariant derivative of Y by:

$$\begin{aligned} \nabla Y : \Gamma(T\mathcal{M}) &\to \Gamma(T\mathcal{M}) \\ X &\mapsto \nabla_X Y. \end{aligned}$$

Since ∇Y is a linear map on $\Gamma(T\mathcal{M})$ over $\mathscr{C}^\infty(\mathcal{M})$, it follows that ∇Y is a tensor of type $(1,1)$. Therefore, we can also think of ∇ as a map

$$\nabla : \Gamma(T\mathcal{M}) \to \mathrm{Hom}_{\mathscr{C}^\infty(\mathcal{M})}\left(\Gamma(T\mathcal{M}), \Gamma(T\mathcal{M})\right).$$

If $S : \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ is a tensor of type $(1,1)$, we can also define its covariant derivative ∇S as a tensor of type $(1,2)$, with the basic idea of guaranteeing that a kind of Leibniz rule is satisfied, that is, so that

$$\nabla_X(S(Y)) = (\nabla_X S)(Y) + S(\nabla_X Y), \ \forall X \in \Gamma(T\mathcal{M}).$$

It is therefore reasonable to define ∇S by

$$\begin{aligned} (\nabla S)(X, Y) &= (\nabla_X S)(Y) \\ &= \nabla_X(S(Y)) - S(\nabla_X Y). \end{aligned}$$

In other words,

$$\nabla_X S = [\nabla_X, S].$$

One easily verifies that $Y \mapsto (\nabla_X S)(Y)$ is tensorial in Y.

In fact, the total covariant derivative ∇ extends naturally even beyond vector fields (which are tensors of type $(1,0)$) and tensors of type $(1,1)$, as we see in the following

Lemma (L.1.3). *Let $\mathcal{M}$ be a Riemannian manifold and ∇ its Levi-Civita connection. Then there exists a unique connection (which by abuse of notation we will also denote by ∇)*

$$\nabla : \Gamma(T\mathcal{M}) \times \mathscr{T}^k_\ell(T\mathcal{M}) \to \mathscr{T}^k_\ell(T\mathcal{M}),$$

on each bundle $\mathscr{T}^k_\ell(T\mathcal{M})$ *such that*

- *On* $\mathscr{T}^0_0(\mathcal{M}) = \mathscr{C}^\infty(\mathcal{M})$,

$$\nabla_X f = X(f),\ \forall X \in \Gamma(T\mathcal{M}),\ \textit{and}\ \forall f \in \mathscr{C}^\infty(\mathcal{M});$$

- *On* $\mathscr{T}^1_0(T\mathcal{M}) = \Gamma(T\mathcal{M})$, ∇ *coincides with the Levi-Civita connection of* $\mathcal{M}$*;*
- ∇ *satisfies the Leibniz rule with respect to tensor products, id est,*

$$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G);$$

- ∇ *commutes with all traces; i.e, if* tr *denotes the trace with respect to any pair of indices, then*

$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F).$$

Proof: See [31].

■

Remark (O.30). It is possible to prove that the connection given by the lemma above also satisfies

$$\nabla_X[\omega(Y)] = (\nabla_X\omega)(Y) + \omega(\nabla_X Y),$$

for all $\omega \in \Gamma(T^*\mathcal{M})$ and $X, Y \in \Gamma(T\mathcal{M})$, and is determined by

$$\begin{aligned}(\nabla_X T)(\omega^1,\dots,\omega^k,X_1,\dots,X_\ell) = {}& \nabla_X\left(T(\omega^1,\dots,\omega^k,X_1,\dots,X_\ell)\right)\\ &- \sum_{1\le i\le \ell} T\left(\omega^1,\dots,\omega^k,X_1,\dots,\nabla_X X_i,\dots,X_\ell,\right)\\ &- \sum_{1\le j\le k} T\left(\omega^1,\dots,\nabla_X\omega^j,\dots\omega^k,X_1,\dots,X_\ell\right),\end{aligned} \tag{1.8}$$

for all $T \in \mathscr{T}^k_\ell(T\mathcal{M})$, $\{X_i\}_{1\le i\le k} \subset \Gamma(T\mathcal{M})$; and $\left\{\omega^j\right\}_{1\le j\le \ell} \subset \Gamma(T^*\mathcal{M})$.

Remark (O.31). Note that in equation (1.8), we interpret ∇_X either as the usual directional derivative (when applied to a real function) or as the covariant derivative of vector fields (when applied to some field).

Definition (D.1.24). A tensor T is said to be *parallel* when $\nabla T \equiv 0$.

Remark (O.32). Note that $\nabla g \equiv 0$ always holds.

Remark (O.33). Using the musical isomorphisms, we can also think of ∇ as a map

$$\nabla : \Gamma(T^*\mathcal{M}) \times \mathscr{T}^k_\ell(T\mathcal{M}) \to \mathscr{T}^k_\ell(T\mathcal{M}),$$

by defining

$$\nabla_\omega T = \nabla_{(\omega)^\sharp} T,$$

A fortiori, recalling Corollary (C.1.1), in local coordinates on a neighborhood $U_p \ni p$ we have the map

$$\nabla_{(\mathrm{d}x^i)^\sharp} = \sum_{1\leq j\leq n} g^{ij}\nabla_{\partial_j} : \mathscr{T}^k_\ell(TU_p) \to \mathscr{T}^k_\ell(TU_p),$$

which we will also denote (as is common in the literature) simply by

$$\nabla^i : \mathscr{T}^k_\ell(TU_p) \to \mathscr{T}^k_\ell(TU_p).$$

Note that, trivially, for a local geodesic frame we have $\nabla^i = \nabla_i$. A natural question then arises: do the musical isomorphisms commute with the connection? As we will see next, the answer is that this happens exactly when $\nabla g = 0$.

Remark (O.34). If $T \in \mathscr{T}^k_\ell(T\mathcal{M})$, it is common to denote

$$T^{s_1\ldots s_k}_{j_1\ldots j_\ell;i} \doteq (\nabla_i T)^{s_1\ldots s_k}_{j_1\ldots j_\ell},$$

and

$$T^{s_1\ldots s_k;i}_{j_1\ldots j_\ell} \doteq (\nabla^i T)^{s_1\ldots s_k}_{j_1\ldots j_\ell}.$$

In particular, when $f \in \mathscr{C}^\infty(\mathcal{M})$, we have the notations

$$\nabla_i f = \partial_i f = f_{;i},$$

and

$$\nabla^i f = (\partial^i)^\sharp(f) = f^{;i} = \sum_{1\leq j\leq n} g^{ij} f_{;j}.$$

Remark (O.35). *Mutatis mutandis*, the musical isomorphisms can also be defined for any bundle $\pi : E \to \mathcal{M}$ endowed with a metric.

Lemma (L.1.4). *The following are equivalent:*

- $(\nabla_X s)_\flat = \nabla_X(s_\flat)$ *and* $(\nabla_X \xi)^\sharp = \nabla_X\left((\xi)^\sharp\right)$ *for all* $X \in \Gamma(T\mathcal{M}), s \in \Gamma(E)$ *and* $\xi \in \Gamma(E^*)$;
- $\nabla g = 0$.

Proof: Suppose that $\nabla g = 0$ and let $\psi \in \Gamma(E)$ be arbitrary. Then

$$(\nabla_X s)_\flat(\psi) = \langle \nabla_X s, \psi\rangle = X(\langle s, \psi\rangle) - \langle s, \nabla_X \psi\rangle = X(s_\flat(\psi)) - s_\flat(\nabla_X \psi) = [\nabla_X(s_\flat)](\psi),$$

where in the second equality we used the hypothesis $\nabla g = 0$ and in the last one we used the definition of the covariant derivative. From the arbitrariness of ψ, we conclude that

$$(\nabla_X s)_\flat = \nabla_X(s_\flat).$$

Analogously, given any $\xi \in \Gamma(E^*)$ and $\sigma \in \Gamma(E)$, we have:

$$\left\langle \nabla_X\left((\xi)^\sharp\right), \sigma\right\rangle = X\left(\left\langle \xi^\sharp, \sigma\right\rangle\right) - \langle \xi^\sharp, \nabla_X \sigma\rangle = X(\xi(\sigma)) - \xi(\nabla_X \sigma) = (\nabla_X \xi)(\sigma) = \langle (\nabla_X \xi)^\sharp, \sigma\rangle,$$

and therefore

$$\flat\left(\nabla_X\left((\xi)^\sharp\right)\right) = \flat\left((\nabla_X\xi)^\sharp\right).$$

It then follows from Proposition **(P.1.3)** that

$$\nabla_X\left((\xi)^\sharp\right) = (\nabla_X\xi)^\sharp,$$

as desired. Conversely, if ∇ commutes with the musical isomorphisms, then for any $X \in \Gamma(T\mathcal{M})$ and $s, \sigma \in \Gamma(E)$, we have

$$\begin{aligned}(\nabla_X g)(s,\sigma) &= X(\langle s,\sigma\rangle) - \langle\nabla_X s,\sigma\rangle - \langle s,\nabla_X\sigma\rangle\\ &= X(s_\flat(\sigma)) - \langle\nabla_X s,\sigma\rangle - s_\flat(\nabla_X\sigma)\\ &= (\nabla_X(s_\flat))(\sigma) - \langle\nabla_X s,\sigma\rangle\\ &= (\nabla_X s)_\flat(\sigma) - \langle\nabla_X s,\sigma\rangle\\ &= 0,\end{aligned}$$

so that $\nabla g = 0$, as claimed.

■

Example (E.1.9). Let $X, Y, Z \in \Gamma(TM)$ be arbitrary fields and $T \in \mathscr{T}^0_2(T\mathcal{M})$ a $(0,2)$ tensor. As we have already seen, T can also be seen as a tensor of type $(1,1)$, determined by the following relation:

$$T(s) = \sharp(T(s,\bullet)),\ \forall s \in \Gamma(T\mathcal{M}).$$

As a consequence of Lemma **(L.1.4)**, we see that

$$\begin{aligned}\langle(\nabla_X T)(Y), Z\rangle &= \langle\nabla_X(T(Y)) - T(\nabla_X Y), Z\rangle\\ &= \langle\nabla_X\left[\sharp(T(Y,\bullet))\right], Z\rangle - T(\nabla_X Y, Z)\\ &= \langle\left[\nabla_X(T(Y,\bullet))\right]^\sharp, Z\rangle - T(\nabla_X Y, Z)\\ &= \left[\nabla_X(T(Y,\bullet))\right](Z) - T(\nabla_X Y, Z)\\ &= \nabla_X(T(Y,Z)) - T(Y,\nabla_X Z) - T(\nabla_X Y, Z)\\ &= (\nabla_X T)(Y,Z).\end{aligned}$$

Note that in the first equality, we are viewing T as a tensor of type $(1,1)$, whereas in the last one we already view it as a tensor of type $(0,2)$. This abuse of notation is very common and is generally not made explicit (*id est*, the explicit identification of tensors of different types via the musical isomorphisms is almost always omitted, and the fact that these commute with ∇ is used implicitly without any mention). *Mutatis mutandis*, the same holds for tensors of type $(1,3)$. To avoid notational overload, we will also commit this same abuse of notation (but now justified as harmless) in several parts of this work (see, for example, the proof of **(1.14)** and equation **(2.12)**).

Definition (D.1.25). Let $\mathcal{M}$ be a Riemannian manifold and ∇ its Levi-Civita connection. We define the Lie derivative map $\mathscr{L}$

$$\mathscr{L} : \Gamma(T\mathcal{M}) \times \mathscr{T}^0_\ell(T\mathcal{M}) \to \mathscr{T}^0_\ell(T\mathcal{M}),$$

on each bundle $\mathscr{T}^0_\ell(T\mathcal{M})$ by

- On $\mathscr{T}^0_0(\mathcal{M}) = \mathscr{C}^\infty(\mathcal{M})$,

$$\mathscr{L}_X(f) = X(f),\ \forall X \in \Gamma(T\mathcal{M}),\ \text{and}\ \forall f \in \mathscr{C}^\infty(\mathcal{M});$$

- On $\mathscr{T}^0_1(T\mathcal{M}) \cong \mathscr{T}^1_0(T\mathcal{M}) = \Gamma(T\mathcal{M})$,

$$\mathscr{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X,\ \forall X, Y \in \Gamma(T\mathcal{M});$$

- On $\mathscr{T}^0_\ell(T\mathcal{M})$,

$$(\mathscr{L}_X T)(Y_1, \ldots, Y_\ell) = X\left(T(Y_1, \ldots, Y_\ell)\right) - \sum_{1 \le i \le \ell} T\left(Y_1, \ldots, \mathscr{L}_X Y_i, \ldots, Y_\ell\right).$$

Remark (O.36). Using the facts that the Levi-Civita connection is compatible with the metric and torsion-free, a simple computation shows that

$$(\mathscr{L}_X g)(Y, Z) = \langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle,$$

for all $X, Y, Z \in \Gamma(T\mathcal{M})$. In local coordinates, we can then write

$$\begin{aligned}(\mathscr{L}_X g)_{ij} &= \langle \nabla_{\partial_i} X, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_j} X\rangle \\ &= (\nabla_{\partial_i} X)_\flat(\partial_j) + (\nabla_{\partial_j} X)_\flat(\partial_i) \\ &= (\nabla_{\partial_i} X_\flat)(\partial_j) + (\nabla_{\partial_j} X_\flat)(\partial_i),\end{aligned}$$

where we used Lemma **(L.1.4)** in the last equality. A common abuse of notation is to identify X with $X_\flat$, drop the use of parentheses, and write the expression above simply as

$$(\mathscr{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i.$$

A fortiori, when $X = \nabla f$ for some $f \in \mathscr{C}^\infty(\mathcal{M})$, then

$$\mathscr{L}_{\nabla f} g = 2 \cdot \nabla^2 f,$$

which in local coordinates is also written as

$$(\mathscr{L}_{\nabla f} g)_{ij} = 2 \cdot (\nabla^2 f)_{ij} = 2 \cdot \nabla_i \nabla_j f.$$

Definition (D.1.26). Let $\mathcal{M}$ be a Riemannian manifold and $T \in \mathscr{T}^0_\ell(T\mathcal{M})$ (resp. $T \in \mathscr{T}^1_\ell(T\mathcal{M})$) . The total covariant derivative of T, denoted by ∇T, is the tensor of type $(0, \ell+1)$ (resp. $(1, \ell+1)$) given by

$$(\nabla T)(X, Y_1, \ldots, Y_\ell) = (\nabla_X T)(Y_1, \ldots, Y_k),$$

for any fields $X, Y_1, \ldots, Y_\ell \in \Gamma(T\mathcal{M})$.

Remark (O.37). A fortiori, if $f \in \mathscr{C}^\infty(\mathcal{M}) = \mathscr{T}_0^0(T\mathcal{M})$, we define the gradient of f, $\nabla f \in \Gamma(T\mathcal{M}) \cong \mathscr{T}_1^0(T\mathcal{M})$, by

$$\nabla f = \sharp(\mathrm{d}f),$$

which in a geodesic frame $\{\mathbf{e}_i\}_{1\leq i\leq n}$ has the coordinate representation

$$\nabla f = \sum_{1\leq i\leq n} \mathbf{e}_i(f)\cdot \mathbf{e}_i = (\mathbf{e}_1(f), \ldots, \mathbf{e}_n(f)),$$

and norm

$$\|\nabla f\| = \sqrt{\sum_{1\leq i\leq n} (\mathbf{e}_i(f))^2}.$$

Lemma (L.1.5). Let $f, h \in \mathscr{C}^\infty(\mathcal{M})$. Then

$$\nabla(f\cdot h) = f\cdot \nabla h + h\cdot \nabla f.$$

Proof: It is an immediate consequence of the fact that for any $X \in \Gamma(T\mathcal{M})$, we have

$$g\left(\nabla(f\cdot h), X\right) = X(f\cdot h) = f\cdot X(h) + h\cdot X(f) = f\cdot g(\nabla h, X) + h\cdot g(\nabla f, X).$$

■

Lemma (L.1.6). Let $h \in \mathscr{C}^\infty(\mathbb{R})$ and $f \in \mathscr{C}^\infty(\mathcal{M})$ be smooth real functions. Then

$$[\nabla(g\circ f)]_p = g'(f(p))\cdot \nabla f.$$

Proof: It is an immediate consequence of the chain rule.

■

Definition (D.1.27). Let $f \in \mathscr{C}^\infty(\mathcal{M})$ be a smooth real function satisfying

$$|f(p) - f(q)| \leq K\cdot \mathrm{dist}(p,q),\ \forall p, q \in \mathcal{M}.$$

In this case we will say that f is K-Lipschitz and we will denote this fact by $\mathrm{Lip}(f) = K$.

Lemma (L.1.7). A smooth real function $f \in \mathscr{C}^\infty(\mathcal{M})$ is K-Lipschitz if, and only if, $\|\nabla f\| \leq K$.

Proof: Suppose that f is K-Lipschitz. Let us arbitrarily fix $p \in \mathcal{M}$ and a unit vector $v \in T_p\mathcal{M}$. We will denote the (unique) geodesic starting from p with initial tangent vector v by γ_v. For $\varepsilon > 0$ sufficiently small, γ_v is given by $(-\varepsilon, \varepsilon) \ni t \mapsto \gamma_v(t) = \exp_p(tv)$. As we see in Riemannian geometry, geodesics are locally minimizing, so we can also assume that $\varepsilon > 0$ is small enough that $\mathrm{dist}(p, \exp_p(tv)) = t$ for every $t \in (-\varepsilon, \varepsilon)$. Then

$$\begin{aligned}
\langle(\nabla f)_p, v\rangle = \mathrm{d}f_p(v) &= \frac{\mathrm{d}}{\mathrm{d}t}(f\circ\gamma)\Big|_{t=0} \\
&= \lim_{t\to 0} \frac{f(\gamma(t)) - f(p)}{t} \\
&\leq \lim_{t\to 0} K\frac{\mathrm{dist}(\exp_p(tv), p)}{t} = K,
\end{aligned}$$

where in the last inequality we used the hypothesis that f is K-Lipschitz. *A fortiori*, taking $v = \dfrac{(\nabla f)_p}{\|\nabla f\|_p}$, we see that

$$\|(\nabla f)_p\| \leq K.$$

From the arbitrariness of p, it follows that $\|\nabla f\| \leq K$, as desired. updated version on April 11

Conversely, suppose that $\|\nabla f\| \leq K$. Let us then arbitrarily fix $\varepsilon > 0$ and $p, q \in M$. By the definition of the distance function and by the definition of infimum, it follows that there exists a smooth curve $\gamma : [0,1] \to \mathcal{M}$ satisfying $\gamma(0) = p$, $\gamma(1) = q$ and

$$\ell(\gamma) = \int_0^1 \|\gamma'(t)\| \, \mathrm{d}t \leq \operatorname{dist}(p,q) + \varepsilon$$

Note also that $f \circ \gamma$ is a smooth curve (with image in $\mathbb{R}$) joining the points $f(p)$ and $f(q)$. Again by the definitions of distance and of infimum, it follows that

$$\begin{aligned} |f(p) - f(q)| &\leq \int_0^1 |(f \circ \gamma)'(t)| \, \mathrm{d}t \\ &= \int_0^1 |\langle (\nabla f)_{\gamma(t)}, \gamma'(t) \rangle| \, \mathrm{d}t \leq K \cdot \ell(\gamma) \leq K \cdot (\operatorname{dist}(p,q) + \varepsilon). \end{aligned}$$

where in the first inequality of the second line we used the Cauchy-Schwarz inequality. Therefore

$$\operatorname{dist}(f(p), f(q)) \leq K \cdot \inf_{\varepsilon > 0} (\operatorname{dist}(p,q) + \varepsilon) = K \cdot \operatorname{dist}(p,q),$$

id. est, f is K-Lipschitz.

■

Lemma (L.1.8). *Let $\mathcal{M}$ be a complete Riemannian manifold and $f : \mathcal{M} \to \mathbb{R}$ (not necessarily differentiable) with $\operatorname{Lip}(f) = K$, and let $\varepsilon > 0$ be arbitrary. Then there exists a differentiable function $h \in \mathscr{C}^\infty(\mathcal{M})$ satisfying $|h(p) - f(p)| < \varepsilon$ and $\operatorname{Lip}(h) \leq K + \varepsilon$.*

Proof: See [2].

■

Lemma (L.1.9). *Let $(\mathcal{M}, g)$ be a complete Riemannian manifold, and $R > 0$ arbitrary. Then there exists a real function $\varphi \in \mathscr{C}^\infty(\mathcal{M})$ such that*

$$\varphi|_{B_p(R)} \equiv 1, \varphi|_{\mathcal{M} \setminus B_p(2R)} \equiv 0, \text{ and } \|\nabla \varphi\| \leq \frac{2}{R}.$$

Proof: We will define the function $z : \mathbb{R} \to \mathbb{R}$ by

$$z(x) = \begin{cases} 1 & x \leq R \\ 2 - \dfrac{x}{R} & R < x < 2R \\ 0 & x \geq 2R \end{cases},$$

and the function $\theta : \mathcal{M} \to \mathbb{R}$ by $\theta(q) = z(\mathrm{dist}(p,q))$ (a known fact of Riemannian geometry is that $\mathrm{dist}(p,\bullet)$ is differentiable away from p and its cut locus and necessarily not differentiable at p and its cut locus, so that θ is not necessarily differentiable). It is easy to see that

$$\theta|_{B(p,R)} = 1, \quad \theta|_{\mathcal{M}\setminus B(p,2R)} = 0, \quad \mathrm{Lip}(\theta) \leq \frac{1}{R}.$$

Lemma **(L.1.8)** then guarantees the existence of a differentiable function $\mathscr{C}^\infty(\mathcal{M}) \ni \theta : \mathcal{M} \to \mathbb{R}$ satisfying

$$|\psi(q) - \theta(q)| < \varepsilon, \quad \mathrm{Lip}(\psi) < \frac{1}{R} + \varepsilon,$$

where $\varepsilon > 0$ will be taken in the following way: given arbitrary $\delta > 0$, take $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{\frac{1}{4}, \frac{\delta}{4\cdot(1+\delta)}, \frac{1}{R}\right\}.$$

Let us now take a smooth function $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$h|_{(1-\varepsilon,\infty)} = 1, \quad h|_{(-\infty,\varepsilon)} = 0, \quad \mathrm{Lip}(h) \leq \frac{1}{1-4\varepsilon}.$$

We now define $\varphi := h \circ \psi$. It is easy to see that φ is identically 1 on the ball of radius R centered at p and identically zero outside the ball of radius $2R$ centered at p. Moreover,

$$\mathrm{Lip}(\varphi) \leq \frac{1}{R}\cdot\frac{1}{1-4\varepsilon} + \frac{\varepsilon}{1-4\varepsilon} \leq \frac{2+2\delta}{R}.$$

The desired result then follows from the arbitrariness of $\delta > 0$ and from Lemma **(L.1.7)**.

■

Remark (O.38). Note that the previous lemma is trivially satisfied in the case where $\mathcal{M}$ is compact and $R > \mathrm{diam}(\mathcal{M})$.

Proposition (P.1.9). Let $T \in \mathscr{T}^0_k(T\mathcal{M})$. Then

$$(\nabla T)_{m\ i_1\ldots i_k} = (\nabla_m T)_{i_1\ldots i_k} = \underbrace{\partial_m\left(T_{i_1\ldots i_k}\right)}_{=\nabla_m(T_{i_1\ldots i_k})} - \sum_{1\leq r,s\leq k} T_{i_1\ldots i_{r-1}s\ i_{r+1}\ldots i_k}\cdot\Gamma^s_{mi_r}.$$

Proof: It follows immediately from the definition of ∇T.

■

Remark (O.39). An immediate consequence of the previous proposition is that in a geodesic frame $\{\mathbf{e}_i\}_{1\leq i\leq n}$ on a neighborhood $U_p \ni p$ (where, a fortiori, all the Christoffel symbols vanish at p), the coordinate representation of ∇T is given by:

$$(\nabla T)_{m\ i_1\ldots i_k} = (\nabla_m T)_{i_1\ldots i_k} = \nabla_m(T_{i_1\ldots i_k}),$$

removing the ambiguity implicit in denoting $(\nabla_m T)_{i_1\ldots i_k}$ by $\nabla_m T_{i_1\ldots i_k}$ or $\nabla_m(T_{i_1\ldots i_k})$.

Remark (O.40). By the previous remark, we see that in a geodesic frame the norm of the total covariant derivative of a tensor $T \in \mathscr{T}^0_k(T\mathcal{M})$ is given by

$$\|\nabla T\|^2 = \sum_{1\leq i_1,\dots,i_k,\ell\leq n} \left(\mathbf{e}_\ell\left[(T_{i_1\dots i_k})\right]\right)^2 = \sum_{1\leq i_1,\dots,i_k\leq n} \|\nabla(T_{i_1\dots i_k})\|^2.$$

Definition (D.1.28). If S is a tensor of type $(0,\ell)$, we define the *second total covariant derivative of* S, denoted by $\nabla^2 S$, as the tensor of type $(0,\ell+2)$ determined by

$$\begin{aligned}[(\nabla\circ\nabla)(S)](X_1,X_2,Y_1,\dots,Y_\ell) &= (\nabla^2 S)(X_1,X_2,Y_1,\dots,Y_\ell)\\ &\doteq (\nabla^2_{X_1,X_2}S)(Y_1,\dots,Y_\ell)\\ &\doteq (\nabla_{X_1}(\nabla S))(X_2,Y_1,\dots,Y_\ell)\\ &= (\nabla_{X_1}(\nabla_{X_2}S))(Y_1,\dots,Y_\ell) - \left(\nabla_{\nabla_{X_1}X_2}S\right)(Y_1,\dots,Y_\ell).\end{aligned}$$

Remark (O.41). The last equality in the previous definition is immediate from the definition of ∇S, but it is worth clarifying it now. We have

$$\begin{aligned}(\nabla_{X_1}(\nabla S))(X_2,Y_1,\dots,Y_\ell) &= X_1\left[(\nabla S)(X_2,Y_1,\dots,Y_\ell)\right] - \sum_{1\leq i\leq\ell}(\nabla S)(X_2,Y_1,\dots,\nabla_{X_1}Y_i,\dots,Y_\ell)\\ &\quad - (\nabla S)(\nabla_{X_1}X_2,Y_1,\dots,Y_\ell)\\ &= X_1\left[(\nabla_{X_2}S)(Y_1,\dots,Y_\ell)\right] - \sum_{1\leq i\leq\ell}(\nabla_{X_2}S)(Y_1,\dots,\nabla_{X_1}Y_i,\dots,Y_\ell)\\ &\quad - (\nabla_{\nabla_{X_1}X_2}S)(Y_1,\dots,Y_\ell)\\ &= (\nabla_{X_1}(\nabla_{X_2}S))(Y_1,\dots,Y_\ell) - \left(\nabla_{\nabla_{X_1}X_2}S\right)(Y_1,\dots,Y_\ell).\end{aligned}$$

Remark (O.42). The total covariant derivatives of order n, denoted by $\nabla^n S = \nabla^{\circ n}$, are defined analogously, for any $n\in\mathbb{N}$. Each total covariant derivative increases the covariant type of the tensor by 1. When $n=1$ and we consider the case of purely covariant tensors, we have a map

$$\nabla : \mathscr{T}^0_k(\mathcal{M}) \to \mathscr{T}^0_{k+1}(\mathcal{M}).$$

When $n=2$, we have a map

$$\nabla^2 : \mathscr{T}^0_k(\mathcal{M}) \to \mathscr{T}^0_{k+2}(\mathcal{M}),$$

which for a 2-tensor h has the local coordinate expression given by

$$\nabla^2 h = \sum_{1\leq i,j,k,\ell\leq n}(\nabla\nabla h)_{ijk\ell}\cdot \mathrm{d}x^i\otimes\mathrm{d}x^j\otimes\mathrm{d}x^k\otimes\mathrm{d}x^\ell,$$

where by definition

$$(\nabla\nabla h)_{ijk\ell} = (\nabla_{\partial_i}\nabla_{\partial_j}h)_{k\ell} = [\nabla_{\partial_i}(\nabla_{\partial_j}h)](\partial_k,\partial_\ell) - (\nabla_{\nabla_{\partial_i}\partial_j}h)(\partial_k,\partial_\ell).$$

It is also clear from the expression above that, in general,

$$\mathscr{T}^0_2(T\mathcal{M}) \ni (\nabla^2 h)(X,Y) \doteq \nabla_X\nabla_Y h \neq \nabla_X(\nabla_Y h),$$

and therefore, a priori the abuse of notation

$$\nabla_i \nabla_j h_{k\ell}$$

is not harmless, since we could reasonably interpret it as the components of the 2-tensor $(\nabla_{\partial_i}\nabla_{\partial_j} h)$ or as the second partial derivatives of the real function $h_{k\ell}$. The solution to this problem, which makes such an abuse harmless (that is, makes both interpretations coincide), is the use of geodesic frames when doing computations locally, as we saw in remark **(O.39)**.

Lemma (L.1.10) (Kato's inequality). *Let E be a bundle over $(\mathcal{M}, g)$ and $s \in \Gamma(E)$ a section of E over $\mathcal{M}$. Then*

$$\|\nabla \|s\|\| \leq \|\nabla s\|. \tag{1.9}$$

Proof: We have

$$2 \cdot \|\nabla\|s\|\| \cdot \|s\| = \|\nabla(\|s\|^2)\| = 2 \cdot \|g(\nabla s, s)\| \leq 2 \cdot \|\nabla s\| \cdot \|s\|.$$

from which the desired inequality follows immediately, where in the last inequality we used Cauchy-Schwarz.

■

Remark (O.43). Using Definition **(D.1.28)** and the fact that ∇ is torsion-free, we see that another way of writing expression **(1.2)** is

$$\mathrm{Rm}(X, Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z.$$

Moreover, it is clear that Rm can be extended to any bundle E over $\mathcal{M}$, that is, that we can view Rm as a map

$$\begin{aligned} \mathrm{Rm} : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(E) &\to \Gamma(E) \\ (X, Y, s) &\mapsto \mathrm{Rm}(X, Y)s \doteq \mathrm{Rm}_{X,Y} s \doteq \nabla^2_{X,Y} s - \nabla^2_{Y,X} s. \end{aligned}$$

Remark (O.44). A particular case of Definition **(D.1.28)** is the following

Definition (D.1.29). The *Hessian* of a real function $f \in \mathscr{C}^\infty(\mathcal{M})$ is the $(0, 2)$ tensor $\nabla^2 f$ determined by

$$\begin{aligned} \nabla^2 f : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\to \mathscr{C}^\infty(\mathcal{M}) \\ (X, Y) &\mapsto (\nabla^2 f)(X, Y) = [\nabla(\nabla f)](X, Y). \end{aligned}$$

Remark (O.45). We can make the previous definition even more explicit. We have

$$\begin{aligned} (\nabla^2 f)(X, Y) &= [\nabla_Y(\nabla f)](X) \\ &= \nabla_Y[\nabla_X f] - (\nabla f)(\nabla_Y X) \\ &= Y(X(f)) - \nabla_{\nabla_Y X} f. \end{aligned}$$

Moreover, since the connection ∇ is torsion-free, we further obtain

$$(\nabla^2 f)(X,Y) - (\nabla^2 f)(Y,X) = -[X,Y](f) - -\nabla_{[X,Y]} f = -([X,Y](f) - [X,Y](f)) = 0.$$

and therefore $\nabla^2 f$ is symmetric. On the other hand, using the compatibility of g with ∇, we also obtain

$$\begin{aligned}(\nabla^2 f)(X,Y) = \nabla^2_{X,Y} f &= \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f \\ &= \nabla_X [g(Y, \nabla f)] - g(\nabla_X Y, \nabla f) \\ &= g(\nabla_X \nabla f, Y).\end{aligned}$$

And therefore, $\nabla^2 f$ seen as a tensor of type $(1,1)$ is given by

$$(\nabla^2 f)(X) = \sharp[(\nabla^2 f)(X, \bullet)] = \nabla_X \nabla f,$$

which is obviously self-adjoint (since $\nabla^2 f$ seen as a $(0,2)$ tensor is symmetric).

Remark (O.46). Let $(\mathcal{M}, g)$ be a Riemannian manifold and $f \in \mathscr{C}^\infty(\mathcal{M})$. $f = \nabla^0 f$ and $\nabla f = \mathrm{d}f$ are obviously tensors of type $(0,0)$ and $(0,1)$, respectively, that do not depend on the metric g (although as a vector field ∇f depends on the metric g, it is obvious that as a differential 1-form $\mathrm{d}f$ is independent of the metric). From order 2 on, this is no longer true: for example, if $\varphi \in \mathscr{C}^\infty(\mathcal{M})$ and we define $\widetilde{g} \doteq e^{2\varphi} g$ then a direct computation shows that

$$\widetilde{\nabla}^2 f = \nabla^2 f + g(\mathrm{d}\varphi, \mathrm{d}f) g - 2\, \mathrm{d}\varphi \odot \mathrm{d}f,$$

where $\odot$ denotes the symmetric product of 1-forms, defined by

$$\mathscr{T}^0_2(T\mathcal{M}) \ni (\omega \odot \eta)(\bullet, \star) \doteq \frac{1}{2} \left(\omega(\bullet)\eta(\star) + \eta(\bullet)\omega(\star)\right).$$

The proof of the Ricci identity below can be found in the **Appendix**.

Lemma (L.1.2.1) (Coordinate-free Ricci identity). *Let $T \in \mathscr{T}^0_k(T\mathcal{M})$. Then:*

$$(\mathrm{Rm}_{x,y} T)(v_1, \ldots, v_k) = - \sum_{1 \le i \le k} T(v_1, \ldots, \mathrm{Rm}_{x,y}(v_i), \ldots, v_k), \tag{1.10}$$

for any $x, y, v_1, \ldots, v_k \in T_p\mathcal{M}$, where $p \in \mathcal{M}$ is arbitrary.

Lemma (L.1.2.2) (Ricci identity in coordinates). *Let $T \in \mathscr{T}^0_k(T\mathcal{M})$. Then:*

$$\begin{aligned}(\mathrm{Rm}_{i,j} T)_{m_1 \ldots m_k} = [(\nabla^2_{\partial_i, \partial_j} - \nabla^2_{\partial_j, \partial_i})(T)]_{m_1 \ldots m_k} &= [(\nabla_i \nabla_j - \nabla_j \nabla_i)(T)]_{m_1 \ldots m_k} \\ &= - \sum_{\substack{1 \le t \le k \\ 1 \le \ell \le n}} \mathrm{Rm}^\ell_{ijm_t} T_{m_1 \ldots m_{t-1} \ell m_{t+1} \ldots m_k}.\end{aligned} \tag{1.11}$$

Proof: Immediate consequence of identity (1.10) and of Example (E.1.4).

■

Remark (O.47). An analogous identity holds for tensors of arbitrary types (k, ℓ). We will not need it in this work, but it is worth observing that it can be obtained using the isomorphism exhibited in the proof of Proposition **(P.1.7)** together with Corollary **(C.1.1)** and the use of a local geodesic frame.

Definition (D.1.30). The *Laplacian* of a real function $f \in \mathscr{C}^\infty(\mathcal{M}^n)$ is the real function $\Delta f \in \mathscr{C}^\infty(\mathcal{M})$ given by

$$\Delta f = \underbrace{(\mathrm{tr} \circ \nabla^2)(f)}_{=\mathrm{tr}_{1,2}(\nabla f)} = \mathrm{div}(\nabla f) = \sum_{1 \leq i \leq n} g\left(\nabla_{\mathbf{e}_i} \nabla f, \mathbf{e}_i\right) = \sum_{1 \leq i \leq n} \nabla^2_{\mathbf{e}_i, \mathbf{e}_i} f,$$

where $\{\mathbf{e}_i\}_{1 \leq i \leq n}$ is any local orthonormal frame.

Remark (O.48). Recalling **(P.1.8)**, we can view the Laplacian as an average of the Hessian over all unit vectors. Concretely, we have that

$$(\Delta f)(p) = \frac{n}{r^2 \cdot \mathrm{vol}(\mathbb{S}_r(p))} \int_{\mathbb{S}_r(p)} (\nabla^2 f)(v, v) \, \mathrm{dVol}_g,$$

where dVol_g is the Riemannian volume form, seen in Proposition **(P.4.11)**.

Definition (D.1.31). The *Laplacian* of a section $s \in \Gamma(E)$ is the section $\Delta s \in \Gamma(E)$ given by

$$\Delta s = \sum_{1 \leq i \leq n} \nabla^2_{\mathbf{e}_i, \mathbf{e}_i} s,$$

where $\{\mathbf{e}_i\}_{1 \leq i \leq n}$ is any local orthonormal frame.

Lemma (L.1.11) (The Second Derivative Test). *Let $(\mathcal{M}^n, g_0)$ be a Riemannian manifold and $u : \mathcal{M} \to \mathbb{R}$ a smooth function. If u has a local minimum point (resp. maximum) at some $p \in \mathcal{M}$, then*

$$(\nabla u)(p) = 0 \text{ and } (\Delta u)(p) \geq 0, \text{ (resp. } (\Delta u)(p) \leq 0). \tag{1.12}$$

Proof: Since the problem is local in nature, we can pass to normal coordinates (x^i) around p to do the computations. By the usual second derivative test for real functions on $\mathbb{R}^n$ and by the definition of the partial derivative induced by a chart, it follows that

$$(\nabla_i u)_p = \frac{\partial u}{\partial x^i}(p) = 0, \ \forall 1 \leq i \leq n,$$

and therefore the gradient of u vanishes at p. Again by the second derivative test on $\mathbb{R}^n$, the entries of the Hessian are all positive and given by

$$\nabla_i \nabla_j u = \frac{\partial^2 u}{\partial x^i \partial x^j}(p) - \sum_{1 \leq k \leq n} \Gamma^k_{ij}(p) \frac{\partial u}{\partial x^k}(p) = \frac{\partial^2 u}{\partial x^i \partial x^j}(p) \geq 0.$$

Since the Laplacian is the trace of the Hessian, the result follows. *Mutatis mutandis,* the same proof can be done in the case where p is a local maximum point.

■

Definition (D.1.32). The f-Laplacian of a section $s \in \Gamma(E)$ is the section $\Delta_f s \in \Gamma(E)$ given by

$$\Delta_f s = \Delta s - \nabla_{\nabla f} s.$$

Lemma (L.1.12). Let $f, h \in \mathscr{C}^\infty(\mathcal{M})$. Then

$$\Delta(f \cdot h) = f \cdot \Delta h + h \cdot \Delta f + 2 \cdot g(\nabla f, \nabla h).$$

Proof: Since all the operators involved are tensorial, it suffices to prove that the identity is satisfied around the arbitrarily fixed center $p \in \mathcal{M}$ of a geodesic frame. This is a direct computation.

■

Lemma (L.1.13). Let $u, v \in \mathscr{C}^\infty(\mathcal{M})$. Then

$$\Delta_f(u \cdot v) = u \cdot \Delta_f v + v \cdot \Delta_f u + 2 \cdot g(\nabla f, \nabla h).$$

Proof: Immediate consequence of Lemmas **(L.1.12)** and **(L.1.5)**.

■

1.3 Exterior algebra

In this section we will recall some important concepts of exterior algebra, based on the reference [33]. Let us fix $k \in \mathbb{N}$ and denote by $\Lambda^k(\mathbb{V})$ the set of k-forms on $\mathbb{V}$. Consider the exterior product $\wedge : \underbrace{\mathbb{V} \times \ldots \times \mathbb{V}}_{\text{k times}} \mapsto \Lambda^k(\mathbb{V})$, defined by

$$\begin{aligned} \wedge(v_1, \ldots, v_k) &\doteq v_1 \wedge \ldots \wedge v_k \\ &= \flat(v_1) \wedge \flat(v_2) \wedge \ldots \wedge \flat(v_k), \end{aligned}$$

for all $v_1, \ldots, v_k \in \mathbb{V}$. Since $\flat(v_i)$ is a 1-form for each $1 \leq i \leq k$, $\wedge$ is well defined. Moreover, as made explicit in [33], such an exterior product is (up to isomorphisms) unique.

Definition (D.1.33). Having fixed $j \in \mathbb{N}$, we will say that a differential j-form $\omega \in \Lambda^j(\mathbb{V})$ is a j-blade on $\mathbb{V}$ when there exist $v_1, \ldots, v_j \in \mathbb{V}$ such that

$$\omega = v_1 \wedge v_2 \wedge \ldots \wedge v_j.$$

Definition (D.1.34). Having fixed $j \in \mathbb{N}$, the vector space generated by linear combinations of j-blades on $\mathbb{V}$ will be denoted by $\mathbb{V}^{\wedge j}$.

Definition (D.1.35). The Grassmannian $\mathrm{Gr_k}(\mathbb{V})$ is the set of all vector subspaces of dimension k of $\mathbb{V}$.

Definition (D.1.36). The *Grassmann bundle* $\mathrm{Gr_k}(\mathcal{M})$ of a differentiable manifold $\mathcal{M}$ is defined by

$$\mathrm{Gr}_k(\mathcal{M}) = \bigcup_{p\in\mathcal{M}} \mathrm{Gr}_k(T_p\mathcal{M})$$

One can prove (see [31]) that the Grassmannian $\mathrm{Gr_k}(\mathcal{M}^n)$ is a differentiable manifold of dimension $n + \binom{n}{k}$.

1.4 The Bianchi identities and their consequences

The main goal of this section is to establish (based on the reference [48]) identity **(1.14)**. Before that, we need to recall some already known identities. As seen in Riemannian geometry (and also proved in [32]), the curvature tensor Rm satisfies the following identities

Identity (I.1.1) (First Bianchi identity).

$$\mathrm{Rm}(x,y)z + \mathrm{Rm}(z,x)y + \mathrm{Rm}(y,z)x = 0,\ \forall x,y,z \in T_p\mathcal{M} \text{ and } \forall p \in \mathcal{M}.$$

Identity (I.1.2) (Second Bianchi identity).

$$(\nabla_z \mathrm{Rm})(x,y)w + (\nabla_x \mathrm{Rm})(y,z)w + (\nabla_y \mathrm{Rm})(z,x)w = 0\ \forall x,y,z,w \in T_p\mathcal{M} \text{ and } \forall p \in \mathcal{M}.$$

These two identities imply the

Identity (I.1.3) (Contracted second Bianchi identity).

$$2\cdot \mathrm{div}(\mathrm{Ric}) = \nabla \mathrm{Scal}. \tag{1.13}$$

Proof: Arbitrarily fix $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$, and consider γ the unique geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. Let us extend v to a field V parallel along γ, just as seen in remark **(O.3)**, and take $\{E_i\}$ a geodesic frame around p. Thus, we have that

$$(\nabla V)|_p = (\nabla E_i)|_p \equiv \nabla_{E_i(p)} E_j = 0,\ \text{for all } 1 \le i,j \le n.$$

Equivalently,

$$\nabla_x V = \nabla_x E_i \equiv \nabla_{E_i(p)} E_j = 0,\ \text{for all } x \in T_p\mathcal{M} \text{ and } 1 \le i,j \le n.$$

Therefore, it follows from the second Bianchi identity and from Lemma **(L.1.4)** that:

$$\begin{aligned}(\mathrm{dScal})(V)(p) = (V(\mathrm{Scal}))(p) &= V\left(\sum_{1\le i\le n} \langle \mathrm{Ric}(E_i), E_i\rangle\right)\\ &= V\left(\sum_{1\le i,j\le n} \langle \mathrm{Rm}(E_i,E_j)E_j, E_i\rangle\right)\\ &= \sum_{1\le i,j\le n} \langle \nabla_V[\mathrm{Rm}(E_i,E_j)E_j], E_i\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\leq i,j\leq n} \langle (\nabla_V \mathrm{Rm})(E_i, E_j)E_j, E_i\rangle \\
&= -\sum_{1\leq i,j\leq n} \langle (\nabla_{E_j} \mathrm{Rm})(V, E_i)E_j, E_i\rangle - \sum_{1\leq i,j\leq n} \langle (\nabla_{E_i} \mathrm{Rm})(E_j, V)E_j, E_i\rangle \\
&= -\sum_{1\leq i,j\leq n} (\nabla_{E_j} \mathrm{Rm})(V, E_i, E_j, E_i) - \sum_{1\leq i,j\leq n} \langle (\nabla_{E_i} \mathrm{Rm})(E_j, V, E_j, E_i)\rangle \\
&= \sum_{1\leq i,j\leq n} (\nabla_{E_j} \mathrm{Rm})(E_j, E_i, E_i, V) + \sum_{1\leq i,j\leq n} (\nabla_{E_i} \mathrm{Rm})(E_i, E_j, E_j, V) \\
&= 2\cdot \sum_{1\leq i,j\leq n} (\nabla_{E_j} \mathrm{Rm})(E_j, E_i, E_i, V) \\
&= 2\cdot \sum_{1\leq i,j\leq n} \nabla_{E_j}(\mathrm{Rm}(E_j, E_i, E_i, V)) \\
&= 2\sum_{1\leq j\leq n} \nabla_{E_j}(\langle \mathrm{Ric}(E_j), V\rangle) \\
&= 2\cdot \sum_{1\leq j\leq n} \nabla_{E_j}(\langle \mathrm{Ric}(V), E_j\rangle) \\
&= 2\cdot \sum_{1\leq j\leq n} \langle (\nabla_{E_j} \mathrm{Ric})(V), E_j\rangle \\
&= 2\cdot (\mathrm{div}(\mathrm{Ric}))(V)(p).
\end{aligned}
$$

The desired result then follows from the arbitrariness of $V(p) = v \in T_p\mathcal{M}$.

■

Remark (O.49). Recalling remark **(O.90)**, we see that the local expression of identity **(1.13)** is given by

$$2\cdot (\mathrm{div}(\mathrm{Ric}))_a = 2\sum_{1\leq b\leq n} (\nabla^b \mathrm{Ric})_{ab} = \nabla_a \mathrm{Scal}.$$

A common abuse in the literature is to drop the use of parentheses and use the Einstein notation convention, so that (denoting Scal by R and Ric_{ij} by R_{ij}) the equation above is written as

$$\nabla^j R_{ij} = \nabla_j R^j_i = \frac{1}{2}\nabla_i R,$$

where the first equality is an immediate consequence of the way we identify Ric as a tensor of type $(1,1)$ and $(0,2)$ simultaneously. Indeed, using the Einstein notation convention, we have

$$
\begin{aligned}
\nabla_j R^j_i &= (\nabla_j \mathrm{Ric})(\mathbf{e}^j, \mathbf{e}_i) \\
&= (\nabla_j \mathrm{Ric})(g^{js}e_s, e_i) \\
&= g^{js}(\nabla_j \mathrm{Ric})_{si} \\
&= (\nabla^s \mathrm{Ric})_{si} \\
&= \nabla^j R_{ij},
\end{aligned}
$$

where in the last equality we made a trivial renaming of indices and used the symmetry of Ric (which implies the symmetry of all its covariant derivatives).

The next two lemmas will also be quite useful to us:

Lemma (L.1.14).

$$(\operatorname{div}\mathrm{Rm})(x,y,z) = (\nabla_x \mathrm{Ric})(y,z) - (\nabla_y \mathrm{Ric})(x,z),\ \forall x,y,z \in T_p\mathcal{M} \text{ and } \forall p \in \mathcal{M}.$$

Proof: Let $\{E_i\}_{1\leq i\leq n}$ be a geodesic frame centered around p (with $E_i(p) \doteq \mathbf{e}_i$) and let us suppose without loss of generality (according to remark (O.3)) that $0 = \nabla_v x = \nabla_v y = \nabla_v z$ for every $v \in T_p\mathcal{M}$. We then have

$$\begin{aligned}
-(\operatorname{div}\mathrm{Rm})(x,y,z) &= -\sum_{1\leq i\leq n} \langle (\nabla_{\mathbf{e}_i}\mathrm{Rm})(x,y,z), \mathbf{e}_i\rangle \\
&= \sum_{1\leq i\leq n} \left[g\left((\nabla_x \mathrm{Rm})(y,\mathbf{e}_i,z),\mathbf{e}_i\right) + g\left((\nabla_y \mathrm{Rm})(\mathbf{e}_i,x,z),\mathbf{e}_i\right)\right] \\
&= \mathrm{tr}((\nabla_x \mathrm{Rm})(y,\bullet,z)) + \mathrm{tr}((\nabla_y \mathrm{Rm})(\bullet,x,z)) \\
&= -(\nabla_x \mathrm{Ric})(y,z) + (\nabla_y \mathrm{Ric})(x,z),
\end{aligned}$$

where in the second equality we used the second Bianchi identity and in the last one we used the fact that ∇ commutes with the trace.

■

Lemma (L.1.15). An Einstein manifold $\mathcal{M}^n$ of dimension $n \geq 3$ (see Definition (D.1.19)) has constant λ. And in the case $n = 3$, $\mathcal{M}$ has constant sectional curvature (and therefore constant Ricci and scalar curvatures as well).

Proof: By the second Bianchi identity, we have $\mathrm{dScal} = 2\operatorname{div}(\mathrm{Ric})$ (where Scal is the scalar curvature of $\mathcal{M}$). In the case where $\mathcal{M}$ is Einstein, we then see that $\mathrm{d}(\lambda \mathrm{tr}(g)) = n \cdot \mathrm{d}\lambda = 2\operatorname{div}(\lambda g)$, which in local coordinates is written as

$$n \cdot \mathrm{d}\lambda = 2\operatorname{div}(\lambda g) = 2\sum_{1\leq i,j,k\leq n} g^{ij}\nabla_i(\lambda g_{jk})\ \mathrm{d}x^k = 2\sum_{1\leq i,j,k\leq n} g^{ij}g_{jk}\left(\nabla_i \lambda\right)\ \mathrm{d}x^k = 2\ \mathrm{d}\lambda,$$

Therefore $n \geq 3 \implies \mathrm{d}\lambda \equiv 0$, and by the connectedness of $\mathcal{M}$, λ is then constant. For the second part, observe that taking an arbitrary $p \in \mathcal{M}$ and a geodesic frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ centered at p, we see that

$$\sum_{1\leq i\leq 3} \mathrm{Rm}(X,\mathbf{e}_i,Y,\mathbf{e}_i) = 2\lambda \cdot g(X,Y),\ \forall X,Y \in \Gamma(T\mathcal{M}).$$

Setting $X = Y = \mathbf{e}_j$ with $j \in \{1,2,3\}$ and evaluating at p we obtain the system

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{\text{determinant}=2} \begin{pmatrix} \mathrm{sec}_{12} \\ \mathrm{sec}_{23} \\ \mathrm{sec}_{13} \end{pmatrix} = \begin{pmatrix} \mathrm{Ric}_{11} \\ \mathrm{Ric}_{22} \\ \mathrm{Ric}_{33} \end{pmatrix} = 2\lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

that is, the sectional curvature is constant and equal to λ.

■

Identity (I.1.4) (Bochner identity). *On a Riemannian manifold $(\mathcal{M}^n, g)$, we have:*

$$[\operatorname{div}(\mathscr{L}_X g)](X) = \frac{1}{2}\cdot\Delta\|X\|^2 - \|\nabla X\|^2 + \operatorname{Ric}(X,X) + X(\operatorname{div} X).$$

And in the case where $X = \nabla f$ for some $f \in \mathscr{C}^\infty(\mathcal{M})$, we have

$$[\operatorname{div}(\mathscr{L}_X g)](Z) = 2\cdot\operatorname{Ric}(Z,X) + 2\cdot Z(\operatorname{div} X),$$

or, equivalently and coordinate-free (viewing the entities above as tensors of type $(1,1)$),

$$\operatorname{div}\nabla^2 f = \operatorname{Ric}(\nabla f) + \nabla\Delta f. \tag{1.14}$$

Proof: We will do the computations in a geodesic frame $\{E_i\}$ around an arbitrarily fixed point $p \in \mathcal{M}$. According to Definition (D.4.11) and Example (E.1.9), we have

$$\begin{aligned}
[\operatorname{div}(\mathscr{L}_X g)](X) &= \sum_{1\le i\le n} g\left([\nabla_{E_i}(\mathscr{L}_X g)](X), E_i\right)\\
&= \sum_{1\le i\le n} [\nabla_{E_i}(\mathscr{L}_X g)](E_i, X)\\
&= \sum_{1\le i\le n} \left[\nabla_{E_i}\left[(\mathscr{L}_X g)(E_i,X)\right] - (\mathscr{L}_X g)(E_i, \nabla_{E_i}X)\right].
\end{aligned}$$

Note that

$$\begin{aligned}
(\mathscr{L}_X g)(E_i, X) &= X(g(E_i,X)) - g([X,E_i],X)\\
&= g(E_i, \nabla_X X) - g(\nabla_X E_i - \nabla_{E_i}X, X)\\
&= g(E_i,\nabla_X X) + g(\nabla_{E_i}X, X),
\end{aligned}$$

and

$$\begin{aligned}
-(\mathscr{L}_X g)(E_i, \nabla_{E_i}X) &= -\left[X(g(E_i,\nabla_{E_i}X)) - g([X,E_i],\nabla_{E_i}X) - g(E_i,[X,\nabla_{E_i}X])\right]\\
&= -[g(E_i,\nabla_X\nabla_{E_i}X) + g(\nabla_{E_i}X,\nabla_{E_i}X) - g(E_i,\nabla_X\nabla_{E_i}X) + g(E_i,\nabla_{\nabla_{E_i}X}X)]\\
&= -g(\nabla_{E_i}X,\nabla_{E_i}X) - g(E_i,\nabla_{\nabla_{E_i}X}X).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\operatorname{div}(\mathscr{L}_X g)](X) &= \sum_{1\le i\le n}\left[\nabla_{E_i}[g(\nabla_{E_i}X,X) + g(E_i,\nabla_X X)] - g(\nabla_{E_i}X,\nabla_{E_i}X) - g(E_i,\nabla_{\nabla_{E_i}X}X)\right]\\
&= \sum_{1\le i\le n}\left[\nabla_{E_i}[g(\nabla_{E_i}X,X)] + \nabla_{E_i}[g(E_i,\nabla_X X)] - g(E_i,\nabla_{\nabla_{E_i}X}X)\right] - \|\nabla X\|^2.
\end{aligned}$$

But since

$$\begin{aligned}
\sum_{1\le i\le n}\nabla_{E_i}[g(\nabla_{E_i}X,X)] &= \sum_{1\le i\le n}[g(\nabla_{E_i}X,\nabla_{E_i}X) + g(X,\nabla_{E_i}\nabla_{E_i}X)]\\
&= \sum_{1\le i,\ell\le n} g([\langle\nabla_{E_i}X,\nabla_{E_\ell}X\rangle + \langle X,\nabla_{E_i}\nabla_{E_\ell}X\rangle]\,E_\ell, E_i)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\leq i,\ell\leq n} g\left(\nabla_{E_i}[\langle X,\nabla_{E_\ell}X\rangle]E_\ell + \langle X,\nabla_{E_\ell}X\rangle \underbrace{\nabla_{E_i}E_\ell}_{=0 \text{ at } p}, E_i\right)\\
&= \sum_{1\leq i\leq n} g\left(\nabla_{E_i}\left(\sum_{1\leq \ell\leq n}\langle X,\nabla_{E_\ell}X\rangle E_\ell\right), E_i\right)\\
&= \frac{1}{2}\sum_{1\leq i\leq n} g\left(\nabla_{E_i}\nabla\left(\|X\|^2\right), E_i\right)\\
&= \frac{1}{2}\Delta\left(\|X\|^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{1\leq i\leq n}\left[\nabla_{E_i}[g(E_i,\nabla_X X)] - g(E_i,\nabla_{\nabla_{E_i}X}X)\right] &= \sum_{1\leq i\leq n}\langle E_i,\nabla_{E_i}\nabla_X X - \nabla_{\nabla_{E_i}X}X\rangle\\
&= \sum_{1\leq i\leq n}\langle \nabla^2_{E_i,X}X, E_i\rangle\\
&= \sum_{1\leq i\leq n}\langle \nabla^2_{E_i,X}X - \nabla^2_{X,E_i}X, E_i\rangle + \sum_{1\leq i\leq n}\langle \nabla^2_{X,E_i}X, E_i\rangle\\
&= \sum_{1\leq i\leq n}\mathrm{Rm}(E_i,X,X,E_i) + \sum_{1\leq i\leq n}\langle \nabla^2_{X,E_i}X, E_i\rangle\\
&= \mathrm{Ric}(X,X) + \sum_{1\leq i\leq n}\langle \nabla^2_{X,E_i}X, E_i\rangle\\
&= \mathrm{Ric}(X,X) + \sum_{1\leq i\leq n}\langle \nabla_X\nabla_{E_i}X, E_i\rangle\\
&= \mathrm{Ric}(X,X) + X\left(\sum_{1\leq i\leq n}\langle \nabla_{E_i}X, E_i\rangle\right)\\
&= \mathrm{Ric}(X,X) + X(\operatorname{div} X),
\end{aligned}
$$

we obtain

$$
[\operatorname{div}(\mathscr{L}_X g)](X) = \frac{1}{2}\cdot\Delta\|X\|^2 - \|\nabla X\|^2 + \mathrm{Ric}(X,X) + X(\operatorname{div} X),
$$

as claimed. *Mutatis mutandis*, it follows from the first part of the proof that

$$
\begin{aligned}
[\operatorname{div}(\mathscr{L}_X g)](Z) &= \sum_{1\leq i\leq n}\left[\nabla_{E_i}(\mathscr{L}_X g)\right](E_i,Z)\\
&= \sum_{1\leq i\leq n}\left[\nabla_{E_i}([\mathscr{L}_X g](E_i,Z)) - (\mathscr{L}_X g)(E_i,\nabla_{E_i}Z)\right]\\
&= \sum_{1\leq i\leq n}\left(\nabla_{E_i}\left[\langle\nabla_{E_i}X,Z\rangle + \langle E_i,\nabla_Z X\rangle\right] - \langle\nabla_{E_i}X,\nabla_{E_i}Z\rangle - \langle E_i,\nabla_{\nabla_{E_i}Z}X\rangle\right).
\end{aligned}
$$

In the case where $X = \nabla f$ (and, a fortiori, the map $Z \mapsto \nabla_Z X$ is self-adjoint, as we saw in remark **(O.45)**), we have

$$
\begin{aligned}
\sum_{1\leq i\leq n}\left(\nabla_{E_i}\left[\langle\nabla_{E_i}X,Z\rangle + \langle E_i,\nabla_Z X\rangle\right]\right) &= 2\cdot\sum_{1\leq i\leq n}\nabla_{E_i}(\langle\nabla_Z X, E_i\rangle)\\
&= 2\cdot\sum_{1\leq i\leq n}\langle\nabla_{E_i}\nabla_Z X, E_i\rangle,
\end{aligned}
$$

and

$$\sum_{1\leq i\leq n}\langle\nabla_{E_i}X,\nabla_{E_i}Z\rangle=\sum_{1\leq i\leq n}\langle E_i,\nabla_{\nabla_{E_i}Z}X\rangle,$$

so that

$$\begin{aligned}[\mathrm{div}(\mathscr{L}_X g)](Z)&=2\cdot\sum_{1\leq i\leq n}\langle E_i,\nabla_{E_i}\nabla_Z X-\nabla_{\nabla_{E_i}Z}X\rangle\\&=2\cdot\sum_{1\leq i\leq n}\langle\nabla^2_{E_i,Z}X-\nabla^2_{Z,E_i}X,E_i\rangle+2\cdot\sum_{1\leq i\leq n}\langle\nabla^2_{Z,E_i}X,E_i\rangle\\&=2\cdot\mathrm{Ric}(Z,X)+2\cdot Z(\mathrm{div}(X))=2\cdot\langle\mathrm{Ric}(Z),X\rangle+2\cdot\langle\nabla\Delta f,Z\rangle.\end{aligned}\tag{1.15}$$

Now, as we already saw in remark **(O.36)**,

$$\mathscr{L}_{\nabla f}g=2\cdot\nabla^2 f.$$

Therefore, equation **(1.15)** is written as:

$$[\mathrm{div}(\nabla^2 f)](\bullet)=\langle\mathrm{Ric}(\nabla f),\bullet\rangle+\langle\nabla\Delta f,\bullet\rangle\,.$$

Or, viewing all the entities above as tensors of type $(1,1)$ (equivalently, endomorphisms),

$$\mathrm{div}(\nabla^2 f)=\sharp\Big([\mathrm{div}(\nabla^2 f)](\bullet)\Big)=\mathrm{Ric}(\nabla f)+\nabla\Delta f.$$

■

Chapter 2

The Ricci flow

What is the Ricci flow? There are several ways, all (necessarily) equivalent, of answering this question. There is no escaping the rigor and exactness of a well-defined formula, but for now let us see how far we can get with words alone. The approaches we present in the next two sections are largely based on the references [52], [12] and [1].

2.1 Motivation and examples

As we mentioned in the **Introduction**, the Ricci flow emerged in the 1980s as the most promising tool for solving the Poincaré conjecture. The plan of attack is the following: let $\mathcal{M}^3$ be a closed simply connected topological 3-manifold. Recall that in dimension $n \leq 3$, the following very remarkable phenomenon occurs:

Lemma (L.2.1). *Let $\mathcal{M}^n$ be a topological manifold of dimension $n \leq 3$. Then $\mathcal{M}^n$ admits a unique differentiable structure compatible with its topology. Consequently,*

$$\mathcal{M}^n \text{ is homeomorphic to } \mathcal{N}^n \iff \mathcal{M}^n \text{ is diffeomorphic to } \mathcal{N}^n.$$

Proof: See [35].

■

We can then endow $\mathcal{M}^3$ with a differentiable structure and consequently with a Riemannian metric, say $g_0 \in \mathscr{T}_0^2(T\mathcal{M}^3)$ (about which, a priori, we know absolutely nothing). Supposing that, miraculously, g_0 happened to be an Einstein metric, the problem would be over: in dimension ≤ 3, Einstein metrics have constant sectional curvature. As a trivial consequence of the Cartan-Hadamard theorem, such curvature would necessarily be positive, and therefore we would be certain (by the Killing-Hopf theorem) that $\mathcal{M}^3$ is isometric (in particular, homeomorphic) to $\mathbb{S}^3$.

The plan of attack developed by R. Hamilton tries to find such a miraculous metric by deforming g_0. Roughly speaking, we will think of the manifold as if it were made of metal, whose temperature varies

between distinct regions, and we will deform the metric in such a way that we make the curvature "flow" from more curved areas to less curved areas, just as heat flows from hotter regions to colder ones. This amounts to changing the metrics of the space so that distances decrease faster in the directions along which the curvature is larger. The best possible scenario (which, as we will see, unfortunately does not happen in general) is that such a process eventually gives us a final metric (a concept that needs to be formalized, as done in [52]) g_∞ that is Einstein.

Despite the initial plausibility and simplicity of such a plan, there are several details that need to be studied carefully. First, in dimension ≥ 3 temperature is a much simpler object than curvature: while, informally, we could define the temperature on the manifold as a real function (which, obviously, assigns to each point its temperature), the curvature, in contrast, is a function $K : \mathrm{Gr}_2(\mathcal{M}) \to \mathbb{R}$, which assigns a value to each planar direction through a point. As we will see in proposition **(P.3.2)**, at each point of $\mathcal{M}$ six different numbers are needed to describe K. In seeking an analogue of the heat equation

$$\frac{\partial}{\partial t}u = \Delta u,$$

for the curvature, we will then need to combine all the numbers that describe the curvature into some geometric quantity that makes sense independently of any choice of coordinates and impose some formula that describes its change.

There is essentially only one way to proceed. As we saw in remark **(O.48)**, the Laplacian is, roughly speaking, an operator that takes averages over small spheres centered around a point, and, analogously, the Ricci tensor is obtained from the Riemannian curvature tensor by averaging curvatures in different directions. Let then $\{g(t)\}_{t\in I}$ be a smooth family of Riemannian metrics on $\mathcal{M}$. Thus motivated, let us arbitrarily fix $X, Y \in \Gamma(T\mathcal{M})$ and $p \in \mathcal{M}$. For each arbitrarily fixed $t_0 \in I$, the geometric object $\mathrm{Ric}_{g(t_0)}$ is well defined, the correspondence $(Z, W) \mapsto \mathrm{Ric}_{g(t_0)}(Z, W)$ is a symmetric 2-tensor and $\mathrm{Ric}_{g(t)}(p)(X(p), Y(p)) \in \mathbb{R}$. Analogously, we have $g(t_0)(p)(X(p), Y(p)) \in \mathbb{R}$. We can therefore take derivatives: we see that $\frac{\partial}{\partial t}\left(g(t)(X(p), Y(p))\right) \in \mathbb{R}$, and the correspondence $(Z, W) \mapsto \frac{\partial}{\partial t}\left(g(t)(X, Y)\right)$ is a symmetric tensor. In particular, fixing a constant $c \in \mathbb{R}$, the equation

$$\frac{\partial}{\partial t}g(t) = c \cdot \mathrm{Ric}_{g(t)}$$

makes sense. In view of the following

Lemma (L.2.2). *In harmonic coordinates with respect to $g(t)$, the Ricci tensor satisfies*

$$c \cdot \mathrm{Ric}_{jk} = -\frac{c}{2} \cdot \Delta(g_{jk}) - \frac{c}{2} \cdot Q_{jk}(\partial g, g^{-1}), \tag{2.1}$$

where Q_{jk} denotes a sum of terms containing components of g^{-1}, and spatial derivatives of order ≤ 1 of the components of the metric g.

Proof: See [50], where **(2.1)** was (equivalently) proved in the case $c = -2$.

■

It is then natural to choose $c = -2$, so that, informally, the flow equation is written as

$$\partial_t g = \Delta g + \dots,$$

fulfilling our goal of finding an analogue of the heat equation for the curvature and justifying the following

Definition (D.2.1). *Let $(\mathcal{M}, g_{\mathcal{M}})$ be a Riemannian manifold and $T > 0$. We will say that a family of metrics $\{g_t\}_{t\in[0,T)}$ is a solution of the Ricci flow on $\mathcal{M}$ with initial condition $g_{\mathcal{M}}$ when*

$$\begin{cases} \dfrac{\partial g_t}{\partial t} = -2 \cdot \mathrm{Ric}_{g_t}, \ \forall t \in (0,T); \\ g_0 = g_{\mathcal{M}}. \end{cases} \tag{2.2}$$

Remark (O.50). In the case $c = 1$, it is common to find in the literature equation (2.1) written as

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{ terms of order } \leq 1.$$

However, it is important to emphasize that in such a situation the notation Δg_{ij} may seem ambiguous, as it can reasonably be interpreted as the Laplacian of the real function g_{ij} or as the components of the Laplacian of the metric. But since trivially $\Delta g = 0$, the context makes such an abuse harmless (in contrast with the situation seen in remark (O.5), where a geodesic frame is used).

Remark (O.51). Although the plan of attack above was initially devised to attack the Poincaré conjecture, Perelman extended it so as to attack the even more general problem of Thurston's geometrization conjecture. One of the several obstacles in this direction is the existence of possible singularities along the evolution of the flow. Initially, to get around this impasse, Hamilton proposed making "cuts" (formally, decomposing the manifold into connected sums) in the manifold and then restarting the flow on each piece. Such a process is commonly called *surgery*.

Ideally, one expects that only a finite number of surgeries has to be performed and that the flow provides on each component a metric modeled on one of Thurston's eight model geometries, so that when we "go back in time", performing the surgery in the opposite direction, both Thurston's geometrization conjecture and the Poincaré conjecture are proved (in the case where the initial manifold is simply connected, one expects each component of the surgery to be a sphere; since connected sums of spheres are spheres, we would prove the Poincaré conjecture).

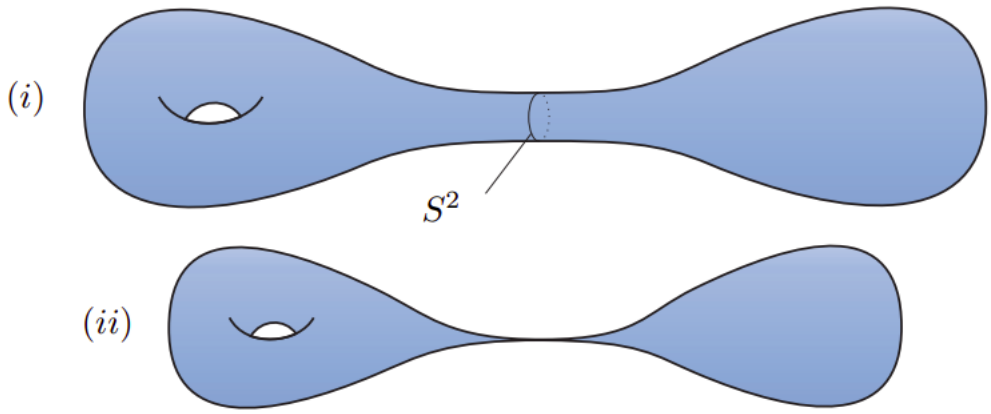


Figure 2.1: a singularity of "neck pinch" type. Figure taken from [52].

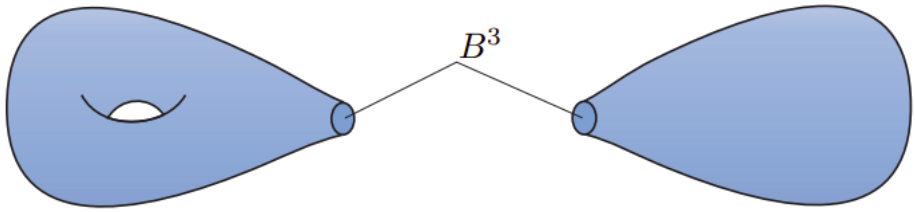


Figure 2.2: surgery. Figure taken from [52].

Hamilton was unable to show that such a process did not get stuck in a situation of the type of Zeno's paradox. *A priori*, it could happen that after 1 second of evolution of the flow we would have to perform such a process for the first time (generating 2 components), after another $\frac{1}{2}$ second we would have to perform it for the second time (generating 4 components), after another $\frac{1}{4}$ of a second, for the third time (generating 8 components), and so on... Perelman showed that such an obstacle could indeed be overcome, that is, that only a finite number of surgeries needs to be performed.

Remark (O.52). Hamilton's idea for classifying the singularities was to apply rescalings in order to decrease the curvature.

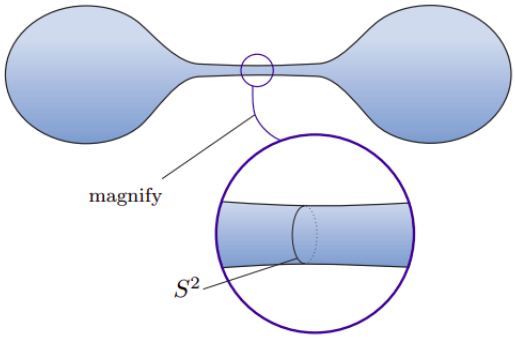


Figure 2.3: strategy for classifying singularities: use a "magnifying glass" near the singularity (apply rescalings) with ever greater "zoom", and then take limits. Figure taken from [52].

Ricci solitons appear as limits of the process we have informally outlined in the figure above (whose formalization can be found in [12] or [52]). Moreover, since solitons (which we will define shortly) are, in a certain sense, stationary solutions of the flow, they "are born" as a major initial obstruction to the plan we sketched in remark **(O.51)**, thus becoming important objects of study.

A priori, the standard theory of PDEs cannot be applied to guarantee short-time existence and uniqueness for the equation above, since it is not strongly parabolic due to the diffeomorphism invariance of the Ricci tensor. Originally, in [26], Hamilton used a Nash-Moser infinite-dimensional inverse function theorem to get around this problem, and established the following

Theorem (T.2.1). *Given a Riemannian metric $g_{\mathcal{M}}$ on a closed manifold $\mathcal{M}$, there exist $\varepsilon > 0$ and a smooth family of metrics $\{g_t\}_{t\in[0,\varepsilon]}$ such that*

$$\begin{cases} \dfrac{\partial g_t}{\partial t} = -2 \cdot \mathrm{Ric}_{g_t}, \ \forall t \in (0, \varepsilon); \\ g_0 = g_{\mathcal{M}}. \end{cases}$$

Proof: See [52] or [50].

■

More recently, Dennis DeTurck successfully proposed a simpler approach, which became known as "DeTurck's trick" in the literature. We also have uniqueness of solutions, in the following sense:

Theorem (T.2.2). *Suppose that $\{g_1(t)\}_{t\in[0,\varepsilon]}$ and $\{g_2(t)\}_{t\in[0,\varepsilon]}$ are solutions of the Ricci flow on a closed manifold $\mathcal{M}$. If there exists $s_0 \in [0, \varepsilon]$ such that $g_1(s_0) = g_2(s_0)$, then $g_1(t) = g_2(t)$ for every $t \in [0, \varepsilon]$.*

Proof: See [52].

■

Remark (O.53). Note the necessity of the instant s_0 at which the solutions coincide. Indeed, the success in dimension 2 of the strategy described earlier (as seen in [25]) shows that (a priori, at least in dimension 2) the flow evolves any geometry of positive curvature to the geometry of the sphere, of constant curvature, and therefore it is not possible to reverse the flow (just as it is not possible to reverse the heat equation). The maximal instant T is commonly called the *singular time*. If such an instant is finite, the curvature necessarily becomes arbitrarily large near T, as we see in the following

Theorem (T.2.3) (The only obstacle to long-time existence). *If g_0 is a smooth metric on a compact manifold $\mathcal{M}$, the Ricci flow with initial condition $g(0) = g_0$ has a unique solution $\{g(t)\}_{t\in I}$ on a maximal interval $0 \leq t < T \leq \infty$. Moreover, if $T < \infty$, then:*

$$\lim_{t\to T^-} \left(\sup_{x\in\mathcal{M}} \left\| \mathrm{Rm}_{g(t)} \right\| (x) \right) = \infty.$$

Proof: See [12].

■

Remark (O.54). Combining theorems **(T.2.1)** and **(T.2.2)**, it then makes sense to talk about ***the*** Ricci flow with initial metric (or initial condition) g_0, on a *maximal* interval $[0, T)$. In this case, "maximal" means that either $T = \infty$, or $T < \infty$ but there is no $\varepsilon > 0$ and no smooth Ricci flow $\{\widetilde{g}(t)\}_{t\in[0,T+\varepsilon)}$ such that $\widetilde{g}(t) = g(t)$ for every $t \in [0, T)$.

Remark (O.55). As a consequence of the uniqueness of solutions, we see that the Ricci flow also preserves isometries: *id est*, if $\psi : (\mathcal{M}, g_{\mathcal{M}}) \to (\mathcal{N}, g_{\mathcal{N}})$ is an isometry of Riemannian manifolds (equivalently, $\psi^*(g_{\mathcal{N}}) = g_{\mathcal{M}}$) and $\{g_1(t)\}_{t\in[0,T_1)}$, $\{g_2(t)\}_{t\in[0,T_2)}$ are solutions of the Ricci flow on $\mathcal{M}$ and $\mathcal{N}$ (with initial conditions $g_{\mathcal{M}}$ and $g_{\mathcal{N}}$), respectively, then $\psi : (\mathcal{M}, g_1(t)) \to (\mathcal{N}, g_2(t))$ is also an isometry.

To prove this fact, first note that equation **(2.2)** is invariant under diffeomorphisms. Indeed, if $\varphi : \mathcal{M} \to \mathcal{N}$ is a time-independent diffeomorphism and $g(t)$ is the metric defined by $g(t) = (\varphi^* g_2)(t)$ for every $t \in \min\{T_1, T_2\}$, we see that

$$\begin{aligned}\frac{\partial}{\partial t} g &= \varphi^* \left(\frac{\partial}{\partial t} g_2 \right) \\ &= \varphi^*(-2 \cdot \mathrm{Ric}(g_2)) \\ &= -2 \cdot \mathrm{Ric}(\varphi^* g_2) \\ &= -2 \cdot \mathrm{Ric}(g),\end{aligned}$$

and therefore g is also a solution of the Ricci flow on $\mathcal{M}$ (where in the second-to-last equality we used the invariance of the Ricci tensor under local isometries). In particular, taking $\varphi = \psi$, we have by the uniqueness of solutions (note that $g(0) = g_1$ trivially) seen in theorem **(T.2.2)** that $g(t) = \psi^*(g_2(t)) = g_1(t)$ for every $t \in \min\{T_1, T_2\}$. Equivalently, ψ remains an isometry along the flow.

Definition (D.2.2). Let $(\mathcal{M}, g_{\mathcal{M}})$ be a Riemannian manifold isometrically immersed in another Riemannian manifold $(\mathcal{N}, g_{\mathcal{N}})$ (via, say, $f_0 : (\mathcal{M}, g_{\mathcal{M}}) \to (\mathcal{N}, g_{\mathcal{N}})$) and suppose that $\{g^{\mathcal{M}}(t)\}_{t\in I}$ is a solution of the Ricci flow on $\mathcal{M}$. When there exists a family of isometric immersions

$$\{f_t : (\mathcal{M}, g^{\mathcal{M}}(t)) \to (\mathcal{N}, g_{\mathcal{N}})\}_{t\in I},$$

we will say that $\{f_t\}_{t\in I}$ is a geometric representation of the Ricci flow on $\mathcal{M}$ with respect to the ambient space $(\mathcal{N}, g_{\mathcal{N}})$.

Remark (O.56). It is common, when giving introductory expositions of the Ricci flow, to present figures of the evolution of the flow without the formalization of definition **(D.2.2)**. Since the Ricci flow is *intrinsic* (in particular, its definition involves no ambient space nor immersions), a priori this is not a harmless abuse. Next, we present one such figure (created by Cameron Slayden, originally appearing in [34], and with use authorized for dissertations, according to [49]), which explicitly portrays the "uniformization" of the curvature brought about by the flow.

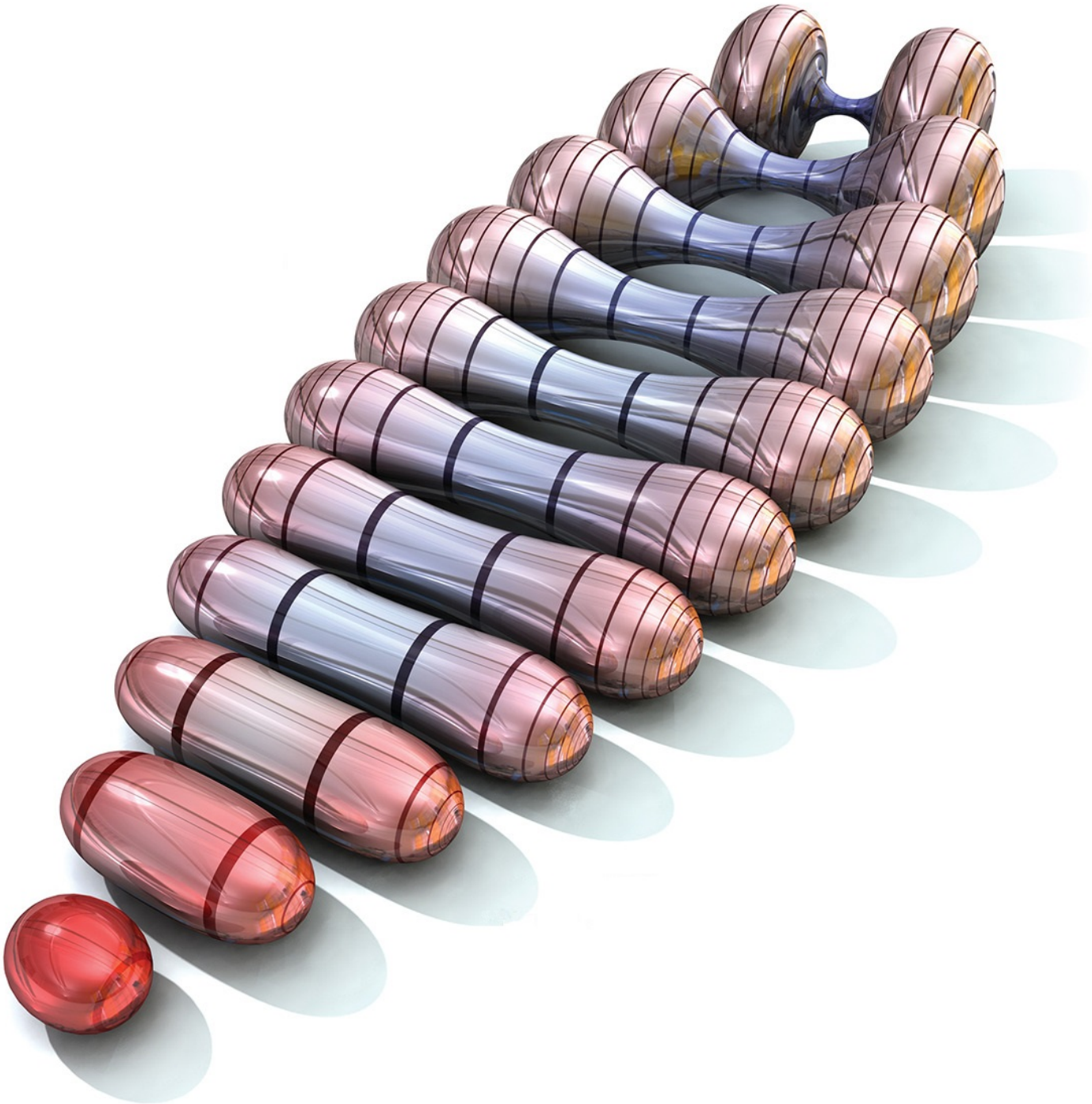


Figure 2.1: visualization of the geometric representation of the Ricci flow of a surface.

Example (E.2.1). Suppose that $(\mathcal{M}, g_0)$ is Einstein, so that

$$\mathrm{Ric}_{ij}(p,0) = \lambda g_{ij}(p,0),\ \forall p \in \mathcal{M},$$

where λ is a constant. Making the guess that

$$g_{ij}(x,t) = \rho^2(t) g_{ij}(x,0),\ \forall x \in \mathcal{M},$$

and using the fact that the Ricci tensor is invariant under diffeomorphisms, we see that $t \mapsto \mathrm{Ric}_{g(t)}$ must satisfy

$$\mathrm{Ric}_{ij}(p,t) = \mathrm{Ric}_{ij}(p,0) = \lambda g_{ij}(p,0),\ \forall p \in \mathcal{M}.$$

In this case, the Ricci flow equation is then written as:

$$\frac{\partial}{\partial t}\left(\rho^2(t) g_{ij}(p,0)\right) = -2\lambda g_{ij}(x,0),$$

which gives us the ODE

$$\frac{d\rho}{dt} = -\frac{\lambda}{\rho},$$

whose solution is given by

$$\rho^2(t) = 1 - 2\lambda t,\ \forall t \in \left[0, (2\lambda)^{-1}\right].$$

In particular, a sphere shrinks/collapses to a point in finite time and its curvature "blows up" near the singular time.

On the other hand, on a manifold $\mathbb{H}^3/\mathcal{G}$ (where $\mathcal{G}$ is a discrete group of isometries) of constant negative sectional curvature, the flow simply expands the manifold forever without ever "blowing up" or collapsing.

If the initial metric is Ricci-flat (*id est*, $\mathrm{Ric} \equiv 0$), then clearly the metric remains stationary forever (concrete examples include $\mathbb{R}^n$ with its usual metric and the flat torus $\mathbb{T}^n$ with the product metric).

Example (E.2.2). If the initial Riemannian manifold $\mathcal{N} = \mathcal{M}/\mathcal{G}$ is a quotient of a Riemannian manifold $\mathcal{M}$ by a discrete group of isometries $\mathcal{G}$, remark **(O.55)** guarantees that this property is preserved by the flow. Some concrete examples include closed surfaces with genus ≥ 1 (which are quotients of $\mathbb{H}^2$), projective spaces (quotients of $\mathbb{S}^n$ by $\mathbb{Z}_2$) and Riemannian manifolds of nonnegative sectional curvature (which are, by the Cartan-Hadamard theorem, quotients of $\mathbb{R}^n$).

Example (E.2.3). If $(\mathcal{M} \times \mathcal{N}, g_{\mathcal{M}} \oplus g_{\mathcal{N}})$ is a product manifold, it is trivial to verify that its metric remains a product metric under the flow (since $\mathrm{Ric}_{\mathcal{M}\times\mathcal{N}} = \mathrm{Ric}_{\mathcal{M}} \oplus \mathrm{Ric}_{\mathcal{N}}$). For example, the Ricci flow on the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ with the usual product metric shrinks the factor $\mathbb{S}^2$ to a point while keeping the factor $\mathbb{S}^1$ fixed, so that the manifold collapses to a circle. Moreover, it is also clear that if $\{g(t)\}_{t\in[0,T)}$ is a solution of the Ricci flow on a Riemannian manifold $(\mathcal{M}, g_{\mathcal{M}})$, then $\{g(t) \oplus g_{\mathbb{R}^n}\}_{t\in[0,T)} = \left\{g(t) \oplus \mathrm{d}r_1^2 + \cdots + \mathrm{d}r_n^2\right\}_{t\in[0,T)}$ is also a solution of the Ricci flow with initial condition $\mathcal{M} \times \mathbb{R}^n$, since the flow remains stationary on the factor $\mathbb{R}^n$.

Remark (O.57). If $\{g(t)\}_{t\in[0,T)}$ is a solution of the Ricci flow on $\mathcal{M}$, then defining

$$\widetilde{g}(t) = \lambda \cdot g\left(x, \frac{t}{\lambda}\right), \ \forall (x,t) \in \mathcal{M} \times [0,T),$$

then $\{\widetilde{g}(t)\}_{t\in[0,\lambda T)}$ is also a solution of the Ricci flow on $\mathcal{M}$. Indeed,

$$\begin{aligned}\frac{\partial \widetilde{g}}{\partial t}(x,t) &= \frac{\partial g}{\partial t}(x, \lambda^{-1}\cdot t) \\ &= -2 \cdot \mathrm{Ric}(g(\lambda^{-1}\cdot t))(x) \\ &= -2 \cdot \mathrm{Ric}(\widetilde{g}(t))(x),\end{aligned}$$

for every $(x,t) \in \mathcal{M} \times [0, \lambda T)$. Note that the Levi-Civita connection, the Ricci tensor and the curvature tensor of type $(1,3)$ are invariant under this rescaling but the sectional curvatures and the scalar curvature are transformed by the factor λ^{-1}.

2.2 Solitons

One of the obstructions to Hamilton's initial program of evolving an arbitrary initial Riemannian metric to a metric of "uniform" curvature (in dimension 3, Einstein, and, *a fortiori*, of constant sectional curvature) are the *Ricci solitons*. Geometrically, these solutions evolve under the flow only by diffeomorphisms and rescalings, and are, in a certain sense, fixed points of the flow (in a sense to be formalized shortly).

Definition (D.2.3). Let $\{g(t)\}_{t\in I}$ be a solution of the Ricci flow on a Riemannian manifold $(\mathcal{M}, g_0)$. If there exist a time-dependent family of diffeomorphisms $\{\varphi_t : \mathcal{M} \to \mathcal{M}\}_{t\in I}$ (with $\varphi_0 = \mathrm{Id}_{\mathcal{M}}$) and a time-dependent scale factor $\sigma \in \mathscr{C}^\infty(I)$ (with $\sigma(0) = 1$) such that

$$g(t) = \sigma(t) \cdot \varphi_t^*(g_0), \tag{2.3}$$

we will say that the solution $\{g(t)\}_{t\in I}$ is a Ricci soliton on $\mathcal{M}$. The triple $(\{g(t)\}_{t\in I}, \{\varphi_t\}_{t\in I}, \sigma)$ is called *a Ricci soliton structure on* $\mathcal{M}$.

Remark (O.58). Since the Ricci tensor is invariant under homotheties and diffeomorphisms, Ricci solitons correspond to fixed points of equation (2.2).

Proposition (P.2.1). A Riemannian manifold $(\mathcal{M}, g_{\mathcal{M}} = g_0)$ admits a Ricci soliton structure if, and only if, there exist a field $V \in \Gamma(T\mathcal{M})$ and a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}_{g_{\mathcal{M}}} + \frac{1}{2} \cdot \mathscr{L}_V\left(g_{\mathcal{M}}\right) = \lambda \cdot g_{\mathcal{M}}. \tag{2.4}$$

Remark (O.59). Given a time-dependent vector field $X : \mathcal{M} \times (-\varepsilon, \varepsilon) \to T\mathcal{M}$, there always exists a time-dependent family of diffeomorphisms (*id est*, a map $\psi : \mathcal{M} \times (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\psi_t = \psi(\bullet, t)$:

$\mathcal{M} \to \mathcal{M}$ is a diffeomorphism for every $t \in (-\varepsilon, \varepsilon)$) satisfying

$$\frac{\partial \psi}{\partial t} = X, \text{ and } \psi_0 = \mathrm{Id}_{\mathcal{M}}.$$

For a proof of this fact, one can consult chapter 9 of [31]. Explicitly, the equation

$$\frac{\partial \psi}{\partial t} = X$$

means that

$$\left.\frac{\partial \psi}{\partial t}\right|_{(p_0,t_0)} = \mathrm{d}\psi_{(p_0,t_0)}\left(\left.\frac{\partial}{\partial t}\right|_{t=t_0}\right) = \underbrace{(X_{t_0} \circ \psi_{t_0})(p_0)}_{\in\, T_{\psi(p_0,t_0)}\mathcal{M}},$$

for all $p_0 \in \mathcal{M}$ and $t_0 \in (-\varepsilon, \varepsilon)$. The family $\{\psi_t\}_{t\in(-\varepsilon,\varepsilon)}$ is called the *family of diffeomorphisms generated by the time-dependent fields* X_t.

Proof: Suppose that definition (D.2.3) is satisfied. Taking the derivative of expression (2.3), we see that

$$\frac{\partial g_t}{\partial t} = \frac{\mathrm{d}\sigma_t}{\mathrm{d}t} \cdot \varphi_t^*(g_0) + \sigma(t) \cdot \frac{\partial(\varphi_t^* g_0)}{\partial t}. \tag{2.5}$$

Evaluating equation (2.5) at $t = 0$ and defining $V \doteq \left.\frac{\partial \varphi_t}{\partial t}\right|_{t=0}$ and $\lambda \doteq -\frac{\sigma'(0)}{2}$, we obtain

$$\mathrm{Ric}_{g_{\mathcal{M}}} + \frac{1}{2} \cdot \mathscr{L}_V\left(g_{\mathcal{M}}\right) = \lambda \cdot g_{\mathcal{M}}.$$

On the other hand, if there exist $V \in \Gamma(T\mathcal{M})$ and $\lambda \in \mathbb{R}$ such that equation (2.4) is satisfied, we can consider the real function defined by

$$\sigma(t) = 1 - 2\lambda t, \tag{2.6}$$

for each $t \in [0, T)$, where $T = \infty$ if $\lambda \leq 0$ and $T = (2\lambda)^{-1}$ if $\lambda > 0$. Defining then for each $t \in [0, T)$ the field

$$\mathcal{M} \ni p \mapsto Y_t(p) = \frac{V(p)}{\sigma(t)},$$

we can then define ψ to be the family of diffeomorphisms generated by the time-dependent fields Y_t, as in remark (O.59). Finally, defining $g(t) = \sigma(t) \cdot \psi_t^*(g_0)$, we see that

$$\begin{aligned}
\frac{\partial g_t}{\partial t} &= \sigma'(t)\psi_t^*(g_0) + \sigma(t) \cdot \psi_t^*\left(\mathscr{L}_{Y(t)} g_0\right) \\
&= \psi_t^*\left([-2\lambda + \mathscr{L}_V] g_0\right) \\
&= \psi_t^*(-2 \cdot \mathrm{Ric}(g_0)) \\
&= -2 \cdot \mathrm{Ric}(\psi_t^*(g_0)) \\
&= -2 \cdot \mathrm{Ric}\left((\sigma(t))^{-1} \cdot g(t)\right) \\
&= -2 \cdot \mathrm{Ric}_{g_t}.
\end{aligned}$$

■

Remark (O.60). In the literature, the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ correspond to *shrinking*, *steady* and *expanding* solitons, respectively. Such nomenclature is justified by the scale factor defined in equation **(2.6)**. Note that in the shrinking case, the solution exists only for $t \in [0, (2\lambda)^{-1})$.

Remark (O.61). Using the notation of proposition **(P.2.1)**, we will say that a Ricci soliton is a gradient Ricci soliton if there exists $f \in \mathscr{C}^\infty(\mathcal{M})$ such that $V = \nabla f$. In this case, it follows from remark **(O.36)** that equation **(2.4)** is written as

$$\mathrm{Ric} + \nabla^2 f = \lambda g. \tag{2.7}$$

Example (E.2.4). Any Einstein manifold admits the trivial null field $V \equiv 0 \in \Gamma(T\mathcal{M})$ and therefore satisfies equation **(2.4)**, thus being a trivial example of a Ricci soliton.

Example (E.2.5) (Hamilton's Cigar). Consider $\mathbb{R}^2$ endowed with the Riemannian metric $g_0 \doteq g_\Sigma(0)$ defined by

$$g_0(x,y) = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{1 + x^2 + y^2}, \ \forall (x,y) \in \mathbb{R}^2.$$

We will define the following family of diffeomorphisms:

$$\begin{aligned} \varphi_t : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x,y) &\mapsto (e^{-2t}x, e^{-2t}y). \end{aligned}$$

It is simple to verify that the family of metrics $\{g(t)\}_{t\geq 0}$ defined by

$$g(t) = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{e^{4t} + x^2 + y^2}$$

is a solution of the Ricci flow, and moreover satisfies

$$g(t) = \varphi_t^*(g_0).$$

Therefore, $\{g(t)\}_{t\geq 0}$ is a Ricci soliton on $\mathbb{R}^2$. In the literature, this soliton is called *Hamilton's cigar*. One can prove that (excluding the trivial example of the plane $\mathbb{R}^2$) it is the only conformally flat steady gradient Ricci soliton.

Example (E.2.6) (Bryant Soliton). Let $g_{\mathbb{S}^{n-1}}$ be the usual metric on the $(n-1)$-dimensional unit round sphere. We will look for steady Ricci solitons that arise as warped products on $(0,\infty) \times \mathbb{S}^{n-1}$. Let us then consider metrics of the form

$$g = \mathrm{d}r^2 + (\phi(r))^2 g_{\mathbb{S}^{n-1}}.$$

Taking a local orthonormal frame on $\mathbb{S}^{n-1}$, say

$$\left\{ \frac{\partial}{\partial \theta^i} \right\}_{1\leq i\leq n-1},$$

we see that $\left\{\frac{\partial}{\partial r}, \frac{1}{\phi}\frac{\partial}{\partial \theta^i}\right\}_{1\leq i\leq n-1}$ is a local orthonormal frame on $(0,\infty)\times_\phi \mathbb{S}^{n-1}$. By item (8) of lemma **(L.4.10)**, we see that the radial curvature of a plane through the radial vector $\frac{\partial}{\partial r}$ is given by

$$\begin{aligned} K_{\text{rad}} = K\left(\frac{\partial}{\partial r}, \frac{1}{\phi}\frac{\partial}{\partial \theta^i}\right) &= g\left(\text{Rm}\left(\frac{\partial}{\partial r}, \frac{1}{\phi}\frac{\partial}{\partial \theta^i}\right)\frac{1}{\phi}\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial r}\right) \\ &= \frac{1}{\phi^2}\cdot g(\text{Rm}(\partial_r, \partial_{\theta^i})\partial_{\theta^i}, \partial_r) \\ &= -\frac{1}{\phi^2}\cdot\frac{\phi^2}{\phi}\cdot\phi'' \\ &= -\frac{\phi''}{\phi}. \end{aligned}$$

Analogously, using item (9) of lemma **(L.4.10)**, we see that the sectional curvature of a plane P_{sph} perpendicular to $\frac{\partial}{\partial r}$ is given by

$$\begin{aligned} K_{\text{sph}} = K\left(\frac{1}{\phi}\partial_{\theta^i}, \frac{1}{\phi}\partial_{\theta^j}\right) &= \frac{1}{\phi^4}\cdot g\left(\text{Rm}(\partial_{\theta^i}, \partial_{\theta^j})\partial_{\theta^j}, \partial_{\theta^i}\right) \\ &= \frac{1}{\phi^4}\cdot\phi^2\cdot g_{\mathbb{S}^{n-1}}\left(\left[\pi^*_{\mathbb{S}^{n-1}}\text{Rm}\right](\partial_{\theta^i}, \partial_{\theta^j})\partial_{\theta^j}\right), \partial_{\theta^i}) \\ &= \frac{1}{\phi^2}\cdot\left(1-\frac{(\phi')^2}{\phi^2}\cdot\phi^2\right) \\ &= \frac{1-(\phi')^2}{\phi^2}. \end{aligned}$$

Therefore,

$$\text{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \sum_{1\leq i\leq n-1} K_{\text{rad}} = -(n-1)\frac{\phi''}{\phi},$$

and

$$\text{Ric}\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right) = \left((n-2)\left(1-(\phi')^2\right) - \phi\phi''\right) g_{\mathbb{S}^{n-1}}\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right)$$

We then conclude that

$$\text{Ric}(g) = -(n-1)\frac{\phi''}{\phi}\cdot \mathrm{d}r^2 + \left((n-2)\left(1-(\phi')^2\right) - \phi\phi''\right) g_{\mathbb{S}^{n-1}}.$$

Moreover, since

$$\nabla^2 f = f''(r)\cdot \mathrm{d}r^2 + \phi\phi' f'\cdot g_{\mathbb{S}^{n-1}},$$

we see that the equation for steady Ricci solitons $\text{Ric}(g) + \nabla^2 f = 0$ in this case is equivalent to the following system of second-order ODEs:

$$\begin{cases} f'' = (n-1)\frac{\phi''}{\phi}; \\ \phi\phi' f' = -(n-2)\left(1-(\phi')^2\right) + \phi\phi''; \end{cases}$$

By studying this system in detail, one can prove the existence of a complete, rotationally symmetric steady gradient Ricci soliton, which is unique up to homotheties.

Example (E.2.7) (Gaussian Soliton). Euclidean space $\mathbb{R}^n$ with the canonical metric $g(t) \equiv g_{\text{can}} \cong \text{Id}_{\mathbb{R}^n}$ is a trivial stationary solution of the Ricci flow, which at first sight may seem uninteresting. But, in fact, the fact that the usual metrics of space forms are invariant under rescalings (more precisely, if g is a metric of constant sectional curvature on a space form $\mathcal{M}$, $cg = (\sqrt{c} \cdot \text{Id}_{\mathcal{M}})^* g$ is isometric to g for every $c > 0$) makes this solution interesting, as it allows $(\mathbb{R}^n, g_{\text{can}})$ to be seen as a gradient Ricci soliton that is not only steady but also possibly expanding or shrinking. Indeed, considering any constant $\lambda \in \mathbb{R}$ and defining $f(x) = \frac{\lambda}{2} \cdot \|x\|^2 \ \forall x \in \mathbb{R}^n$, we see that equation **(2.7)** is trivially satisfied. This example shows that it is possible for a solution of the Ricci flow to be a gradient Ricci soliton that is either shrinking, steady or expanding, depending on how the potential function f is chosen.

Example (E.2.8). Consider the product $\mathbb{R}^m \times \mathbb{S}^n$ with the usual product metric $g = g_{\mathbb{R}^m} \oplus g_{\mathbb{S}^n}$. It is simple to verify that equation **(2.7)** is satisfied by taking $\lambda = n - 1$ and $f(x, \theta) = \frac{\lambda}{2} \cdot \|x\|^2 \ \forall (x, \theta) \in \mathbb{R}^m \times \mathbb{S}^n$.

The following lemmas will be very useful later on.

Lemma (L.2.3). Consider the square of the curvature operator $\text{Rm} : \Lambda^2(\mathcal{M}^n) \to \Lambda^2(\mathcal{M})$ (defined in **Chapter 3**), given by

$$\text{Rm}^{\circ 2} = \text{Rm} \circ \text{Rm} : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M}).$$

Then in any local orthonormal frame $\{E_i\}_{1 \leq i \leq n}$ (which at an arbitrarily fixed point u we will denote by $\mathbf{e}_i$), we have

$$-\sum_{1 \leq i \leq n} g\left(\text{Rm}(w, z)\mathbf{e}_i, \text{Rm}(x, y)\mathbf{e}_i\right) = -2 \cdot \text{Rm}^{\circ 2}(x, y, z, w).$$

Proof: First, note that the equality

$$-\sum_{1 \leq i \leq n} g\left(\text{Rm}(w, z)\mathbf{e}_i, \text{Rm}(x, y)\mathbf{e}_i\right) = -2 \cdot \text{Rm}^{\circ 2}(x, y, z, w) \tag{2.8}$$

is satisfied if, and only if,

$$\text{Rm}^{\circ 2}_{pqrs} = \frac{1}{2} \sum_{1 \leq i, j \leq n} \text{Rm}_{srij} \text{Rm}_{pqij}. \tag{2.9}$$

Indeed, equality **(2.8)** is satisfied if, and only if,

$$\begin{aligned} \text{Rm}^{\circ 2}_{pqrs} &= \frac{1}{2} \sum_{1 \leq i \leq n} \langle \text{Rm}(E_p, E_q)E_i, \text{Rm}(E_s, E_r)E_i \rangle \\ &= \frac{1}{2} \sum_{1 \leq i \leq n} \text{Rm}(E_s, E_r, E_i, \text{Rm}(E_p, E_q)E_i) \\ &= \frac{1}{2} \sum_{1 \leq i, j \leq n} \text{Rm}_{srij} \text{Rm}_{pqij}. \end{aligned}$$

On the other hand, since the operator Rm is self-adjoint, it follows (by the construction of the identification between curvature-type tensors and endomorphisms of $\Lambda^2(\mathcal{M})$ seen in **Chapter 3**) that

$$\mathrm{Rm}^{\circ 2}_{pqrs} = \langle \mathrm{Rm}^{\circ 2}(\mathbf{e}^p \wedge \mathbf{e}^q), \mathbf{e}^s \wedge \mathbf{e}^r \rangle = \langle \mathrm{Rm}(\mathbf{e}^p \wedge \mathbf{e}^q), \mathrm{Rm}(\mathbf{e}^s \wedge \mathbf{e}^r) \rangle.$$

Moreover, from

$$\begin{aligned}(\mathrm{Rm}(\mathbf{e}^p \wedge \mathbf{e}^q))_{ij} &= \frac{1}{2} \cdot \sum_{1 \leq k,\ell \leq n} (\mathbf{e}^p \wedge \mathbf{e}^q)_{k\ell} \mathrm{Rm}_{ij\ell k} \\ &= \frac{1}{2} \cdot \sum_{1 \leq k,\ell \leq n} \delta^{pq}_{k\ell} \cdot \mathrm{Rm}_{ij\ell k} = \mathrm{Rm}_{ijqp},\end{aligned}$$

we have that

$$\mathrm{Rm}(\mathbf{e}^p \wedge \mathbf{e}^q) = \frac{1}{2} \cdot \sum_{1 \leq i,j \leq n} \mathrm{Rm}_{ijqp} \mathbf{e}^i \wedge \mathbf{e}^j,$$

and analogously

$$\mathrm{Rm}(\mathbf{e}^s \wedge \mathbf{e}^r) = \frac{1}{2} \cdot \sum_{1 \leq k,\ell \leq n} \mathrm{Rm}_{k\ell rs} \mathbf{e}^k \wedge \mathbf{e}^\ell.$$

Hence,

$$\begin{aligned}\mathrm{Rm}^{\circ 2}_{pqrs} &= \langle \mathrm{Rm}(\mathbf{e}^p \wedge \mathbf{e}^q), \mathrm{Rm}(\mathbf{e}^s \wedge \mathbf{e}^r) \rangle \\ &= \left\langle \frac{1}{2} \cdot \sum_{1 \leq i,j \leq n} \mathrm{Rm}_{ijpq} \mathbf{e}^i \wedge \mathbf{e}^j, \frac{1}{2} \cdot \sum_{1 \leq k,\ell \leq n} \mathrm{Rm}_{k\ell sr} \mathbf{e}^k \wedge \mathbf{e}^\ell \right\rangle \\ &= \frac{1}{4} \sum_{1 \leq i,j,k,\ell \leq n} \mathrm{Rm}_{pqij}\, \mathrm{Rm}_{srk\ell} \cdot \delta^{ij}_{k\ell} \\ &= \frac{1}{2} \sum_{1 \leq k,\ell \leq n} \mathrm{Rm}_{pqij}\, \mathrm{Rm}_{srij},\end{aligned}$$

and therefore equality **(2.9)** is verified.

■

In the following lemma, we will prove some important properties satisfied by shrinking gradient Ricci solitons (following [24]). We will use these properties in an essential way in the next chapter.

Lemma (L.2.4). *Let $(\mathcal{M}^n, g, f)$ be a shrinking gradient Ricci soliton. Then we have:*

(a) $\mathrm{Scal} + \Delta f = \frac{n}{2}$;

(b) $\nabla \mathrm{Scal} = 2 \cdot \mathrm{Ric}(\nabla f)$;

(c) $\Delta_f \mathrm{Scal} = \mathrm{Scal} - 2 \cdot \|\mathrm{Ric}\|^2$;

(d) $\mathrm{Scal} + \|\nabla f\|^2 = f$;

(e) $\Delta_f \mathrm{Rm} = \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}$.

Moreover, in a geodesic frame we have:

(f) $(\Delta_f \mathrm{Ric})_{ij} = \Delta_f(\mathrm{Ric}_{ij}) = \mathrm{Ric}_{ij} - 2 \sum_{1 \leq k,\ell \leq n} \mathrm{Rm}_{ik\ell j} \mathrm{Ric}_{k\ell}$;

(g)

$$\sum_{1\leq \ell\leq n}(\nabla_\ell \mathrm{Rm})_{ijk\ell} = \sum_{1\leq \ell\leq n}\nabla_\ell(\mathrm{Rm}_{ijk\ell}) = (\nabla_j \mathrm{Ric})_{ik} - (\nabla_i \mathrm{Ric})_{jk} = \nabla_j(\mathrm{Ric}_{ik}) - \nabla_i(\mathrm{Ric}_{jk})$$
$$= \sum_{1\leq \ell\leq n}\mathrm{Rm}_{ijk\ell} f_\ell.$$

Proof: By hypothesis (assuming without loss of generality $2\lambda = 1$), we have

$$\mathrm{Ric} + \nabla^2 f = \frac{1}{2}g. \tag{2.10}$$

Taking the g-trace of both sides of the equation above, we obtain

$$\mathrm{Scal} + \Delta f = \frac{1}{2}\cdot n = \frac{n}{2},$$

and therefore **(a)** is proved. It also follows that

$$\nabla\mathrm{Scal} + \nabla\Delta f = 0, \text{ and } \mathrm{div}(\mathrm{Ric}) + \mathrm{div}(\nabla^2 f) = \frac{1}{2}\,\mathrm{tr}(\nabla g) = 0.$$

On the other hand, we have by identities **(1.13)** and **(1.14)** that:

$$\nabla\mathrm{Scal} = 2\cdot\mathrm{div}(\mathrm{Ric}) \text{ and } \mathrm{div}(\nabla^2 f) = \mathrm{Ric}(\nabla f) + \nabla\Delta f.$$

Thus, it follows that:

$$\begin{aligned}\nabla\mathrm{Scal} &= 2\cdot\mathrm{div}(\mathrm{Ric})\\ &= -2\cdot\mathrm{div}(\nabla^2 f)\\ &= -2\cdot\mathrm{Ric}(\nabla f) - 2\cdot\nabla\Delta f\\ &= -2\cdot\mathrm{Ric}(\nabla f) + 2\cdot\nabla\mathrm{Scal},\end{aligned}$$

so that

$$\nabla\mathrm{Scal} = 2\cdot\mathrm{Ric}(\nabla f),$$

and therefore **(b)** is proved. From here on, we will do all the computations using a geodesic frame $\{\mathbf{e}_i\}_{1\leq i\leq n}$ around an arbitrarily fixed point $p\in\mathcal{M}$ (as seen in lemma **(L.1.1)**). We have:

$$\begin{aligned}g(\nabla(\mathrm{Scal} + \|\nabla f\|^2), \mathbf{e}_i) &= g\left(2\cdot\mathrm{Ric}(\nabla f) + 2\cdot\underbrace{g(\nabla^2 f, \nabla f)}_{=(\nabla^2 f)(\nabla f)}, \mathbf{e}_i\right)\\ &= 2\cdot[\mathrm{Ric}(\nabla f, \mathbf{e}_i) + (\nabla^2 f)(\nabla f, \mathbf{e}_i)]\\ &= g(\nabla f, \mathbf{e}_i).\end{aligned}$$

Since i and p are arbitrary, it follows that

$$\nabla(\mathrm{Scal} + \|\nabla f\|^2 - f) = 0,$$

and therefore the real function $\mathrm{Scal} + \|\nabla f\|^2 - f$ is constant. Without loss of generality (replacing, if necessary, f by $\tilde{f} \doteq f + c$ for some appropriate constant c, whose existence has just been demonstrated), we then have that

$$\mathrm{Scal} + \|\nabla f\|^2 = f,$$

so that **(d)** is proved. In a geodesic frame, it is clear that the first two equalities in **(f)** and **(g)** are satisfied, as we already saw in remark **(O.39)**. We will now prove in two different ways the last equality of **(g)**. Again from equation **(2.10)**, we have that

$$\begin{aligned}\mathrm{Ric}_{jk} + (\nabla^2 f)_{jk} &= \frac{1}{2} g_{jk};\\ \mathrm{Ric}_{ik} + (\nabla^2 f)_{ik} &= \frac{1}{2} g_{ik},\end{aligned}$$

whence it follows that:

$$(\nabla_i \mathrm{Ric})_{jk} + (\nabla^3 f)_{ijk} = (\nabla_j \mathrm{Ric})_{ik} + (\nabla^3 f)_{jik} = 0,$$

and therefore

$$\begin{aligned}(\nabla_j \mathrm{Ric})_{ik} - (\nabla_i \mathrm{Ric})_{jk} &= (\nabla^3 f)_{ijk} - (\nabla^3 f)_{jik}\\ &= \langle \partial_i \circ \partial_j \circ \partial_k - \partial_j \circ \partial_i \circ \partial_k, \nabla f \rangle\\ &= \sum_{1 \le a \le n} \mathrm{Rm}_{ijk}{}^a f_a\\ &= \sum_{1 \le \ell, a \le n} \underbrace{g^{a\ell}}_{=\delta^{a\ell}} \mathrm{Rm}_{ijk\ell} f_a\\ &= \sum_{1 \le \ell \le n} \mathrm{Rm}_{ijk\ell} f_\ell.\end{aligned}$$

On the other hand, given $x, y, z \in T_p\mathcal{M}$ we have by definition that

$$\begin{aligned}(\nabla_x \mathrm{Ric})(y,z) &= \nabla_x(\mathrm{Ric}(y,z)) - \mathrm{Ric}(\nabla_x y, z) - \mathrm{Ric}(y, \nabla_x z)\\ &= \nabla_x \left(\frac{1}{2} g(y,z) - (\nabla^2 f)(y,z)\right) - \left(\frac{1}{2} g(\nabla_x y, z) - (\nabla^2 f)(\nabla_x y, z)\right)\\ &- \left(\frac{1}{2} g(y, \nabla_x z) - (\nabla^2 f)(y, \nabla_x z)\right)\\ &= (\nabla^2 f)(\nabla_x y, z) + (\nabla^2 f)(y, \nabla_x z) - \nabla_x(g(\nabla_y \nabla f, z))\\ &= g\left(\nabla_{\nabla_x y} \nabla f, z\right) + g\left(\nabla_y \nabla f, \nabla_x z\right) - g\left(\nabla_x \nabla_y \nabla f, z\right) - g\left(\nabla_y \nabla f, \nabla_x z\right)\\ &= g\left(\nabla_{\nabla_x y} \nabla f - \nabla_x \nabla_y \nabla f, z\right),\end{aligned}$$

and therefore

$$\begin{aligned}(\nabla_x \mathrm{Ric})(y,z) - (\nabla_y \mathrm{Ric})(x,z) &= g\left(\nabla_{\nabla_x y} \nabla f - \nabla_{\nabla_y x} \nabla f + \nabla_y \nabla_x \nabla f - \nabla_x \nabla_y \nabla f, z\right)\\ &= -g\left(\nabla_x \nabla_y \nabla f - \nabla_y \nabla_x \nabla f + \nabla_{\nabla_y x - \nabla_x y} \nabla f, z\right) \qquad \textbf{(2.11)}\\ &= -g(\mathrm{Rm}(x,y)\nabla f, z) = \mathrm{Rm}(x,y,z,\nabla f),\end{aligned}$$

which is a coordinate-free version of the last equality in (g). Note that in the last equality we used the fact that ∇ is torsion-free and therefore satisfies

$$\nabla_x y - \nabla_y x = [x, y].$$

Finally, now using lemma **(L.1.14)**, we have

$$(\nabla_x \mathrm{Ric})(y,z) - (\nabla_y \mathrm{Ric})(x,z) = (\operatorname{div} \mathrm{Rm})(x,y,z) = \sum_{1\leq \ell \leq n} (\nabla_{\mathbf{e}_\ell} \mathrm{Rm})(x,y,z,\mathbf{e}_\ell), \tag{2.12}$$

and therefore (g) is proved. Now, by definition (see (D.1.32)), we have that:

$$(\Delta \mathrm{Rm})(x,y,z,w) = \sum_{1\leq i\leq n} (\nabla^2_{\mathbf{e}_i,\mathbf{e}_i} \mathrm{Rm})(x,y,z,w).$$

But note that by definition (see (D.1.26)), we have

$$\begin{aligned}(\nabla^2_{\mathbf{e}_i,x} \mathrm{Rm})(\mathbf{e}_i,y,z,w) &= [\nabla_{\mathbf{e}_i}(\nabla_x \mathrm{Rm})](\mathbf{e}_i,y,z,w) - \underbrace{(\nabla_{\nabla_{e_i} x} \mathrm{Rm})(\mathbf{e}_i,y,z,w)}_{=0,\text{ since } \nabla_{\mathbf{e}_i} x = 0} \\ &= \nabla_{\mathbf{e}_i}\left[(\nabla_x \mathrm{Rm})(\mathbf{e}_i,y,z,w)\right] \\ &= \nabla_{\mathbf{e}_i}\left[(\nabla_y \mathrm{Rm})(\mathbf{e}_i,x,z,w) + (\nabla_{\mathbf{e}_i} \mathrm{Rm})(x,y,z,w)\right],\end{aligned}$$

where we repeatedly used that $\{\mathbf{e}_i\}_{1\leq i\leq n}$ is a geodesic frame and in the last equality we used the second Bianchi identity. We then have that:

$$(\nabla^2_{\mathbf{e}_i,x} \mathrm{Rm})(\mathbf{e}_i,y,z,w) - (\nabla^2_{\mathbf{e}_i,y} \mathrm{Rm})(\mathbf{e}_i,x,z,w) = (\nabla^2_{\mathbf{e}_i,\mathbf{e}_i} \mathrm{Rm})(x,y,z,w).$$

Thus, we see that:

$$\begin{aligned}(\Delta \mathrm{Rm})(x,y,z,w) &= \sum_{1\leq i\leq n} \left[(\nabla^2_{\mathbf{e}_i,x} \mathrm{Rm})(\mathbf{e}_i,y,z,w) - (\nabla^2_{\mathbf{e}_i,y} \mathrm{Rm})(\mathbf{e}_i,x,z,w)\right] \\ &= \sum_{1\leq i\leq n} \left[(\nabla^2_{x,\mathbf{e}_i} \mathrm{Rm})(\mathbf{e}_i,y,z,w) + (\mathrm{Rm}_{\mathbf{e}_i,x} \mathrm{Rm})(\mathbf{e}_i,y,z,w)\right] \\ &\quad - \sum_{1\leq i\leq n} \left[(\nabla^2_{y,\mathbf{e}_i} \mathrm{Rm})(\mathbf{e}_i,x,z,w) - (\mathrm{Rm}_{\mathbf{e}_i,y} \mathrm{Rm})(\mathbf{e}_i,x,z,w)\right].\end{aligned}$$

But again using the fact that we are working in a geodesic frame, we see that:

$$\begin{aligned}\sum_{1\leq i\leq n} (\nabla^2_{x,\mathbf{e}_i} \mathrm{Rm})(\mathbf{e}_i,y,z,w) &= \nabla_x \left(\sum_{1\leq i\leq n} (\nabla_{\mathbf{e}_i} \mathrm{Rm})(\mathbf{e}_i,y,z,w)\right) \\ &= \nabla_x(\mathrm{Rm}(\nabla f,y,z,w)),\end{aligned}$$

where we used equations (2.11) and (2.12) in the last equality. In an entirely analogous way, we have

$$\sum_{1\leq i\leq n} (\nabla^2_{y,\mathbf{e}_i} \mathrm{Rm})(\mathbf{e}_i,x,z,w) = \nabla_y \left(\mathrm{Rm}(\nabla f,x,z,w)\right).$$

Therefore:

$$\begin{aligned}(\Delta\mathrm{Rm})(x,y,z,w) &= \nabla_x(\mathrm{Rm}(\nabla f,y,z,w)) - \nabla_y\left(\mathrm{Rm}(\nabla f,x,z,w)\right)\\ &\quad+\sum_{1\le i\le n}\left[(\mathrm{Rm}_{\mathbf{e}_i,x}\mathrm{Rm})(\mathbf{e}_i,y,z,w) - (\mathrm{Rm}_{\mathbf{e}_i,y}\mathrm{Rm})(\mathbf{e}_i,x,z,w)\right]\\ &= (\nabla_x\mathrm{Rm})(\nabla f,y,z,w) + \mathrm{Rm}(\nabla_x\nabla f,y,z,w) - (\nabla_y\mathrm{Rm})(\nabla f,x,z,w) - \mathrm{Rm}(\nabla_y\nabla f,x,z,w)\\ &\quad+\sum_{1\le i\le n}\left[(\mathrm{Rm}_{\mathbf{e}_i,x}\mathrm{Rm})(\mathbf{e}_i,y,z,w) - (\mathrm{Rm}_{\mathbf{e}_i,y}\mathrm{Rm})(\mathbf{e}_i,x,z,w)\right].\end{aligned}$$

Now, from the second Bianchi identity, we have

$$(\nabla_{\nabla f}\mathrm{Rm})(x,y,z,w) = (\nabla_x\mathrm{Rm})(\nabla f,y,z,w) - (\nabla_y\mathrm{Rm})(\nabla f,x,z,w).$$

Note also that:

$$\begin{aligned}\mathrm{Rm}(\nabla_x\nabla f,y,z,w) &= g\left(\mathrm{Rm}(\nabla_x\nabla f,y)z,w\right)\\ &= g\left(\mathrm{Rm}((\nabla^2 f)(x),y,z),w\right)\\ &= g\left(\mathrm{Rm}\left(\frac{1}{2}x-\mathrm{Ric}(x),y,z\right),w\right)\\ &= \frac{1}{2}\mathrm{Rm}(x,y,z,w) - \mathrm{Rm}(\mathrm{Ric}(x),y,z,w).\end{aligned}$$

From this we obtain:

$$\begin{aligned}(\Delta\mathrm{Rm})(x,y,z,w) &= (\nabla_{\nabla f}\mathrm{Rm})(x,y,z,w) + \frac{1}{2}\left(\mathrm{Rm}(x,y,z,w) - \mathrm{Rm}(y,x,z,w)\right)\\ &\quad-\mathrm{Rm}(\mathrm{Ric}(x),y,z,w) + \mathrm{Rm}(\mathrm{Ric}(y),x,z,w)\\ &\quad+\sum_{1\le i\le n}\left[(\mathrm{Rm}_{\mathbf{e}_i,x}\mathrm{Rm})(\mathbf{e}_i,y,z,w) - (\mathrm{Rm}_{\mathbf{e}_i,y}\mathrm{Rm})(\mathbf{e}_i,x,z,w)\right],\end{aligned}$$

whence it follows that

$$\begin{aligned}(\Delta_f\mathrm{Rm})(x,y,z,w) &= \mathrm{Rm}(x,y,z,w) - \mathrm{Rm}(\mathrm{Ric}(x),y,z,w) + \mathrm{Rm}(\mathrm{Ric}(y),x,z,w)\\ &\quad+\sum_{1\le i\le n}\left[(\mathrm{Rm}_{\mathbf{e}_i,x}\mathrm{Rm})(\mathbf{e}_i,y,z,w) - (\mathrm{Rm}_{\mathbf{e}_i,y}\mathrm{Rm})(\mathbf{e}_i,x,z,w)\right].\end{aligned}$$

Now, as seen in **(1.10)**, we have

$$\begin{aligned}(\mathrm{Rm}_{\mathbf{e}_i,x}\mathrm{Rm})(\mathbf{e}_i,y,z,w) &= -\mathrm{Rm}(\mathrm{Rm}(\mathbf{e}_i,x)\mathbf{e}_i,y,z,w) - \mathrm{Rm}(\mathbf{e}_i,\mathrm{Rm}(\mathbf{e}_i,x)y,z,w)\\ &\quad-\mathrm{Rm}(\mathbf{e}_i,y,\mathrm{Rm}(\mathbf{e}_i,x)z,w) - \mathrm{Rm}(\mathbf{e}_i,y,z,\mathrm{Rm}(\mathbf{e}_i,x)w),\end{aligned}$$

and since

$$\begin{aligned}-\mathrm{Rm}(\mathbf{e}_i,y,z,\mathrm{Rm}(\mathbf{e}_i,x)w) &= -g\left(\mathrm{Rm}(\mathbf{e}_i,y)z,\mathrm{Rm}(\mathbf{e}_i,x)w\right)\\ &= -\mathrm{Rm}(\mathbf{e}_i,x,w,\mathrm{Rm}(\mathbf{e}_i,y)z)\end{aligned}$$

$$= \mathrm{Rm}(\mathbf{e}_i, x, \mathrm{Rm}(\mathbf{e}_i, y)z, w),$$

it follows that

$$\begin{aligned}(\mathrm{Rm}_{\mathbf{e}_i,x}\mathrm{Rm})(\mathbf{e}_i, y, z, w) = \mathrm{Rm}(\mathbf{e}_i, x, \mathrm{Rm}(\mathbf{e}_i, y)z, w) - \mathrm{Rm}(\mathrm{Rm}(\mathbf{e}_i, x)\mathbf{e}_i, y, z, w)\\ - \mathrm{Rm}(\mathbf{e}_i, \mathrm{Rm}(\mathbf{e}_i, x)y, z, w) - \mathrm{Rm}(\mathbf{e}_i, y, \mathrm{Rm}(\mathbf{e}_i, x)z, w).\end{aligned}$$

Now, recalling that

$$\sum_{1\leq i\leq n} \mathrm{Rm}(v, \mathbf{e}_i)\mathbf{e}_i = \mathrm{Ric}(v)\ \forall v \in T_p\mathcal{M},$$

and denoting

$$A(x, y, z, w) \doteq \sum_{1\leq i\leq n} \left[(\mathrm{Rm}_{\mathbf{e}_i,x}\mathrm{Rm})(\mathbf{e}_i, y, z, w) - (\mathrm{Rm}_{\mathbf{e}_i,y}\mathrm{Rm})(\mathbf{e}_i, x, z, w)\right],$$

we then have that

$$\begin{aligned}A(x, y, z, w) &= \sum_{1\leq i\leq n} [\mathrm{Rm}(\mathbf{e}_i, x, \mathrm{Rm}(\mathbf{e}_i, y)z, w) - \mathrm{Rm}(\mathbf{e}_i, y, \mathrm{Rm}(\mathbf{e}_i, x)z, w)]\\ &+ \sum_{1\leq i\leq n} [\mathrm{Rm}(\mathrm{Rm}(\mathbf{e}_i, y)\mathbf{e}_i, x, z, w) - \mathrm{Rm}(\mathrm{Rm}(\mathbf{e}_i, x)\mathbf{e}_i, y, z, w)]\\ &+ \sum_{1\leq i\leq n} [\mathrm{Rm}(\mathbf{e}_i, \mathrm{Rm}(\mathbf{e}_i, y)x, z, w) - \mathrm{Rm}(\mathbf{e}_i, \mathrm{Rm}(\mathbf{e}_i, x)y, z, w)]\\ &+ \sum_{1\leq i\leq n} [\mathrm{Rm}(\mathbf{e}_i, x, \mathrm{Rm}(\mathbf{e}_i, y)z, w) - \mathrm{Rm}(\mathbf{e}_i, y, \mathrm{Rm}(\mathbf{e}_i, x)z, w)]\\ &= \mathrm{Rm}(\mathrm{Ric}(x), y, z, w) - \mathrm{Rm}(\mathrm{Ric}(y), x, z, w) + \sum_{1\leq i\leq n} \mathrm{Rm}(\mathbf{e}_i, \mathrm{Rm}(x, y)\mathbf{e}_i, z, w)\\ &+ 2\sum_{1\leq i\leq n} [\mathrm{Rm}(y, \mathbf{e}_i, \mathrm{Rm}(\mathbf{e}_i, x)z, w) - \mathrm{Rm}(x, \mathbf{e}_i, \mathrm{Rm}(\mathbf{e}_i, y)z, w)].\end{aligned}$$

Now, note that

$$\begin{aligned}\sum_{1\leq i\leq n} \mathrm{Rm}(\mathbf{e}_i, \mathrm{Rm}(x, y)\mathbf{e}_i, z, w) &= \sum_{1\leq i\leq n} \mathrm{Rm}(z, w, \mathbf{e}_i, \mathrm{Rm}(x, y)\mathbf{e}_i)\\ &= -\sum_{1\leq i\leq n} \mathrm{Rm}(w, z, \mathbf{e}_i, \mathrm{Rm}(x, y)\mathbf{e}_i)\\ &= -\sum_{1\leq i\leq n} g\left(\mathrm{Rm}(w, z)\mathbf{e}_i, \mathrm{Rm}(x, y)\mathbf{e}_i\right)\\ &= -2\cdot \mathrm{Rm}^2(x, y, z, w),\end{aligned} \tag{2.13}$$

where in the last equality we used lemma **(L.2.3)**. Moreover,

$$2\sum_{1\leq i\leq n} [\mathrm{Rm}(y, \mathbf{e}_i, \mathrm{Rm}(\mathbf{e}_i, x)z, w) - \mathrm{Rm}(x, \mathbf{e}_i, \mathrm{Rm}(\mathbf{e}_i, y)z, w)] = -2\cdot \mathrm{Rm}^{\#}(x, y, z, w). \tag{2.14}$$

By the arbitrariness of x, y, z, w and of $p \in \mathcal{M}$, it follows that

$$\Delta_f \mathrm{Rm} = \mathrm{Rm} - 2\cdot(\mathrm{Rm}^2 + \mathrm{Rm}^{\#}), \tag{2.15}$$

and therefore (e) is proved. Another way of writing (g) is:

$$(\nabla_j \mathrm{Ric})_{ik} - (\nabla_i \mathrm{Ric})_{jk} = \sum_{1\leq s\leq n} \mathrm{Rm}_{ijks}\nabla^s f.$$

Thus, we have that:

$$\begin{aligned}(\Delta \mathrm{Ric})_{ik} = \sum_{1\leq j\leq n} \nabla^j(\nabla_j \mathrm{Ric}_{ik}) &= \sum_{1\leq j\leq n} \nabla^j\left(\nabla_i \mathrm{Ric}_{jk} + \sum_{1\leq s\leq n} \mathrm{Rm}_{ijks}\nabla^s f\right)\\ &= \sum_{1\leq i,s\leq n}\left[\nabla^j\nabla_i \mathrm{Ric}_{jk} + (\nabla^j(\mathrm{Rm}_{ijks}))\cdot\nabla^s f + \mathrm{Rm}_{ijks}\cdot\nabla^j\nabla^s f\right].\end{aligned} \tag{2.16}$$

On the other hand, now using the notation introduced in remark (O.33), it follows from the Ricci identity (see (1.11)) that

$$\begin{aligned}\sum_{1\leq j\leq n}\left[\nabla^j\nabla_i \mathrm{Ric}_{jk} - \nabla_i\nabla^j \mathrm{Ric}_{jk}\right] &= -\sum_{1\leq j,\ell,m\leq n}\left[g^{j\ell}\mathrm{Rm}_{i\ell j}{}^m \mathrm{Ric}_{mk} + g^{j\ell}\mathrm{Rm}_{i\ell k}{}^m \mathrm{Ric}_{jm}\right]\\ &= \sum_{1\leq j,\ell,m,t\leq n}\left[g^{j\ell}g^{mt}(\mathrm{Rm}_{\ell ijt}\mathrm{Ric}_{mk} + \mathrm{Rm}_{\ell ikt}\mathrm{Ric}_{mt})\right]\\ &= \sum_{1\leq j,t\leq n}\left[\mathrm{Rm}^j{}_{ijt}\mathrm{Ric}^t_k + \mathrm{Rm}^j{}_{ikt}\mathrm{Ric}^t_j\right] = \sum_{1\leq j,s\leq n}\left[\mathrm{Rm}^j{}_{ijs}\mathrm{Ric}^s_k + \mathrm{Rm}^j{}_{iks}\mathrm{Ric}^s_j\right].\end{aligned}$$

Now, by the second Bianchi identity, we also have:

$$\begin{aligned}&\nabla_k \mathrm{Rm}_{\ell sij} + \nabla_s \mathrm{Rm}_{k\ell ij} + \nabla_\ell \mathrm{Rm}_{skij} = 0;\\ \Longrightarrow\ &\nabla_k \mathrm{Rm}_{\ell sij} - \nabla_s \mathrm{Rm}_{\ell kij} = \nabla_\ell \mathrm{Rm}_{ijks};\\ \Longrightarrow\ &\sum_{1\leq j\leq n}\left[\nabla_k \mathrm{Rm}^j{}_{sij} - \nabla_s \mathrm{Rm}^j{}_{kij}\right] = \sum_{1\leq j\leq n}\nabla^j \mathrm{Rm}_{ijks}.\end{aligned}$$

Equation (2.16) is then written as:

$$\begin{aligned}(\Delta \mathrm{Ric})_{ik} &= \sum_{1\leq j,s\leq n}\left[\nabla_i\nabla^j \mathrm{Ric}_{jk} + \mathrm{Rm}^j{}_{ijs}\mathrm{Ric}^s_k + (\nabla_k \mathrm{Rm}^j{}_{sij})\cdot\nabla^s f - (\nabla_s \mathrm{Rm}^j{}_{kij})\cdot\nabla^s f + \mathrm{Rm}_{ijks}\cdot\nabla^j\nabla^s f\right]\\ &= \sum_{1\leq j,s\leq n}\left[\nabla_i\nabla^j \mathrm{Ric}_{jk} + \mathrm{Ric}_{is}\mathrm{Ric}^s_k + \mathrm{Rm}^j{}_{iks}\mathrm{Ric}^s_j - (\nabla_k \mathrm{Ric}_{si})\cdot\nabla^s f + \mathrm{Rm}_{ijks}\cdot\nabla^j\nabla^s f\right].\end{aligned} \tag{2.17}$$

By the contracted second Bianchi identity seen in the previous chapter (see remark (O.49)), we have that

$$\begin{aligned}(\mathrm{div}\,\mathrm{Ric})_i = \nabla^j \mathrm{Ric}_{ij} &= \frac{1}{2}\nabla_i \mathrm{Scal};\\ \Longrightarrow \nabla_i\nabla^j \mathrm{Ric}_{jk} &= \frac{1}{2}\nabla_i\nabla_k \mathrm{Scal}.\end{aligned}$$

And by the equation defining a gradient Ricci soliton, we have

$$\sum_{1\leq j,s\leq n}\mathrm{Rm}_{ijks}\nabla^j\nabla^s f = \sum_{1\leq j,s,\ell,t\leq n}\left[\mathrm{Rm}_{ijks}g^{j\ell}g^{st}\left(\frac{1}{2}g_{\ell t} - \mathrm{Ric}_{\ell t}\right)\right]$$

$$= \sum_{1\leq j,s,\ell,t\leq n} \mathrm{Rm}_{ijks}\left(\frac{1}{2}g^{j\ell}\delta^s_\ell - \mathrm{Ric}^{js}\right) = \sum_{1\leq j,s\leq n} \mathrm{Rm}_{ijks}\left(\frac{1}{2}g^{js} - \mathrm{Ric}^{js}\right)$$
$$= \sum_{1\leq j,s\leq n}\left[\frac{1}{2}\mathrm{Ric}_{ik} - \mathrm{Rm}_{ijks}\mathrm{Ric}^{js}\right].$$

Note also that:

$$(\nabla_{\nabla f}\mathrm{Ric})_{ik} = \nabla_{\nabla f}(\mathrm{Ric}_{ik}) = \sum_{1\leq s\leq n} \nabla_{(\nabla^s f)\cdot \mathbf{e}_s}\mathrm{Ric}_{ik} = (\nabla^s f)\cdot\nabla_s \mathrm{Ric}_{ik}.$$

Thus, equation **(2.17)** is written as

$$(\Delta\mathrm{Ric})_{ik} = (\nabla_{\nabla f}\mathrm{Ric})_{ik} + \frac{1}{2}\nabla_i\nabla_k\mathrm{Scal} + \frac{1}{2}\mathrm{Ric}_{ik} + \sum_{1\leq j,s\leq n}\left[\mathrm{Ric}_{is}\mathrm{Ric}^s_k + \underbrace{\mathrm{Rm}^j{}_{iks}\mathrm{Ric}^s_j}_{=-\mathrm{Rm}_{ijks}\mathrm{Ric}^{js}} - (\nabla_k\mathrm{Ric}_{si})\cdot\nabla^s f\right]$$
$$- \sum_{1\leq j,s\leq n}\mathrm{Rm}_{ijks}\mathrm{Ric}^{js}. \tag{2.18}$$

By the coordinate representation of **(b)**, we also see that

$$\nabla_i\mathrm{Scal} = 2\sum_{1\leq j\leq n}\mathrm{Ric}_{ij}\nabla^j f \implies 0 = \frac{1}{2}\nabla_k\left(\nabla_i\mathrm{Scal} - \sum_{1\leq s\leq n} 2\mathrm{Ric}_{is}\nabla^s f\right)$$
$$\implies \frac{1}{2}\nabla_k\nabla_i\mathrm{Scal} - \sum_{1\leq s\leq n}(\nabla_k\mathrm{Ric}_{is})\cdot\nabla^s f = \sum_{1\leq s\leq n}\mathrm{Ric}_{is}\cdot\nabla^s\nabla_k f.$$

Hence:

$$\frac{1}{2}\nabla_k\nabla_i\mathrm{Scal} - \sum_{1\leq s\leq n}(\nabla_k\mathrm{Ric}_{is})\cdot\nabla^s f = \sum_{1\leq s,\ell\leq n}\mathrm{Ric}_{is}g^{s\ell}\nabla_\ell\nabla_k f$$
$$= \sum_{1\leq s,\ell\leq n}\mathrm{Ric}_{is}g^{s\ell}\left(\frac{1}{2}g_{\ell k} - \mathrm{Ric}_{\ell k}\right)$$
$$= \sum_{1\leq s\leq n}\mathrm{Ric}_{is}\left(\frac{1}{2}\delta^s_k - \mathrm{Ric}^s_k\right)$$
$$= \frac{1}{2}\mathrm{Ric}_{ik} - \sum_{1\leq s\leq n}\mathrm{Ric}_{is}\mathrm{Ric}^s_k.$$

Therefore, equation **(2.18)** is written as:

$$(\Delta\mathrm{Ric})_{ik} = (\nabla_{\nabla f}\mathrm{Ric})_{ik} + \frac{1}{2}\mathrm{Ric}_{ik} - \sum_{1\leq j,s\leq n}\left[2\mathrm{Rm}_{ijks}\mathrm{Ric}^{js} + \mathrm{Ric}_{is}\mathrm{Ric}^s_k + \mathrm{Ric}_{is}\cdot\nabla^s\nabla_k f\right]$$
$$= (\nabla_{\nabla f}\mathrm{Ric})_{ik} + \frac{1}{2}\mathrm{Ric}_{ik} - \sum_{1\leq j,s\leq n}\left[2\mathrm{Rm}_{ijks}\mathrm{Ric}^{js} + \mathrm{Ric}_{is}\mathrm{Ric}^s_k - \mathrm{Ric}_{is}\mathrm{Ric}^s_k\right] + \frac{1}{2}\mathrm{Ric}_{ik}$$
$$= (\nabla_{\nabla f}\mathrm{Ric})_{ik} + \mathrm{Ric}_{ik} - 2\sum_{1\leq j,s\leq n}\mathrm{Rm}_{ijks}\mathrm{Ric}^{js}.$$

Equivalently,

$$(\Delta_f \mathrm{Ric})_{ij} = \Delta_f(\mathrm{Ric}_{ij}) = \mathrm{Ric}_{ij} - 2\sum_{1\leq k,\ell\leq n} \mathrm{Rm}_{ik\ell j}\mathrm{Ric}_{k\ell},$$

and therefore **(f)** is proved. Finally, taking the trace in the equation above, we obtain

$$\begin{aligned}\sum_{1\leq i,j\leq n} g^{ij}(\Delta_f \mathrm{Ric})_{ij} = \sum_{1\leq i,j\leq n} \Delta_f(g^{ij}\mathrm{Ric}_{ij}) = \Delta_f \mathrm{Scal} &= \sum_{1\leq i,j\leq n} g^{ij}\mathrm{Ric}_{ij} - 2\sum_{1\leq i,j,k,\ell\leq n} g^{ij}\mathrm{Rm}_{ik\ell j}\mathrm{Ric}_{k\ell} \\ &= \mathrm{Scal} - 2\sum_{1\leq k,\ell\leq n} (\mathrm{Ric}_{k\ell})^2 \\ &= \mathrm{Scal} - 2\,\|\mathrm{Ric}\|^2,\end{aligned}$$

so that **(c)** is proved.

■

Chapter 3

Curvature estimates

In this chapter we will present the proofs of Theorems **(T.3.1)** and **(T.3.2)** (due to Huai-Dong Cao, Ernani Ribeiro Jr and Detang Zhou, originally presented in [6]). The first Theorem, **(T.3.1)**, gives us a kind of classification (in dimension 4) of complete shrinking gradient Ricci solitons satisfying suitable curvature estimates, while the second Theorem, **(T.3.2)**, gives us some curvature estimates for Ric and Rm under the hypothesis of a certain control on Scal.

At first sight, the following subsections may seem somewhat "dry" and out of place in this work. However, they play a crucial role in establishing the items outlined below, which in turn are extremely important to the proofs of Theorems **(T.3.1)** and **(T.3.2)**:

- the decomposition of the Riemannian curvature tensor into its components corresponding to the Weyl tensor and to the Einstein tensor, and its important specialization to dimension 4, which is closely tied to the decomposition mentioned in the next item;
- the decomposition of the bundle of 2-forms of a four-dimensional manifold into eigenbundles invariant under the Hodge star operator.

3.1 Prologue

There are several reasons that make four dimensions a very special case. One of them, as we will soon see, is tied to the extremely important equality

$$2 + 2 = 4.$$

Indeed, this equality allows us a decomposition of the bundle of 2-forms of a differentiable manifold into its self-dual and anti-self-dual components (for example, the curvature 2-form of a Riemannian manifold, which takes values in a certain bundle, admits such a decomposition). Before that, we will establish some important decompositions, which hold in any dimension.

3.1 Some important decompositions

The main goal of this section is to establish (based on the expositions in [13], [54] and [3]) the decompositions of the curvature tensor in arbitrary dimension and of the bundle of 2-forms in dimension 4. We will see that these decompositions play a crucial role in the proofs of all our results. We will need some definitions and lemmas first.

Definition (D.3.1). A curvature-type tensor on a real vector space $\mathbb{V}$ is a tensor $R \in \mathscr{T}^0_4(\mathbb{V})$ satisfying the symmetries:

(a) $R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z)$;

(b) $R(x,y,z,\cdot) + R(y,z,x,\cdot) + R(z,x,y,\cdot) = 0$.

for all $x,y,z,w \in \mathbb{V}$. We will denote the vector space of all curvature-type tensors on $\mathbb{V}$ by $\mathscr{R}(\mathbb{V})$.

Remark (O.62). We will assume from now on that $\mathbb{V}$ is endowed with an inner product, denoted by g or $\langle \cdot, \cdot \rangle$. In this context, we define the norm $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ by

$$\|v\| = \sqrt{g(v,v)}, \ \forall v \in \mathbb{V}.$$

Definition (D.3.2). Given $R \in \mathscr{R}(\mathbb{V})$, we define the R-sectional curvature of a two-dimensional plane $\sigma = \operatorname{span}(u,v) \subset \mathbb{V}$ as

$$K^R(\sigma) = \frac{R(u,v,v,u)}{\|u\|^2\|v\|^2 - \langle u,v\rangle^2}.$$

Note that the linear independence of u and v forces the denominator of the expression above to be nonzero. Moreover, it is simple to check that if $x,y \in \mathbb{V}$ are such that $\operatorname{span}(x,y) = \sigma$, then $K^R(x,y) = K^R(u,v)$, so that $K^R(\sigma)$ is well defined.

Remark (O.63). On a Riemannian manifold $(\mathcal{M}^n, g)$, we define the bundle of curvature-type tensors on $\mathcal{M}^n$ as

$$\mathscr{R}(T\mathcal{M}) \doteq \bigcup_{p \in \mathcal{M}} \mathscr{R}(T_p\mathcal{M}).$$

Remark (O.64). It is common to call **(b)** the *"first Bianchi identity"* (known from Riemannian geometry). Together, **(a)** and **(b)** also imply the following *pair symmetry*:

(c) $R(x,y,z,w) = R(z,w,x,y)$ $\forall x,y,z,w \in \mathbb{V}$.

Indeed, given any $x,y,z,w \in \mathbb{V}$, we have

$$\begin{aligned}
2 \cdot R(x,y,z,w) &\overset{(a)}{=} R(x,y,z,w) + R(y,x,w,z) \\
&\overset{(b)}{=} -R(y,z,x,w) - R(z,x,y,w) - R(x,w,y,z) - R(w,y,x,z) \\
&\overset{(a)}{=} -R(z,y,w,x) - R(x,z,w,y) - R(w,x,z,y) - R(y,w,z,x) \\
&\overset{(b)}{=} R(w,z,y,x) + R(z,w,x,y) \\
&\overset{(a)}{=} 2 \cdot R(z,w,x,y).
\end{aligned}$$

Consider now the 4-tensor $g \owedge g$, defined by

$$(g \owedge g)(x,y,z,w) \doteq g(y,z)g(x,w) - g(x,z)g(y,w) \ \forall z,w \in \mathbb{V}.$$

This is our first example of a curvature-type tensor: $g \owedge g \in \mathscr{R}(\mathbb{V})$. Symmetry **(a)** of Definition **(D.3.1)** is immediate, while **(b)** of the same definition follows from a direct computation, with the six initial terms canceling in pairs. We will call $g \owedge g$ the *fundamental curvature* of $(\mathbb{V}, g)$. In order to extract even more examples from this idea, we will view $g \owedge g$ as a quadratic form in the variable g and use polarization again. Recall that if Q is a quadratic form, we can recover the associated bilinear form B through the relation

$$B(x,y) = \frac{Q(x+y) - Q(x) - Q(y)}{2},$$

so that $Q(x) = B(x,x)$. Let then $T, S \in \mathscr{T}^0_2(\mathbb{V})$. We then have:

$$\begin{aligned}((T+S) \owedge (T+S))(x,y,z,w) - (T \owedge T)(x,y,z,w) - (S \owedge S)(x,y,z,w) &= (T+S)(y,z)\cdot(T+S)(x,w)\\ - (T+S)(x,z)\cdot(T+S)(y,w) - T(y,z)\cdot T(x,w) + T(x,z)\cdot T(y,w) &- S(y,z)\cdot S(x,w)\\ + S(x,z)\cdot S(y,w)&\\ = T(y,z)\cdot S(x,w) + S(y,z)\cdot T(x,w) - T(x,z)\cdot S(y,w) &- S(x,z)\cdot T(y,w).\end{aligned}$$

This motivates the

Definition (D.3.3). Let $T, S \in \mathscr{T}^0_2(\mathbb{V})$. The *Kulkarni-Nomizu* product of T and S is the tensor $T \owedge S \in \mathscr{T}^0_4(\mathbb{V})$ defined by

$$2\cdot(T \owedge S)(x,y,z,w) = T(y,z)S(x,w) - T(x,z)S(y,w) + S(y,z)T(x,w) - S(x,z)T(y,w), \ \forall x,y,z,w \in \mathbb{V}.$$

We then have a symmetric map $\owedge : \mathscr{T}^0_2(\mathbb{V}) \times \mathscr{T}^0_2(\mathbb{V}) \to \mathscr{T}^0_4(\mathbb{V})$.

Remark (O.65).

- Note that the factor 2 on the left-hand side appeared as a natural consequence of the polarization process.
- Fixing a basis $\{\mathbf{e}_i\}_{1\leq i\leq n}$ of $\mathbb{V}$, we have the coordinate expression:

$$2\cdot(T \owedge S)_{ijk\ell} = T_{jk}S_{i\ell} - T_{ik}S_{j\ell} + S_{jk}T_{i\ell} - S_{ik}T_{j\ell}.$$

It is easy to see that the product $T \owedge S$ does not always satisfy symmetry **(a)** of Definition **(D.3.1)**, but it does satisfy symmetry **(c)**. We will now investigate which conditions on T and S guarantee that $T \owedge S$ is a curvature-type tensor. First, note that, by brute force, we see that (twice) the left-hand side of the Bianchi identity for $(T \owedge S)(x,y,z,w)$ is actually equal to the sum of the following six terms:

$$\begin{aligned}&(T(y,z) - T(z,y))S(x,w) + (T(z,x) - T(x,z))S(y,w)\\ &+ (T(x,y) - T(y,x))S(z,w) \ + \ \text{(three more terms, now swapping } T \leftrightarrow S),\end{aligned}$$

which vanish when both T and S are symmetric tensors. We have thus proved the:

Proposition (P.3.1). Let $T, S \in \mathscr{T}^0_2(\mathbb{V})$. If T and S are both symmetric, then the tensor $T \owedge S \in \mathscr{R}(\mathbb{V})$ is of curvature type.

Remark (O.66).

- We will denote by $\mathcal{S}(\mathbb{V})$ the space of all symmetric tensors of type $(0,2)$ on $\mathbb{V}$, and by $\mathcal{S}^2(\mathbb{V})$ the space of symmetric bilinear forms on $\mathbb{V}$.
- Focusing on these symmetries allows us to give another interpretation of the Kulkarni-Nomizu product. If $R \in \mathscr{R}(\mathbb{V})$, symmetry **(a)** tells us that R induces a map $R' : \mathbb{V}^{\wedge 2} \times \mathbb{V}^{\wedge 2} \to \mathbb{R}$, characterized by its action on pairs of 2-blades as

$$R'(x \wedge y, z \wedge w) \doteq R(x, y, w, z).$$

Once an inner product on $\mathbb{V}^{\wedge 2}$ is fixed, a self-adjoint endomorphism $\widetilde{R} \in \mathrm{End}(\mathbb{V}^{\wedge 2})$ is then determined, characterized by

$$\widetilde{R}(x \wedge y) = \sharp(R'(x \wedge y, \bullet)).$$

Now, symmetry **(c)** induces yet another map $R'' : \mathbb{V}^{\wedge 2} \odot \mathbb{V}^{\wedge 2} \to \mathbb{R}$ (where here $\odot$ denotes the symmetric product) characterized by $R''((x \wedge y) \odot (z \wedge w)) = R(x, y, z, w)$. On the other hand, if we endow $\mathbb{V}$ with an inner product g, we have an induced inner product on $\mathbb{V}^{\wedge 2}$ (also denoted by g), which acts on 2-blades via

$$g(x \wedge y, z \wedge w) = \det \begin{pmatrix} g(x,z) & g(x,w) \\ g(y,z) & g(y,w) \end{pmatrix} = g(x,z)g(y,w) - g(x,w)g(y,z).$$

This induced inner product is exactly $-(g \owedge g)''$. It follows that $\widetilde{g \owedge g} = -\mathrm{Id}$. Therefore, $\owedge$ is in a certain sense the identity map (up to sign, depending on the conventions adopted).

With this "mini-factory" of examples that $\owedge$ provides us in mind, let us move on and establish some algebraic properties of curvature-type tensors.

Proposition (P.3.2). If $\dim(\mathbb{V}) = n$, then $12 \cdot \dim(\mathscr{R}(\mathbb{V})) = n^2(n^2-1)$.

Proof: Let $R \in \mathscr{R}(\mathbb{V})$. Since

$$R_{ijk\ell} = R_{k\ell ij},$$

then denoting $p = (i,j), q = (k,\ell)$, we can view R as a symmetric matrix $(R_{pq})_{1 \le p,q \le n}$ where the indices p, q range over an antisymmetric matrix, in view of the fact that

$$R_{jik\ell} = -R_{ijk\ell};$$
$$R_{ij\ell k} = -R_{ijk\ell}.$$

Since the dimension of the subspace $\left(\text{of } \mathbb{R}^{n^2}\right)$ of antisymmetric $n \times n$ matrices (in other words, the number of independent components of an antisymmetric $n \times n$ matrix) is

$$\frac{n(n-1)}{2},$$

and the subspace of symmetric $n \times n$ matrices has dimension

$$\frac{n(n+1)}{2},$$

it follows that $(R_{pq})_{1\leq p,q\leq n}$ viewed as a symmetric matrix has

$$\frac{m(m+1)}{2} \text{ independent components, where } m = \frac{n(n-1)}{2}.$$

Therefore, without taking into account the symmetry of the Bianchi identity, we have

$$\frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}+1\right)}{2} = \frac{n^4-2n^3+3n^2-2n}{8}$$

independent components to be considered. In the general case, since there are $\binom{n}{4}$ ways of choosing 4 distinct indices and the Bianchi identity eliminates one of these, the number of independent components is given by

$$\frac{n^4-2n^3+3n^2-2n}{8} - \frac{n!}{(n-4)!\cdot 24} = \frac{n^2(n^2-1)}{12}.$$

■

Remark (O.67). We then have that in dimension 1, $\mathscr{R}(\mathbb{V}) = \{0\}$ (this is also obvious from the antisymmetries that a curvature-type tensor must satisfy, which in dimension 1 force it to vanish). In other words, it is not possible to distinguish two curves by purely geometric properties (all of them are locally isometric to $\mathbb{R}$). In dimension 2 we have $\dim(\mathscr{R}(\mathbb{V})) = 1$, so that $\mathscr{R}(\mathbb{V}) = \mathbb{R}(g \owedge g)$, which implies that $\mathrm{Rm} = Kg \owedge g$ for some constant $K \in \mathbb{R}$. Taking an orthonormal basis of $\mathbb{V}$, say $\{e_1, e_2\}$, we see that $K = \mathrm{Rm}(e_1, e_2, e_1, e_2)$, which is the sectional curvature of the plane spanned by e_1 and e_2, that is, the Gaussian curvature at p (once we fix a surface S, a point $p \in S$ and set $\mathbb{V} = T_pS$).

Our goal now will be to decompose (in any dimension) the space of curvature-type tensors as something of the form

$$\mathscr{R}(\mathbb{V}) = \mathbb{R}(g \owedge g) \ \oplus \ (\text{some extra factors}).$$

In order to describe these extra factors, we will need to recall the notion of *contraction*, and focus in particular on the contraction known as the *Ricci contraction,* to be defined shortly.

We can analyze what all the possible contractions of a curvature-type tensor $R \in \mathscr{R}(\mathbb{V})$ are. Since R is antisymmetric both in the first and in the last pair, it follows that $\mathrm{tr}_{1,2}(R) = \mathrm{tr}_{3,4}(R) = 0$. Moreover, applying symmetry **(a)** of Definition **(D.3.1)** we see that $\mathrm{tr}_{2,4}(R) = \mathrm{tr}_{1,3}(R)$. Finally, we also have $\mathrm{tr}_{2,3}(R) = -\mathrm{tr}_{1,3}(R) = \mathrm{tr}_{1,4}(R)$. In other words, $\mathrm{tr}_{1,4}(R)$ is (up to sign) the only nontrivial contraction of R.

Definition (D.3.4). Let $R \in \mathscr{R}(\mathbb{V})$. The *Ricci contraction* of R is the tensor $\mathrm{Ric}_{\mathbb{V}}(R) \in \mathscr{T}^0_2(\mathbb{V})$ given by $\mathrm{Ric}_{\mathbb{V}}(R) \doteq \mathrm{tr}_{1,4}(R)$. The *scalar curvature* of R is the number $\mathrm{SC}(R) \doteq \mathrm{tr}_{1,2}(\mathrm{Ric}_{\mathbb{V}}(R))$.

Remark (O.68).

- Let $\mathcal{M}$ be a Riemannian manifold. Extending $\mathrm{Ric}_{\mathbb{V}}$ to all of $\mathscr{R}(T\mathcal{M})$ in the obvious way, it is clear that $\mathrm{Ric}_{\mathbb{V}}(\mathrm{Rm}) = \mathrm{Ric}$ and $\mathrm{SC}(\mathrm{Rm}) = \mathrm{Scal}$, which are the Ricci tensor and the scalar curvature, respectively.

- Since $R(z,x,y,w) = R(x,z,w,y) = R(w,y,x,z)$, the symmetry of g allows us to take the trace with respect to the pair (z,w) and obtain $\mathrm{Ric}_{\mathbb{V}}(R)(x,y) = \mathrm{Ric}_{\mathbb{V}}(R)(y,x)$, so that $\mathrm{Ric}_{\mathbb{V}}(R) \in \mathcal{S}(\mathbb{V})$. In coordinates, $(\mathrm{Ric}_{\mathbb{V}}(R))_{ij} = \sum_{1\leq \ell,m\leq n} g^{\ell m} R_{\ell ijm}$ and $\mathrm{SC}(R) = \sum_{1\leq i,j\leq n} g^{ij}(\mathrm{Ric}_{\mathbb{V}}(R))_{ij}$.

Example (E.3.1). Consider the fundamental curvature $g \owedge g$, given in terms of a basis by the expression $(g \owedge g)_{ijk\ell} = g_{jk}g_{i\ell} - g_{ik}g_{j\ell}$. The Ricci contraction is then given by

$$[\mathrm{Ric}_{\mathbb{V}}(g \owedge g)]_{jk} = \sum_{1\leq i,\ell\leq n} \left(g^{i\ell}g_{jk}g_{i\ell} - g^{i\ell}g_{ik}g_{j\ell}\right) = ng_{jk} - \sum_{1\leq \ell\leq n} \delta^{\ell}_{k} g_{j\ell} = (n-1)g_{jk},$$

and therefore

$$\mathrm{Ric}_{\mathbb{V}}(g \owedge g) = (n-1)g.$$

Next, we also have

$$\mathrm{SC}(g \owedge g) = \sum_{1\leq i,j\leq n} g^{ij}(n-1)g_{ij} = n(n-1).$$

And for a general product, we have the following:

Proposition (P.3.3). Let $T, S \in \mathcal{S}(\mathbb{V})$, and consider $T \owedge S \in \mathscr{R}(\mathbb{V})$. Then:

$$2\cdot[\mathrm{Ric}_{\mathbb{V}}(T \owedge S)](x,y) = \mathrm{tr}_{1,2}(S)T(x,y) - \mathrm{tr}_{1,4}(T\otimes S)(y,x) + \mathrm{tr}_{1,2}(T)S(x,y) - \mathrm{tr}_{1,4}(S\otimes T)(y,x).$$

Proof: A direct computation gives us

$$\begin{aligned}
2\cdot[\mathrm{Ric}_{\mathbb{V}}(T \owedge S)]_{jk} &= \sum_{1\leq i,\ell\leq n} \left(g^{i\ell}T_{jk}S_{i\ell} - g^{i\ell}T_{ik}S_{j\ell} + g^{i\ell}S_{jk}T_{i\ell} - g^{i\ell}S_{ik}T_{j\ell}\right) \\
&= \left[\sum_{1\leq i,\ell\leq n} (g^{i\ell}S_{i\ell})\right]\cdot T_{jk} + \sum_{1\leq i,\ell\leq n}\left[-g^{i\ell}(T\otimes S)_{ikj\ell} + (g^{i\ell}T_{i\ell})S_{jk} - g^{i\ell}(S\otimes T)_{ikj\ell}\right] \\
&= \mathrm{tr}_{1,2}(S)T_{jk} - \mathrm{tr}_{1,4}(T\otimes S)_{kj} + \mathrm{tr}_{1,2}(T)S_{jk} - \mathrm{tr}_{1,4}(S\otimes T)_{kj}.
\end{aligned}$$

■

As a consequence, we have the following important particular case:

Corollary (C.3.1). *Let $T \in \mathcal{S}(\mathbb{V})$. Then*

$$2\cdot\mathrm{Ric}_{\mathbb{V}}(T \owedge g) = (n-2)T + \mathrm{tr}_{1,2}(T)g,$$

and

$$\mathrm{SC}(T \owedge g) = (n-1)\,\mathrm{tr}_{1,2}(T).$$

Proof: As in the previous proof, we have

$$\begin{aligned}
2\cdot[\mathrm{Ric}_{\mathbb{V}}(T \owedge g)]_{jk} &= \sum_{1\leq i,\ell\leq n} \left(g^{i\ell}T_{jk}g_{i\ell} - g^{i\ell}T_{ik}g_{j\ell} + g^{i\ell}g_{jk}T_{i\ell} - g^{i\ell}g_{ik}T_{j\ell}\right) \\
&= nT_{jk} - \left(\sum_{1\leq i\leq n}\delta^{i}_{j}T_{ik}\right) + \mathrm{tr}_{1,2}(T)g_{jk} - \left(\sum_{1\leq \ell\leq n}\delta^{\ell}_{k}T_{j\ell}\right)
\end{aligned}$$

$$= (n-2)T_{jk} + \mathrm{tr}_{1,2}(T)g_{jk}.$$

And for the second part, it suffices to note that

$$2 \cdot \mathrm{SC}(T \owedge g) = (n-2)\,\mathrm{tr}_{1,2}(T) + \mathrm{tr}_{1,2}(T)n = (2n-2)\,\mathrm{tr}_{1,2}(T).$$

■

Remark (O.69). Just as the space $\mathbb{R}(g \owedge g)$ appeared earlier, let us also fix the notation

$$g \owedge \mathcal{S}(\mathbb{V}) \doteq \{g \owedge T \in \mathscr{R}(\mathbb{V}) \mid T \in \mathcal{S}(\mathbb{V})\},$$

and

$$g \owedge \mathcal{S}(\mathbb{V})_0 \doteq \{g \owedge T \in \mathscr{R}(\mathbb{V}) \mid T \in \mathcal{S}(\mathbb{V}) \text{ satisfies } \mathrm{tr}_{1,2}(T) = 0\}.$$

Corollary (C.3.2). *If $n \geq 3$, the maps $g \owedge _ : \mathcal{S}(\mathbb{V}) \to g \owedge \mathcal{S}(\mathbb{V})$ and $\mathrm{Ric}_{\mathbb{V}} : g \owedge \mathcal{S}(\mathbb{V}) \to \mathcal{S}(\mathbb{V})$ are isomorphisms.*

Proof: We will first verify that the composition $(\mathrm{Ric}_{\mathbb{V}} \circ (g \owedge _)) : \mathcal{S}(\mathbb{V}) \to \mathcal{S}(\mathbb{V})$ is an isomorphism. For this, it suffices to verify that it is injective. Suppose then that $T \in \mathcal{S}(\mathbb{V})$ is a tensor satisfying $\mathrm{Ric}_{\mathbb{V}}(g \owedge T) = 0$. Then $(n-2)T + \mathrm{tr}_{1,2}(T)g = 0$. Taking the trace, we obtain $(2n-2)\,\mathrm{tr}_{1,2}(T) = 0$, so that $\mathrm{tr}_{1,2}(T) = 0$ and $(n-2)T = 0$. Hence $T = 0$, as desired. It follows that both $g_$ and $\mathrm{Ric}_{\mathbb{V}}$ are injective. But $g \owedge _$ is surjective by definition, and therefore it is also an isomorphism. Since $g \owedge _$ and the composition with $\mathrm{Ric}_{\mathbb{V}}$ are isomorphisms, it follows that $\mathrm{Ric}_{\mathbb{V}}$ is an isomorphism as well.

■

Before moving on, we recall that given any endomorphism $A \in \mathrm{End}(\mathbb{V})$, the *trace-free part* of A is defined as the endomorphism A_0 given by

$$A_0 = A - \frac{\mathrm{tr}(A)}{n}\mathrm{Id}_{\mathbb{V}}.$$

Keeping the same notation as in Example (E.1.7), we can convert this into a relation between bilinear maps through the relation

$$\alpha_0 = \alpha - \frac{\mathrm{tr}_{1,2}(\alpha)}{n}g,$$

so that $\mathrm{tr}(A_0) = \mathrm{tr}_{1,2}(\alpha_0) = 0$. This idea will allow us to construct simpler objects associated with a curvature-type tensor $R \in \mathscr{R}(\mathbb{V})$. For example, we have the

Definition (D.3.5). Let $R \in \mathscr{R}(\mathbb{V})$. The *Einstein tensor* of R is the tensor $E(R) \in \mathcal{S}(\mathbb{V})$ defined by

$$E(R) = \mathrm{Ric}_{\mathbb{V}}(R) - \frac{\mathrm{SC}(R)}{n}g.$$

Remark (O.70). As suggested by its name, the tensor E was born in physics. In the words of Carl Sagan, the Cosmos is all that is or ever was or ever will be; and Einstein's theory of general relativity models the Cosmos as a pseudo-Riemannian manifold of dimension 4 (with 3 spatial dimensions and 1

temporal one), whose gravitational effects can be explained by the curvature of this manifold, which is commonly called *spacetime*. Explicitly, as seen in [21], in this model gravity cannot be regarded as a force in the conventional sense: in fact, it is an intrinsic property of spacetime and "arises" as a result of its local curvature, which in turn reacts to the distribution of mass-energy, momentum and stress of a spatial region via the following equation:

$$\mathrm{Ric} - \frac{1}{2} \cdot \mathrm{Scal} \cdot g + \Lambda g = \frac{8\pi G}{c^4} T, \tag{3.1}$$

where T denotes the energy-momentum tensor of spacetime and the constants Λ, G and c denote, respectively, the *cosmological constant* of the model, the Newtonian gravitational constant, and the speed of light in vacuum. In fact, the convention of denoting the Riemannian metric by g originated with Einstein and Grossman around 1910-13, who noticed that the components $\{g_{\mu\nu}\}_{1\leq\mu,\nu\leq 4}$ that determine the geometry of spacetime seemed to depend on the amount of gravitating matter in the region under study. And as also seen in [21], there is a precise sense in which the quantities $\{g_{\mu\nu}\}_{1\leq\mu,\nu\leq 4}$ can be viewed as gravitational potentials. Usually, equation (3.1) is presented in local coordinates, in the form

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} \cdot \mathrm{Scal} \cdot g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

which altogether constitute a system of 10 partial differential equations (where the 10 comes from the fact that the tensors involved have 10 independent components, as seen in remark (O.73)). Einstein originally introduced the constant Λ in 1917, aiming to counterbalance the effect of gravity and to model a static universe, which was the most accepted notion at the time. After Edwin Hubble's confirmation that the universe was expanding, Einstein removed this constant, writing equation (3.1) as

$$\mathrm{Ric} - \frac{1}{2} \mathrm{Scal}\, g = \frac{8\pi G}{c^4} T.$$

In the particular case $T \equiv 0$ the equation reduces to *Einstein's vacuum field equation* and we obtain (after taking the trace of both sides) $\mathrm{Scal} \equiv 0$, $\mathrm{Ric} \equiv 0$ and $E \equiv 0$. A Riemannian manifold satisfying $E \equiv 0$ (an equality motivated, roughly speaking, by the idea "curvature = gravity") is called an *Einstein manifold*. It is easy to see that this definition coincides with the usual definition of Einstein manifolds found in the literature.

Example (E.3.2). For the fundamental curvature of $(\mathbb{V}, g)$, we have

$$E(g \owedge g) = \mathrm{Ric}_{\mathbb{V}}(g \owedge g) - \frac{\mathrm{SC}(g \owedge g)}{n} g = (n-1)g - \frac{n(n-1)}{n} g = 0.$$

In what follows, in the same way that we constructed the trace-free part of $\mathrm{Ric}_{\mathbb{V}}(R)$, we would like to construct the Ricci-trace-free part of R. Thus, to motivate the next definition, we will look for a tensor $W(R)$ in the kernel of $\mathrm{Ric}_{\mathbb{V}}$ of the particular form

$$W(R) = R - \lambda(g \owedge h(R)).$$

It will also be convenient to look for $h(R)$ of the form $h(R) = \mathrm{Ric}_{\mathbb{V}}(R) - \mu g$ for some $\mu \in \mathbb{R}$. Our goal then is to find λ and μ that carry out this construction. Taking the trace in the expression for $W(R)$,

we obtain $0 = \mathrm{Ric}_{\mathbb{V}}(W(R)) = \mathrm{Ric}_{\mathbb{V}}(R) - \lambda\,\mathrm{Ric}_{\mathbb{V}}(g \owedge h(R))$, and therefore

$$\begin{aligned}\mathrm{Ric}_{\mathbb{V}}(R) = \lambda\,\mathrm{Ric}_{\mathbb{V}}(g \owedge h(R)) &= \frac{\lambda}{2}((n-2)h(R) + \mathrm{tr}_{1,2}(h(R))g) \\ &= \frac{\lambda}{2}((n-2)(\mathrm{Ric}_{\mathbb{V}}(R) - \mu g) + (\mathrm{SC}(R) - n\mu)g) \\ &= \frac{\lambda}{2}((n-2)\,\mathrm{Ric}_{\mathbb{V}}(R) + (\mathrm{SC}(R) - 2(n-1)\mu)g).\end{aligned}$$

We then see that by defining

$$\mu = \frac{\mathrm{SC}(R)}{2n-2},$$

and

$$\lambda = \frac{2}{n-2}.$$

we achieve the desired purpose.

Definition (D.3.6). Let $R \in \mathscr{R}(\mathbb{V})$.

(a) The *Schouten tensor* of R is $h(R) \in \mathcal{S}(\mathbb{V})$ given by

$$h(R) = \mathrm{Ric}_{\mathbb{V}}(R) - \frac{\mathrm{SC}(R)}{2(n-1)}g.$$

(b) The *Weyl tensor* of R is $W(R) \in \mathscr{R}(\mathbb{V})$ given (when $n > 2$) by

$$W(R) = R - \frac{2}{n-2}(g \owedge h(R)).$$

We will also define $\mathscr{W}(\mathbb{V}) = \{W \in \mathscr{R}(\mathbb{V}) \mid \mathrm{Ric}_{\mathbb{V}}(W) = 0\}$.

Remark (O.71). By its construction, the Weyl tensor has the same symmetries as the Riemannian curvature tensor, so that:

$$\begin{aligned}&W_{ijk\ell} = -W_{jik\ell} = -W_{ijlk} = W_{ji\ell k} = W_{k\ell ij}; \\ &W_{ijk\ell} + W_{ik\ell j} + W_{i\ell jk} = 0.\end{aligned}$$

Moreover, all of its traces vanish, hence

$$\sum_{1\le i,k\le n} g^{j\ell}W_{ijk\ell} = 0,$$

for any fixed $1 \le i, k \le n$, from which it follows that

$$W_{1111} + W_{1212} + W_{1313} = 0,$$

and therefore

$$W_{1212} = -W_{1313} = W_{2323} = -W_{2121} = -W_{1212} \implies W_{1212} = 0.$$

We also have

$$W_{1213} + W_{2223} + W_{3233} = 0 \implies W_{1213} = 0,$$

thus showing that $W_{ijk\ell} = 0$ whenever any two indices are equal. Since in dimension 3 there are only 3 possible choices for each index, $W \equiv 0$ on a three-dimensional manifold. By Proposition **(P.3.4)**, it follows that in dimension ≤ 3, Ric completely determines Rm.

Remark (O.72).

- It is common to denote the Weyl tensor of a Riemannian manifold $(\mathcal{M}^n, g)$ by C instead of W. As mentioned in [15], this is justified by the fact that W is a conformal invariant of the metric and determines whether $\mathcal{M}$ is (locally) conformally flat: in dimension $n \geq 4$, a necessary and sufficient condition for $\mathcal{M}$ to be (locally) conformally flat is that W vanish.
- When $n \geq 3$, we have $W = \mathrm{Rm}$ if and only if $\mathrm{Ric} = 0$. Indeed, if $\mathrm{Ric} = 0$, it is clear that $\mathrm{Scal} = 0$ and $h = 0$, so that $W = \mathrm{Rm}$. Conversely, if $W = \mathrm{Rm}$, Corollary **(C.3.2)** guarantees that $h = 0$, so we have $\mathrm{tr}_{1,2}(h) = 0$ and therefore $\mathrm{Scal} = 0$, guaranteeing (by the definition of h) that $\mathrm{Ric} = 0$.

Example (E.3.3). Take $K \in \mathbb{R}$ and consider $R = K\, g \owedge g$. We already saw in Example **(E.3.1)** that in this case $\mathrm{Ric}_{\mathbb{V}}(R) = (n-1)Kg$, and therefore $\mathrm{SC}(R) = n(n-1)K$. Hence, we have

$$h(R) = (n-1)Kg - \frac{n(n-1)K}{2(n-1)}g = \frac{n-2}{2}Kg,$$

and also

$$W = Kg \owedge g - \frac{2}{n-2}g \owedge \left(\frac{n-2}{2}Kg\right) = Kg \owedge g - Kg \owedge g = 0.$$

Example (E.3.4). Given $T \in \mathcal{S}(\mathbb{V})$, consider $R = g \owedge T$. We already saw in Corollary **(C.3.1)** that, in this case,

$$\mathrm{Ric}_{\mathbb{V}}(R) = \frac{1}{2}((n-2)T + \mathrm{tr}_{1,2}(T)g),$$

and

$$\mathrm{SC}(R) = (n-1)\,\mathrm{tr}_{1,2}(T).$$

Therefore, it follows that

$$h(R) = \frac{1}{2}((n-2)T + \mathrm{tr}_{1,2}(T)g) - \frac{(n-1)\,\mathrm{tr}_{1,2}(T)}{2(n-1)}g = \frac{n-2}{2}\,T,$$

and

$$W(R) = g \owedge T - \frac{2}{n-2}g \owedge \left(\frac{n-2}{2}\,T\right) = g \owedge T - g \owedge T = 0,$$

as well as

$$E(R) = \left(\frac{n-2}{2}\right)\left(T - \frac{g}{n}\right).$$

The two previous examples are particular cases of the following:

Proposition (P.3.4). *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold of dimension n. Then $\mathscr{R}(T\mathcal{M})$ admits the following orthogonal decomposition:*

$$\mathscr{R}(T\mathcal{M}) = \mathbb{R}(g \owedge g) \;\oplus\; (g \owedge \mathcal{S}(T\mathcal{M})_0) \;\oplus\; \mathscr{W}(T\mathcal{M}).$$

A fortiori, the explicit decomposition of the curvature tensor of $\mathcal{M}$ is given by:

$$\mathrm{Rm} = \frac{\mathrm{Scal}}{n(n-1)}(g \owedge g) + \frac{2}{n-2}(g \owedge E) + W.$$

Proof: By construction, we have

$$\begin{aligned} R &= \frac{2}{n-2} g \owedge h + W \\ &= \frac{2}{n-2} g \owedge \left(\mathrm{Ric}_{\mathbb{V}} - \frac{\mathrm{SC}}{2(n-1)g} \right) + W \\ &= \frac{2}{n-2} g \owedge \left(E + \frac{\mathrm{SC}}{n} g - \frac{\mathrm{SC}}{2(n-1)g} \right) + W \\ &= \frac{2}{n-2} g \owedge \left(E + \frac{(n-2)\,\mathrm{SC}}{2n(n-1)} g \right) + W \\ &= \frac{\mathrm{SC}}{n(n-1)} g \owedge g + \frac{2}{n-2} g \owedge E + W. \end{aligned}$$

By Corollary (C.3.2), we also have

$$\mathbb{R}(g \owedge g) \cap (g \owedge \mathcal{S}(T_p\mathcal{M})_0) = \{0\} \ \forall p \in \mathcal{M};$$

and

$$(g \cap \mathcal{S}(T_p\mathcal{M})) \cap \mathscr{W}(T_p\mathcal{M}) = \{0\} \ \forall p \in \mathcal{M};$$

so that the sums are direct. Orthogonality follows easily by doing the computation in an orthonormal frame using the fact that E and W are trace-free.

■

Remark (O.73). Equivalently,

$$\mathscr{R}(\mathbb{V}) = (g \wedge \mathcal{S}^2(\mathbb{V})) \oplus \mathscr{W}(\mathbb{V}),$$

and therefore

$$\frac{n^2(n^2-1)}{12} = \frac{n(n+1)}{2} + \dim \mathscr{W}(\mathbb{V}).$$

Hence,

$$\dim \mathscr{W}(\mathbb{V}) = \frac{n(n+1)(n+2)(n-3)}{12}.$$

In particular, in dimension 4 the 20 independent components of the curvature tensor come from 10 independent components of the Ricci tensor and 10 of the Weyl tensor.

Remark (O.74). As we have already proved, in dimension $n = 3$ we have $W = 0$, so that Ric completely determines Rm. As seen in [12], this is a crucial fact for the proof of the Poincaré conjecture.

Remark (O.75). An immediate consequence of Proposition (P.3.4) is that locally conformally flat Einstein manifolds (or, equivalently, those locally conformal to a manifold of constant sectional curvature) have constant sectional curvature. For an alternative proof of this fact, one may consult [29].

3.1 Some important particularities of dimension 4

Proposition (P.3.5). *Let $(\mathcal{M}^4, g)$ be an oriented Riemannian manifold of dimension 4. Then the bundle $\Lambda^2(\mathcal{M})$ of 2-forms on $\mathcal{M}$ admits the following conformally invariant decomposition into $\star$-invariant eigenbundles:*

$$\Lambda^2(\mathcal{M}) = \Lambda^2_+(\mathcal{M}) \oplus \Lambda^2_-(\mathcal{M}),$$

where $\Lambda^2_\pm(\mathcal{M}) = \{\omega \in \Lambda^2(\mathcal{M}) \mid \star\,\omega = \pm\omega\}$.

Proof: It is easy to verify that in dimension 4 we have $\star^2 = \star \circ \star = \mathrm{Id}$. Therefore an eigenvalue of $\star$ is necessarily ± 1. Given $\omega \in \Lambda^2(\mathcal{M})$, let us now define the 2-forms:

$$\omega^+ = \frac{1}{2}(\omega + \star\omega) \in \Lambda^2_+(\mathcal{M}),$$

and

$$\omega^- = \frac{1}{2}(\omega - \star\omega) \in \Lambda^2_-(\mathcal{M}).$$

It is clear that $\omega = \omega^+ + \omega^-$. This decomposition is conformally invariant, since if g and $\tilde{g}$ are two conformal metrics on $\mathcal{M}$, we see by a direct computation using Proposition **(P.4.15)** that $\star_g = \star_{\tilde{g}}$. Finally, it is obvious that $\Lambda^2_+(\mathcal{M}) \cap \Lambda^2_-(\mathcal{M}) = \{0\}$, from which the desired result follows.

■

Remark (O.76). Elements of $\Lambda^2_+(\mathcal{M})$ are called self-dual, while elements of $\Lambda^2_-(\mathcal{M})$ are called anti-self-dual.

We will now see how to identify tensors of type $(0,4)$ with self-adjoint operators on $\Lambda^2(\mathcal{M})$. Let $(\mathcal{M}^4, g)$ be an oriented Riemannian manifold of dimension 4. Let us fix a point $p \in \mathcal{M}$, an open set $U \ni p$ of $\mathcal{M}$ and a local orthonormal frame $\{\mathbf{e}_i \in \Gamma(TU)\}_{1\leq i\leq 4}$ on U, with $\{\mathbf{e}^i \in \Gamma(T^*U)\}_{1\leq i\leq 4}$ its associated coframe of 1-forms, determined by $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$ for each $1 \leq i, j \leq 4$. We define:

$$\begin{aligned}
\omega_1^\pm &= \mathbf{e}^1 \wedge \mathbf{e}^2 \pm \mathbf{e}^3 \wedge \mathbf{e}^4;\\
\omega_2^\pm &= \mathbf{e}^1 \wedge \mathbf{e}^3 \pm \mathbf{e}^4 \wedge \mathbf{e}^2;\\
\omega_3^\pm &= \mathbf{e}^1 \wedge \mathbf{e}^4 \pm \mathbf{e}^2 \wedge \mathbf{e}^3.
\end{aligned}$$

Note that $*\omega_i^\pm = \pm\omega_i^+$, and $\frac{1}{\sqrt{2}}\{\omega_i^\pm\}_{1\leq i\leq 4}$ is a basis of $\Lambda^2_\pm(U)$. For a 2-form ω, we can then write

$$\begin{aligned}
\omega|_U = \sum_{i<j} \omega_{ij}\mathbf{e}^i \wedge \mathbf{e}^j &= \sum_{i<j} \frac{(w_{ij} - w_{ji})}{2}\ \mathbf{e}^i \wedge \mathbf{e}^j\\
&= \frac{1}{2}\sum_{i<j} \omega_{ij}\mathbf{e}^i \wedge \mathbf{e}^j - \frac{1}{2}\sum_{j<i} \omega_{ij}\mathbf{e}^j \wedge \mathbf{e}^i\\
&= \frac{1}{2}\sum_{1\leq i,j\leq 4} \omega_{ij}\mathbf{e}^i \wedge \mathbf{e}^j.
\end{aligned}$$

Noting also that:

$$\mathbf{e}^p \wedge \mathbf{e}^q = \sum_{1\leq i,j\leq n} \frac{1}{2}(\mathbf{e}^p \wedge \mathbf{e}^q)_{ij}\mathbf{e}^i \wedge \mathbf{e}^j = \sum_{1\leq i,j\leq n} \frac{1}{2}\delta_{ij}^{pq}\mathbf{e}^i \wedge \mathbf{e}^j,$$

we see that the components of $\mathbf{e}^p \wedge \mathbf{e}^q$ are given by $(\mathbf{e}^p \wedge \mathbf{e}^q)_{ij} = \delta_{ij}^{pq}$, which is the generalized Kronecker delta (defined as 1 if $(p,q) = (i,j)$, -1 if $(p,q) = (j,i)$ and 0 otherwise).

Let now P be a $(0,4)$ tensor on $\mathcal{M}^4$ of curvature type. We can define an operator

$$\widetilde{P} : \Lambda^2(\mathcal{M}) \times \Lambda^2(\mathcal{M}) \to \mathscr{C}^\infty(\mathcal{M}),$$

by requiring that locally $\widetilde{P}\left(\mathbf{e}^i \wedge \mathbf{e}^j, \mathbf{e}^k \wedge \mathbf{e}^\ell\right) = P_{ij\ell k}$ and (by multilinearity) determining it on all of $\Lambda^2(\mathcal{M}) \times \Lambda^2(\mathcal{M})$ using this relation (or, equivalently, requiring that for any $x, y, v, w \in T_p\mathcal{M}$ we have $\widetilde{P}(x \wedge y, v \wedge w) = P(x, y, w, v)$, with $p \in \mathcal{M}$ arbitrary). The relation:

$$g(\widehat{P}(x \wedge y), v \wedge w) = \widetilde{P}(x \wedge y, v \wedge w)\ \forall x, y, v, w \in T_p\mathcal{M},\ \forall p \in \mathcal{M};$$

then determines an operator $\widehat{P} : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M})$. Indeed $\widehat{P}$ is well defined (that is, $P(\omega) \in \Lambda^2(\mathcal{M})$ for each $\omega \in \Lambda^2(\mathcal{M})$), since:

$$\begin{aligned} g(\widehat{P}(y \wedge x), v \wedge w) = \widetilde{P}(y \wedge x, v \wedge w) &= -\widetilde{P}(x \wedge y, v \wedge w) \\ &= -g(\widehat{P}(x \wedge y), v \wedge w), \end{aligned}$$

for all $x, y, v, w \in T_p\mathcal{M}$. From now on we will identify P with its respective associated operators, committing the (reasonable) abuse of notation of writing $P = \widehat{P} = \widetilde{P}$. In local coordinates, given $\omega \in \Lambda^2(\mathcal{M})$, we have:

$$P(\omega) = \sum_{1\leq i,j\leq n} \frac{1}{2}[P\omega]_{ij}\mathbf{e}^i \wedge \mathbf{e}^j.$$

Note that by definition:

$$g(P(\omega), \mathbf{e}^p \wedge \mathbf{e}^q) = \sum_{1\leq i,j\leq n} (P\omega)_{ij}\delta_{pq}^{ij},$$

while on the other hand:

$$\begin{aligned} g(P(\omega), \mathbf{e}^p \wedge \mathbf{e}^q) &= g\left(\sum_{1\leq k,\ell\leq n} P\left(\frac{1}{2}\omega_{k\ell}\mathbf{e}^k \wedge \mathbf{e}^\ell\right), \mathbf{e}^p \wedge \mathbf{e}^q\right) \\ &= \frac{1}{2}\sum_{1\leq k,\ell\leq n} \omega_{k\ell}P_{k\ell qp}. \end{aligned}$$

Setting $(p,q) = (i,j)$ we then see that

$$\frac{1}{2}\sum_{1\leq k,\ell\leq n} \omega_{k\ell}P_{k\ell ji} = \frac{1}{2}\sum_{1\leq k,\ell\leq n} \omega_{k\ell}P_{ij\ell k} = [P(\omega)]_{ij},$$

and therefore

$$P(\omega) = \frac{1}{2}\sum_{1\leq i,j\leq n} [P(\omega)]_{ij}\mathbf{e}^i \wedge \mathbf{e}^j = \frac{1}{4}\sum_{1\leq i,j,k,\ell\leq n} P_{ij\ell k}\omega_{k\ell}\mathbf{e}^i \wedge \mathbf{e}^j.$$

The relation above completely determines $P : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M})$. Although here we started with a $(0,4)$ tensor of curvature type and obtained such an operator on $\Lambda^2(\mathcal{M})$, it is clear that we could have gone the other way around: that is, start with an operator $P : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M})$ and obtain a $(0,4)$ tensor determined by the relation:

$$P_{pqrs} = g(P(\mathbf{e}^p \wedge \mathbf{e}^q), \mathbf{e}^s \wedge \mathbf{e}^r).$$

Evidently this construction forces P to be self-adjoint. Indeed, this follows from the fact that decomposable 2-forms constitute a basis of $\Lambda^2(\mathcal{M})$ and that for any $x, y, v, w \in T_p\mathcal{M}$ we have:

$$\begin{aligned} g(x \wedge y, P(v \wedge w)) &= P(v \wedge w, x \wedge y) \\ &= P(v, w, y, x) \\ &= P(x, y, w, v) \\ &= g(P(x \wedge y), v \wedge w). \end{aligned}$$

With these identifications in mind, we now define the *Weyl curvature operator*, $\mathcal{W} : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M})$ by

$$[\mathcal{W}(\omega)]_{ij} = \frac{1}{2} \sum_{1 \le k,\ell \le n} W_{ij\ell k}\, \omega_{k\ell}.$$

The operators associated with each factor of the decomposition of Rm seen earlier are defined in an entirely analogous way. We define the projections $\pi_\pm : \Lambda^2(\mathcal{M}) \to \Lambda^2_\pm(\mathcal{M})$ onto the $\star$-invariant eigenbundles of $\Lambda^2(\mathcal{M})$ by

$$\pi_\pm = \frac{1}{2} \left(\mathrm{Id} \pm \star\right).$$

Thus, given $P \in \Lambda^2(\mathcal{M})$, we have the decomposition $P = P^+ + P^-$, where $P^\pm = \pi_\pm(P) \in \Lambda^2_\pm(\mathcal{M})$. For example, the operator $\mathcal{W}^+$ corresponds to a curvature-type tensor W^+, determined by its components defined by the following relation:

$$\begin{aligned} W^+_{pqrs} &= g\left(\mathcal{W}^+(\mathbf{e}^p \wedge \mathbf{e}^q), \mathbf{e}^s \wedge \mathbf{e}^r\right) \\ &= \frac{1}{2} g\left(W^+(\mathbf{e}^p \wedge \mathbf{e}^q + \star(\mathbf{e}^p \wedge \mathbf{e}^q)), \mathbf{e}^s \wedge \mathbf{e}^r\right). \end{aligned}$$

Explicitly, we have that

$$\begin{aligned} W^+_{1234} &= \frac{1}{2} g\left(\mathcal{W}(\mathbf{e}^1 \wedge \mathbf{e}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4), \mathbf{e}^4 \wedge \mathbf{e}^3\right) \\ &= \frac{1}{2}(W_{1234} + W_{4343}). \end{aligned}$$

The factors W^+ and W^- of the decomposition

$$W = W^+ + W^-$$

are called the self-dual and anti-self-dual components of the Weyl curvature, respectively. Therefore, in dimension 4, we also have the following orthogonal decomposition of the curvature tensor:

$$\mathrm{Rm} = W^+ + W^- + g \owedge E + \frac{\mathrm{Scal}}{12}(g \owedge g).$$

The *traceless Ricci curvature operator*, which we will denote by $\mathcal{E}$, is then defined as the self-adjoint operator on $\Lambda^2(\mathcal{M})$ associated with the curvature-type tensor $g \owedge E$, and the *scalar curvature operator* $\mathcal{S}$ is the operator associated with $\mathrm{Scal} \cdot (g \owedge g)$.

We are now ready to establish the following proposition, stated in [54] (as Proposition 26.1) but proved there in less detail than we will present here. The decomposition given by Proposition **(P.3.6)** will be one of the cornerstones of the proofs of Theorems **(T.3.1)** and **(T.3.2)**.

Proposition (P.3.6). *The Weyl curvature operator preserves the type of forms, id est,*

$$\mathcal{W} : \Lambda^2_\pm(\mathcal{M}) \to \Lambda^2_\pm(\mathcal{M}),$$

and the trace-free Ricci curvature operator reverses types,

$$\mathcal{E} : \Lambda^2_\pm(\mathcal{M}) \to \Lambda^2_\mp(\mathcal{M}).$$

Explicitly, $\mathcal{W}(\omega) \in \Lambda^2_\pm(\mathcal{M})$ whenever $\omega \in \Lambda^2_\pm(\mathcal{M})$, and $\mathcal{E}(\omega) \in \Lambda^2_\mp(\mathcal{M})$ whenever $\omega \in \Lambda^2_\pm(\mathcal{M})$. Moreover, the scalar curvature operator satisfies

$$\mathcal{S} = \mathrm{Scal} \cdot \mathrm{Id}\,.$$

Finally, the block decomposition of the curvature operator corresponding to the decomposition of $\Lambda^2(\mathcal{M})$ via the Hodge star $\star$ is given by

$$\mathrm{Rm} = \left(\begin{array}{c|c} \left(\mathcal{W}^+ + \dfrac{\mathcal{S}}{12}\right)\Big|_{\Lambda^2_+(\mathcal{M})} & \mathcal{E}|_{\Lambda^2_-(\mathcal{M})} \\ \hline \mathcal{E}|_{\Lambda^2_+(\mathcal{M})} & \left(\mathcal{W}^- + \dfrac{\mathcal{S}}{12}\right)\Big|_{\Lambda^2_-(\mathcal{M})} \end{array}\right).$$

Proof: For the first part, it suffices to prove that

$$(\pi_- \circ \mathcal{W})(\omega_i^+) = (\pi_+ \circ \mathcal{W})(\omega_i^-) = 0,$$

for each $1 \leq i \leq 3$. And since $\{\omega_i^\pm\}_{1\leq i\leq 3}$ is an orthogonal basis of $\Lambda^2(\mathcal{M})$ and $\mathcal{W}$ is self-adjoint, it is then enough to show the following sequence of equalities:

$$0 = \underbrace{g(\mathcal{W}(w_1^+), \omega_1^-)}_{=g(\mathcal{W}(\omega_1^-),\ \omega_1^+)} = \underbrace{g(\mathcal{W}(\omega_1^+), \omega_2^-)}_{=g(\mathcal{W}(\omega_2^-),\ \omega_1^+)} = \underbrace{g(\mathcal{W}(\omega_1^+), \omega_3^-)}_{=g(\mathcal{W}(\omega_3^-),\ \omega_1^+)};$$
$$0 = \underbrace{g(\mathcal{W}(w_2^+), \omega_1^-)}_{=g(\mathcal{W}(\omega_1^-),\ \omega_2^+)} = \underbrace{g(\mathcal{W}(w_2^+), \omega_2^-)}_{=g(\mathcal{W}(\omega_2^-),\ \omega_2^+)} = \underbrace{g(\mathcal{W}(w_2^+), \omega_3^-)}_{=g(\mathcal{W}(\omega_3^-),\ \omega_2^+)};$$

$$0 = \underbrace{g(\mathcal{W}(w_3^+), \omega_1^-)}_{=g(\mathcal{W}(\omega_1^-),\ \omega_3^+)} = \underbrace{g(\mathcal{W}(w_3^+), \omega_2^-)}_{=g(\mathcal{W}(\omega_2^-),\ \omega_3^+)} = \underbrace{g(\mathcal{W}(w_3^+), \omega_3^-)}_{=g(\mathcal{W}(\omega_3^-),\ \omega_3^+)}.$$

Indeed, we have:

$$\begin{aligned}[\mathcal{W}(\omega_1^+)]_{ij} &= -\frac{1}{2}\sum_{1\le k,\ell\le 4} W_{ijk\ell}(\delta^{12}_{k\ell} + \delta^{34}_{k\ell}) \\ &= -(W_{ij12} + W_{ij34}),\end{aligned}$$

hence,

$$\begin{aligned}-\frac{1}{2}g(\mathcal{W}(\omega_1^+), \omega_1^-) &= -\frac{1}{2}\sum_{1\le i,j\le 4} [\mathcal{W}(\omega_1^+)]_{ij}(\omega_1^-)_{ij} \\ &= \frac{1}{2}\sum_{1\le i,j\le 4} (W_{ij12} + W_{ij34})(\delta^{12}_{ij} - \delta^{34}_{ij}) \\ &= W_{1212} - W_{3412} + W_{1234} - W_{3434} \\ &= W_{1212} - W_{3434}.\end{aligned}$$

Recalling that $\mathrm{tr}(W) = 0$ and that the trace is independent of the basis, we have that

$$\sum_{1\le \ell\le 4} W_{i\ell j\ell} = \sum_{1\le \ell\le 4} W_{\ell i\ell j} = 0,$$

for any fixed $1 \le i, j \le 4$. And therefore:

$$\begin{aligned}W_{1212} + W_{1313} + W_{1414} &= 0; \\ W_{1212} + W_{3232} + W_{4242} &= 0.\end{aligned}$$

Adding the two equations above, we see that:

$$2\,W_{1212} = -W_{1313} - W_{1414} - W_{3232} - W_{4242}.$$

We also have:

$$\begin{aligned}W_{1414} + W_{2424} + W_{3434} &= 0; \\ W_{3131} + W_{3232} + W_{3434} &= 0,\end{aligned}$$

from which it follows that

$$2\,W_{3434} = -W_{1414} - W_{2424} - W_{3131} - W_{3232} = 2W_{1212},$$

and therefore

$$g(\mathcal{W}(w_1^+), \omega_1^-) = 0.$$

Next, we also have:

$$\begin{aligned}-\frac{1}{2}g(\mathcal{W}(\omega_1^+),\omega_1^-) &= \frac{1}{2}\sum_{1\leq i,j\leq 4}(W_{ij12}+W_{ij34})(\delta_{ij}^{13}-\delta_{ij}^{42})\\ &= W_{1312}-W_{4212}+W_{1334}-W_{4234}\\ &= -W_{1231}-W_{4212}-W_{4313}-W_{4234},\end{aligned}$$

but recalling again that $\mathrm{tr}(W)=0$, we see that:

$$\begin{aligned}W_{4212}+W_{4313}&=0;\\ W_{1231}+W_{4234}&=0,\end{aligned}$$

and therefore

$$g(\mathcal{W}(w_1^+),\omega_2^-)=0.$$

Once more,

$$\begin{aligned}-\frac{1}{2}g(\mathcal{W}(\omega_1^+),\omega_3^-) &= \frac{1}{2}\sum_{1\leq i,j\leq 4}(W_{ij12}+W_{ij34})(\delta_{ij}^{14}-\delta_{ij}^{23})\\ &= W_{1412}+W_{1434}-W_{2312}-W_{2334},\end{aligned}$$

but again, we have

$$\begin{aligned}W_{1434}+W_{1232}=W_{1434}-W_{2312}&=0;\\ W_{4323}+W_{4121}=W_{1412}-W_{2334}&=0,\end{aligned}$$

so that

$$g(\mathcal{W}(w_1^+),\omega_3^-)=0,$$

and therefore the first row of desired equalities is proved. The other equalities follow in an entirely analogous way, but for completeness we will prove all of them. We have:

$$\begin{aligned}[\mathcal{W}(\omega_1^-)]_{ij} &= -\frac{1}{2}\sum_{1\leq k,\ell\leq 4}W_{ijk\ell}(\delta_{k\ell}^{12}-\delta_{k\ell}^{34})\\ &= W_{ij34}-W_{ij12},\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{2}g(\mathcal{W}(\omega_1^-),\omega_2^+) &= \frac{1}{2}\sum_{1\leq i,j\leq 4}(W_{ij34}-W_{ij12})(\delta_{ij}^{13}+\delta_{ij}^{42})\\ &= W_{1334}-W_{1312}+W_{4234}-W_{4212},\end{aligned}$$

but since

$$W_{4212}+W_{4313}=W_{4212}-W_{1334}=0;$$

$$W_{3424} + W_{3121} = -W_{4234} + W_{1312} = 0,$$

it follows that

$$g(\mathcal{W}(\omega_1^-), \omega_2^+) = 0.$$

Analogously, from

$$\begin{aligned}\frac{1}{2} g(\mathcal{W}(\omega_1^-), \omega_3^+) &= \frac{1}{2} \sum_{1 \le i,j \le 4} (W_{ij34} - W_{ij12})(\delta_{ij}^{14} + \delta_{ij}^{23}) \\ &= W_{1434} - W_{1412} + W_{2334} - W_{2312},\end{aligned}$$

and from

$$\begin{aligned}W_{4121} + W_{4323} = W_{1412} - W_{2334} = 0; \\ W_{1434} + W_{1232} = -W_{1434} + W_{2312} = 0,\end{aligned}$$

it follows that

$$g(\mathcal{W}(\omega_1^-), \omega_3^+) = 0.$$

Since we also have

$$\begin{aligned}[\mathcal{W}(\omega_2^-)]_{ij} &= -\frac{1}{2} \sum_{1 \le k,\ell \le 4} W_{ijk\ell}(\delta_{k\ell}^{13} - \delta_{k\ell}^{42}) \\ &= W_{ij42} - W_{ij13},\end{aligned}$$

we then have

$$\begin{aligned}\frac{1}{2} g(\mathcal{W}(\omega_2^-), \omega_2^+) &= \frac{1}{2} \sum_{1 \le i,j \le 4} (W_{ij42} - W_{ij13})(\delta_{ij}^{13} + \delta_{ij}^{42}) \\ &= W_{1342} - W_{1313} + W_{4242} - W_{4213} \\ &= W_{4242} - W_{1313}.\end{aligned}$$

But from

$$\begin{aligned}W_{4242} + W_{4343} + W_{4141} = 0; \\ W_{4242} + W_{3232} + W_{1212} = 0; \\ \implies 2\, W_{4242} = -W_{4343} - W_{4141} - W_{3232} - W_{1212},\end{aligned}$$

and from

$$\begin{aligned}W_{1313} + W_{1212} + W_{1414} = 0; \\ W_{1313} + W_{2323} + W_{4343} = 0; \\ \implies 2\, W_{1313} = -W_{1212} - W_{1414} - W_{2323} - W_{4343} = 2\, W_{4242},\end{aligned}$$

it follows that

$$g(\mathcal{W}(\omega_2^-), \omega_2^+) = 0.$$

Once more, we have

$$\begin{aligned}\frac{1}{2}g(\mathcal{W}(\omega_2^-), \omega_3^+) &= \frac{1}{2}\sum_{1\leq i,j\leq 4}(W_{ij42} - W_{ij13})(\delta_{ij}^{14} + \delta_{ij}^{23}) \\ &= W_{1442} - W_{1413} + W_{2342} - W_{2313}.\end{aligned}$$

And from

$$\begin{aligned}W_{4142} + W_{3132} &= -W_{1442} + W_{2313} = 0; \\ W_{1413} + W_{2423} &= W_{1413} - W_{2342} = 0,\end{aligned}$$

it follows that

$$g(\mathcal{W}(\omega_2^-), \omega_3^+) = 0.$$

We also have:

$$\begin{aligned}[\mathcal{W}(\omega_3^-)]_{ij} &= -\frac{1}{2}\sum_{1\leq k,\ell\leq 4} W_{ijk\ell}(\delta_{k\ell}^{14} - \delta_{k\ell}^{23}) \\ &= W_{ij23} - W_{ij14},\end{aligned}$$

hence,

$$\begin{aligned}\frac{1}{2}g(\mathcal{W}(\omega_3^-), \omega_2^+) &= \frac{1}{2}\sum_{1\leq i,j\leq 4}(W_{ij23} - W_{ij14})(\delta_{ij}^{13} + \delta_{ij}^{42}) \\ &= W_{1323} - W_{1314} + W_{4223} - W_{4214}.\end{aligned}$$

But since

$$\begin{aligned}W_{1323} + W_{1424} &= W_{1323} - W_{4214} = 0; \\ W_{1314} + W_{2324} &= W_{1314} - W_{4223} = 0,\end{aligned}$$

we obtain

$$g(\mathcal{W}(\omega_3^-), \omega_2^+) = 0.$$

Finally, note that

$$\begin{aligned}\frac{1}{2}g(\mathcal{W}(\omega_3^-), \omega_3^+) &= \frac{1}{2}\sum_{1\leq i,j\leq 4}(W_{ij23} - W_{ij14})(\delta_{ij}^{14} + \delta_{ij}^{23}) \\ &= W_{1423} - W_{1414} + W_{2323} - W_{2314},\end{aligned}$$

and since

$$\begin{aligned} W_{1414} + W_{1313} + W_{1212} &= 0; \\ W_{1414} + W_{2424} + W_{3434} &= 0; \\ \implies 2\,W_{1414} &= -W_{1313} - W_{1212} - W_{2424} - W_{3434}, \end{aligned}$$

and also

$$\begin{aligned} W_{2323} + W_{2424} + W_{2121} &= 0; \\ W_{2323} + W_{1313} + W_{4343} &= 0; \\ \implies 2\,W_{2323} &= -W_{2424} - W_{2121} - W_{1313} - W_{4343} = 2\,W_{1414}, \end{aligned}$$

we have

$$g(\mathcal{W}(\omega_3^-), \omega_3^+) = 0,$$

thus proving that $\mathcal{W}$ preserves types. For the second part, we may without loss of generality assume that $\{\mathbf{e}_i\}_{1\leq i\leq 4}$ is an orthonormal basis that diagonalizes E (since at each fixed point p, E is a symmetric bilinear form on $T_p\mathcal{M}$). Thus, we have $E_{ij} = g(E(\mathbf{e}_i), \mathbf{e}_j) = \lambda_i\delta_{ij}$ for each $1 \leq i, j \leq 4$, and it follows that:

$$\begin{aligned} 4[\mathcal{E}(\omega)]_{ij} &= \sum_{k,\ell} 2(g \owedge E)_{ij\ell k}\omega_{k\ell} \\ &= -\sum_{k,\ell}(E_{i\ell}g_{jk} - E_{j\ell}g_{ik} - E_{ik}g_{j\ell} + E_{jk}g_{i\ell})\omega_{k\ell} \\ &= -\sum_{k,\ell}(\lambda_i\delta_{i\ell}\delta_{jk} - \lambda_j\delta_{j\ell}\delta_{ik} - \lambda_i\delta_{ik}\delta_{j\ell} + \lambda_j\delta_{jk}\delta_{i\ell})\omega_{k\ell} \\ &= -(\lambda_i\omega_{ji} - \lambda_j\omega_{ij} - \lambda_i\omega_{ij} + \lambda_j\omega_{ji}) \\ &= 2(\lambda_i + \lambda_j)\omega_{ij}. \end{aligned}$$

A fortiori,

$$\begin{aligned} 2[\mathcal{E}(\omega_1^+)]_{ij} &= (\lambda_i + \lambda_j)(\delta_{ij}^{12} + \delta_{ij}^{34}) \\ &= (\lambda_1 + \lambda_2)\delta_{ij}^{12} + (\lambda_3 + \lambda_4)\delta_{ij}^{34}. \end{aligned}$$

But since $\mathrm{tr}(E) = 0$, it follows that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. Therefore,

$$[\mathcal{E}(\omega_1^+)]_{ij} = \frac{1}{2}(\lambda_1 + \lambda_2)(\delta_{ij}^{12} - \delta_{ij}^{34}) = \frac{1}{2}(\lambda_1 + \lambda_2)(\omega_1^-)_{ij}.$$

Equivalently,

$$\mathcal{E}(\omega_1^+) = \frac{1}{2}(\lambda_1 + \lambda_2)\omega_1^- \in \Lambda^2_-(\mathcal{M}).$$

In an entirely analogous way, we see that

$$\mathcal{E}(w_2^+) = \frac{1}{2}(\lambda_1 + \lambda_3)\omega_2^- \in \Lambda^2_-(\mathcal{M});$$

$$\mathcal{E}(w_3^+) = \frac{1}{2}(\lambda_1+\lambda_4)\omega_3^- \in \Lambda^2_-(\mathcal{M});$$
$$\mathcal{E}(w_1^-) = \frac{1}{2}(\lambda_1+\lambda_2)\omega_1^+ \in \Lambda^2_+(\mathcal{M});$$
$$\mathcal{E}(w_2^-) = \frac{1}{2}(\lambda_1+\lambda_3)\omega_2^+ \in \Lambda^2_+(\mathcal{M});$$
$$\mathcal{E}(w_3^-) = \frac{1}{2}(\lambda_1+\lambda_4)\omega_3^+ \in \Lambda^2_+(\mathcal{M}).$$

In short, defining $c_i \doteq \frac{1}{2}(\lambda_1+\lambda_{i+1})$, we have

$$\mathcal{E}(\omega_i^\pm) = c_i\,\omega_i^\mp \in \Lambda^2_\mp(\mathcal{M}), \text{ for every } 1\leq i\leq 3,$$

which proves that $\mathcal{E}(\omega)\in\Lambda^2_\mp(\mathcal{M})$ whenever $\omega\in\Lambda^2_\pm(\mathcal{M})$, as desired. Lastly, note that

$$\begin{aligned}-4[(g\owedge g)(\omega)]_{ij} &= \sum_{k,\ell} 2(g\owedge g)_{ijk\ell}\omega_{k\ell}\\ &= \sum_{k,\ell} -(\delta_{ik}\delta_{j\ell}-\delta_{jk}\delta_{i\ell}-\delta_{i\ell}\delta_{jk}+\delta_{j\ell}\delta_{ik})\omega_{k\ell}\\ &= -4\cdot\omega_{ij},\end{aligned}$$

hence,

$$\mathcal{S} = \mathrm{Scal}\cdot(g\owedge g) = \mathrm{Scal}\cdot\mathrm{Id}\,.$$

It then follows from everything done so far that:

$$\mathrm{Rm}(\omega_i^+) = \overbrace{\mathcal{W}^+(\omega_i^+)}^{\in\Lambda^2_+(\mathcal{M})} + \overbrace{\mathcal{W}^-(\omega_i^+)}^{=0} + \overbrace{\mathcal{E}(\omega_i^+)}^{\in\Lambda^2_-(\mathcal{M})} + \overbrace{\frac{\mathrm{Scal}}{12}\cdot\omega_i^+}^{\in\Lambda^2_+(\mathcal{M})} = \underbrace{\left(\mathcal{W}^+(\omega_i^+)+\frac{\mathrm{Scal}}{12}\cdot\omega_i^+\ ,\ \mathcal{E}(\omega_i^+)\right)}_{\text{in the basis } \{\omega_i^\pm\}_{1\leq i\leq 3}};$$

$$\mathrm{Rm}(\omega_i^-) = \overbrace{\mathcal{W}^+(\omega_i^-)}^{=0} + \overbrace{\mathcal{W}^-(\omega_i^-)}^{\in\Lambda^2_-(\mathcal{M})} + \overbrace{\mathcal{E}(\omega_i^-)}^{\in\Lambda^2_+(\mathcal{M})} + \overbrace{\frac{\mathrm{Scal}}{12}(\omega_i^-)}^{\in\Lambda^2_-(\mathcal{M})} = \underbrace{\left(\mathcal{E}(\omega_i^-)\ ,\ \mathcal{W}^-(\omega_i^-)+\frac{\mathrm{Scal}}{12}\cdot\omega_i^-\right)}_{\text{in the basis } \{\omega_i^\pm\}1\leq i\leq 3}.$$

Equivalently, the matrix representation of Rm is given by:

$$\begin{aligned}\mathrm{Rm} &= \begin{pmatrix}\mathrm{Rm}(\omega_1^+) & \mathrm{Rm}(\omega_2^+) & \mathrm{Rm}(\omega_3^+) & \mathrm{Rm}(\omega_1^-) & \mathrm{Rm}(\omega_2^-) & \mathrm{Rm}(\omega_3^-)\end{pmatrix}\\ &= \left(\begin{array}{c|c} \left[\left(\mathcal{W}^+ + \frac{\mathcal{S}}{12}\right)\Big|_{\Lambda^2_+(\mathcal{M})}(\omega_i^+)\right]_{1\leq i\leq 3} & \left[\mathcal{E}|_{\Lambda^2_-(\mathcal{M})}(\omega_i^-)\right]_{1\leq i\leq 3}\\ \hline \left[\mathcal{E}|_{\Lambda^2_+(\mathcal{M})}(\omega_i^+)\right]_{1\leq i\leq 3} & \left[\left(\mathcal{W}^- + \frac{\mathcal{S}}{12}\right)\Big|_{\Lambda^2_-(\mathcal{M})}(\omega_i^-)\right]_{1\leq i\leq 3}\end{array}\right).\end{aligned}$$

Note that each matrix in the block representation above is 3 by 3, making up a total of 36 entries, which agrees with the fact that $\dim[\mathrm{End}(\Lambda^2(\mathcal{M}))] = (\dim[\Lambda^2(\mathcal{M})])^2 = 36$.

■

Remark (O.77). Writing

$$\mathrm{Rm} = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

and recalling that the matrix representation in an orthonormal basis of the adjoint of an operator is given by its transpose, we have, since Rm is self-adjoint, that:

$$\mathrm{Rm} = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} = \begin{pmatrix} A & B^T \\ C^T & D \end{pmatrix},$$

and therefore A and D are symmetric and $B = C^*$, the adjoint of C. And since we have already seen that the correspondence

$$\mathscr{T}^0_2(\mathcal{M}) \ni h \mapsto h \owedge g \in \mathscr{T}^0_4(T\mathcal{M})$$

is injective, a reasonable abuse of notation is to write

$$\mathrm{Rm} = \left(\begin{array}{c|c} \mathcal{W}^+ + \dfrac{\mathcal{S}}{12} & E \\ \hline E^* & \mathcal{W}^- + \dfrac{\mathcal{S}}{12} \end{array}\right),$$

as is done in [6].

Note also that the self-dual and anti-self-dual components of the Weyl tensor must also be self-adjoint and trace-free. Indeed, we have

$$g\left(\mathcal{W}^\pm(\omega_i^\pm), \omega_j^\pm\right) = g\left(\mathcal{W}(\omega_i^\pm), \omega_j^\pm\right) = g\left(\mathcal{W}(\omega_j^\pm), \omega_i^\pm\right) = g\left(\mathcal{W}^\pm(\omega_j^\pm), \omega_i^\pm\right),$$

for each $1 \leq i \leq 3$. Moreover, denoting by $\{\beta_i^\pm\}$ the eigenvectors of $\mathcal{W}^\pm$ (which are obviously eigenvectors of $\mathcal{W}$ as well, since $\mathcal{W}^\pm$ preserves types) and by $\{\alpha_i^\pm\}_{1\leq i\leq 3}$ the corresponding eigenvalues, we have by construction

$$\begin{aligned} 0 = 2 \cdot \mathcal{W}^-(\beta_i^+) &= \mathcal{W}(\beta_i^+) - (\star \circ \mathcal{W})(\beta_i^+), \\ \implies \mathcal{W}(\beta_i^+) &= (\star \circ \mathcal{W})(\beta_i^+), \end{aligned}$$

and therefore β_i^+ is also an eigenvector of $\star \circ \mathcal{W}$. Moreover, from

$$\begin{aligned} 0 = 2 \cdot \mathcal{W}^+(\beta_i^-) &= \mathcal{W}(\beta_i^-) - (\star \circ \mathcal{W})(\beta_i^-), \\ \implies \mathcal{W}(\beta_i^-) &= (\star \circ \mathcal{W})(\beta_i^-), \end{aligned}$$

it follows that β_i^- is also an eigenvector of $\star \circ \mathcal{W}$. Then, since

$$\begin{aligned} \star\,\mathcal{W} &= 2\mathcal{W}^+ - \mathcal{W}; \\ \star\,\mathcal{W} &= \mathcal{W} - 2\mathcal{W}^-, \end{aligned}$$

it follows that

$$0 = \operatorname{tr}(\mathcal{W}) = \operatorname{tr}(\star \circ \mathcal{W}) = 2\operatorname{tr}(\mathcal{W}^+) = -2\operatorname{tr}(\mathcal{W}^-).$$

We then see that

$$0 = \sum_{1\leq i\leq 3} \alpha_i^+ = \operatorname{tr}(\mathcal{W}^+) = \operatorname{tr}(\mathcal{W}^-) = \sum_{1\leq i\leq 3} \alpha_i^-,$$

as claimed.

3.2 Auxiliary results

Remark (O.78). Several of the constants in our estimates differ by a multiplicative factor from the constants in [6]. This happens due to different conventions in the definition of the Kulkarni-Nomizu product. While our definition (see (D.3.3)) adopts a normalization factor 2^{-1} inherited from the polarization formula, it is common in the literature not to make this normalization, as is done in [6].

Let $(\mathcal{M}^4, g)$ be an oriented Riemannian manifold. As we already saw in the previous chapter, we can view the Weyl tensor W as a self-adjoint operator $\mathcal{W} \in \operatorname{End}(\Lambda^2(\mathcal{M}))$ of the bundle $\Lambda^2(\mathcal{M}) = \Lambda^2_+(\mathcal{M}) + \Lambda^2_-(\mathcal{M})$, which decomposes as

$$\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^-,$$

and then fix a point $p \in \mathcal{M}^4$ and diagonalize $\mathcal{W}^\pm$ so that $\alpha_i^\pm$, with $1 \leq i \leq 3$, are its respective eigenvalues. Assuming without loss of generality that they are ordered, we have already seen that these eigenvalues satisfy:

$$\alpha_1^\pm \leq \alpha_2^\pm \leq \alpha_3^\pm \text{ and } \sum_{1\leq i\leq 3} \alpha_i^- = \sum_{1\leq i\leq 3} \alpha_i^+ = 0.$$

Proposition (P.3.7). *We have the following inequalities for $\mathcal{W}$:*

- $$\frac{3}{2}(\alpha_3^+)^2 \leq \|\mathcal{W}^+\|^2, \text{ and}$$
- $$\det \mathcal{W}^+ \leq \frac{1}{6}(\alpha_3^+)\|\mathcal{W}^+\|^2 \leq \frac{\sqrt{6}}{18}\|\mathcal{W}^+\|^3.$$

Moreover, equality holds if and only if $\alpha_1^+ = \alpha_2^+$.

Proof: First, we have by construction that

$$\begin{aligned}\|\mathcal{W}^+\|^2 &= (\alpha_1^+)^2 + (\alpha_2^+)^2 + (\alpha_3^+)^2 \\ &= \frac{1}{2}(\underbrace{\alpha_1^+ + \alpha_2^+}_{=-\alpha_3^+})^2 + \frac{1}{2}(\alpha_1^+ - \alpha_2^+)^2 + (\alpha_3^+)^2 \\ &\geq \frac{1}{2}(\alpha_3^+)^2 + (\alpha_3^+)^2 = \frac{3}{2}(\alpha_3^+)^2,\end{aligned}$$

where equality holds if and only if $\alpha_1^+ - \alpha_2^+ = 0$, as claimed. Lastly, we compute that

$$\begin{aligned}(\alpha_3^+)\|\mathcal{W}^+\|^2 - 6\det\mathcal{W}^+ &= (\alpha_3^+)\left[(\alpha_3^+)^2 + (\alpha_2^+)^2 + \underbrace{(\alpha_1^+)^2}_{=(-\alpha_2^+-\alpha_3^+)^2}\right] - 6\cdot\alpha_1^+\alpha_2^+\alpha_3^+\\ &= 2(\alpha_3^+)\left[(\alpha_3^+)^2 + (\alpha_2^+)^2 + (\alpha_3^+)(\alpha_2^+)\right] - 6\cdot\alpha_1^+\alpha_2^+\alpha_3^+\\ &= 2(\alpha_3^+)^3 + \underbrace{2(\alpha_3^+)(\alpha_2^+)^2 + 2\cdot(\alpha_3^+)^2(\alpha_2^+)}_{=-2\cdot\alpha_1^+\alpha_2^+\alpha_3^+} - 6\cdot\alpha_1^+\alpha_2^+\alpha_3^+\\ &= 2(\alpha_3^+)\underbrace{\left[(\alpha_3^+)^2 - 4\cdot\alpha_1^+\alpha_2^+\right]}_{=(\alpha_1^+-\alpha_2^+)^2}\\ &= 2(\alpha_3^+)(\alpha_1^+ - \alpha_2^+)^2,\end{aligned}$$

from which it follows that

$$\det\mathcal{W}^+ \leq \frac{1}{6}(\alpha_3^+)\|\mathcal{W}^+\|^2.$$

Finally, from the first part of the proof, it follows that

$$\frac{\sqrt{3}}{\sqrt{2}}\alpha_3^+ \leq \frac{\sqrt{3}}{\sqrt{2}}|\alpha_3^+| \leq \|\mathcal{W}^+\|,$$

and therefore

$$\frac{1}{6}\cdot\alpha_3^+\|\mathcal{W}^+\|^2 = \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{2}}\cdot\alpha_3^+\cdot\frac{1}{6}\|\mathcal{W}^+\|^2 \leq \frac{\sqrt{2}}{6\sqrt{3}}\|\mathcal{W}^+\|^3 = \frac{\sqrt{6}}{18}\|\mathcal{W}^+\|^3,$$

as desired. The equality case is again obvious.

■

Next, we will present an estimate for $(E \owedge E)^+$ on oriented Riemannian 4-manifolds. Recall that the Einstein tensor E of $\left(\mathcal{M}^4, g\right)$ is given by

$$E = \mathrm{Ric} - \frac{\mathrm{Scal}}{4}\,g$$

Proposition (P.3.8). *Let $(\mathcal{M}^4, g)$ be an oriented Riemannian 4-manifold. Then we have:*

(a) $\|(E \owedge E)\|^2 \leq \frac{3}{2}\|E\|^4$. *Moreover, equality holds if and only if*

$$4\cdot\|E^2\|^2 = \|E\|^4,$$

where $(E^2)_{ik} = \sum_{1\leq p\leq 4} E_{ip}E_{kp}$.

(b) $\|(E \owedge E)^+\|^2 \leq \frac{3}{2}\|E\|^4$. *Moreover, equality holds if and only if* $\|(E \owedge E)^-\| = 0$.

Proof: We will do all the computations in an orthonormal frame that diagonalizes E (so that $E_{ij} = \lambda_i \delta_{ij}$). First, we have by the definition of the Kulkarni-Nomizu product that:

$$\begin{aligned} -2(E \owedge E)_{ijk\ell} &= 2(E_{ik}E_{j\ell} + E_{j\ell}E_{ik} - E_{i\ell}E_{jk} - E_{jk}E_{i\ell}) \\ &= 2(E_{ik}E_{j\ell} - E_{i\ell}E_{jk}) \\ &= 2\lambda_i\lambda_j(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}). \end{aligned}$$

Now, by Proposition **(P.3.1)**, we have that $(E \owedge E)_{ijk\ell} = 0$ whenever $i = j$ or $k = \ell$. Therefore:

$$\begin{aligned} \|E \owedge E\|^2 &= \sum_{i\neq j, k\neq \ell} [\lambda_i\lambda_j(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})]^2 \\ &= \sum_{i\neq j, k\neq \ell} \lambda_i^2\lambda_j^2(\delta_{ik}\delta_{j\ell} - 2\cdot \underbrace{\delta_{ik}\delta_{j\ell}\delta_{i\ell}\delta_{jk}}_{=0 \text{ if } k=i \text{ or } k\neq i} + \delta_{i\ell}\delta_{jk}) \\ &= 2\sum_{i\neq j}\lambda_i^2\lambda_j^2 = 2\sum_{1\leq i\leq 4}\left(\sum_{j\neq i}\lambda_i^2\lambda_j^2\right) \\ &= 2\sum_{1\leq i\leq 4}\lambda_i^2\left(\sum_{j\neq i}\lambda_j^2\right) = 2\sum_{1\leq i\leq 4}\lambda_i^2\left[\left(\sum_{1\leq j\leq 4}\lambda_j^2\right) - \lambda_i^2\right] \\ &= 2\left[\left(\sum_{1\leq i\leq 4}\lambda_i^2\right)^2 - \sum_{1\leq i\leq 4}\lambda_i^4\right] \\ &\overset{(*)}{\leq} 2\left[\left(\sum_{1\leq i\leq 4}\lambda_i^2\right)^2 - \frac{1}{4}\left(\sum_{1\leq i\leq 4}\lambda_i^2\right)^2\right] = \frac{3}{2}\|E\|^4, \end{aligned}$$

where in $(*)$ we used Cauchy-Schwarz, and equality holds if and only if

$$\|E^2\|^2 = \sum_{1\leq i\leq 4}\lambda_i^4 = \frac{1}{4}\left(\sum_{1\leq i\leq 4}\lambda_i^2\right)^2 = \frac{1}{4}\|E\|^4,$$

as claimed. Finally, since the decomposition into self-dual and anti-self-dual components is orthogonal by construction, we have

$$\|(E \owedge E)^{\pm}\|^2 \leq \|E \owedge E\|^2 \leq \frac{3}{2}\|E\|^4,$$

and consequently, equality holds if and only if

$$\frac{3}{2}\|E\|^4 = \|(E \owedge E)\|^2 = \|(E \owedge E)^+\|^2 + \|(E \owedge E)^-\|^2 \leq \frac{3}{2}\|E\|^4 + \|(E \owedge E)^-\|^2,$$

or, equivalently, if and only if

$$\|(E \owedge E)^-\| = 0,$$

as claimed.

■

Remark (O.79). It is clear that by replacing the self-dual components with the anti-self-dual ones in **(P.3.7)** and **(P.3.8)**, we will have entirely analogous results (that is, it will suffice to swap $+$ for $-$).

It will be enlightening to analyze the example of the shrinking gradient Ricci soliton on $\mathbb{S}^2 \times \mathbb{R}^2$.

Example (E.3.5). We will denote by $\mathcal{M} = (\mathbb{S}^2(\sqrt{2}) \times \mathbb{R}^2, g, f)$ the gradient Ricci soliton on the round cylinder $\mathbb{S}^2(\sqrt{2}) \times \mathbb{R}^2$, where g denotes the usual product metric and the potential function f is given by

$$f(x,y) = \frac{\|y\|^2}{4} + 1, \text{ for each } (x,y) \in \mathbb{S}^2(\sqrt{2}) \times \mathbb{R}^2.$$

Since $\mathbb{S}^2(\sqrt{2})$ has constant sectional curvature $\frac{1}{2}$ $\left(\text{and therefore constant scalar curvature } 2 \cdot (2-1) \cdot \frac{1}{2} = 1\right)$ and $\mathbb{R}^2$ is flat, we see that $\mathbb{S}^2(\sqrt{2}) \times \mathbb{R}^2$ has constant scalar curvature $\mathrm{Scal} = 1$. We will fix a local orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ so that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is tangent to $\mathbb{S}^2(\sqrt{2})$ and $\{\mathbf{e}_3, \mathbf{e}_4\}$ is tangent to $\mathbb{R}^2$. In these coordinates, we then have that the matrix representations of the Ricci curvature and of the trace-free Ricci curvature are all diagonal and given by:

$$\mathrm{Ric} = \begin{pmatrix} \mathrm{Ric}(\mathbf{e}_1) & \mathrm{Ric}(\mathbf{e}_2) & \mathrm{Ric}(\mathbf{e}_3) & \mathrm{Ric}(\mathbf{e}_4) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$E = \mathrm{Ric} - \frac{1}{4}\,\mathrm{Id} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}. \tag{3.2}$$

Next, now again viewing Rm as an endomorphism of $\Lambda^2(\mathcal{M})$, we claim that its matrix representation is given by:

$$\begin{aligned} \mathrm{Rm} &= \begin{pmatrix} \mathrm{Rm}(\omega_1^+) & \mathrm{Rm}(\omega_2^+) & \mathrm{Rm}(\omega_3^+) & \mathrm{Rm}(\omega_1^-) & \mathrm{Rm}(\omega_2^-) & \mathrm{Rm}(\omega_3^-) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Indeed, this is a direct consequence of the expression for the curvature tensor of a product manifold. We have that

$$\left[\mathrm{Rm}(\mathbf{e}^a \wedge \mathbf{e}^b)\right]_{ij} = -\frac{1}{2}\sum_{k,\ell} \mathrm{Rm}_{ijk\ell}\delta^{ab}_{k\ell} = -\mathrm{Rm}_{ijab};$$

$$\implies \mathrm{Rm}(\mathbf{e}^1 \wedge \mathbf{e}^2) = \frac{1}{2}\,\mathrm{Rm}_{1221}\mathbf{e}^1 \wedge \mathbf{e}^2 = \frac{1}{4}\,\mathbf{e}^1 \wedge \mathbf{e}^2, \text{ and analogously}$$

$$\mathrm{Rm}(\mathbf{e}^1\wedge\mathbf{e}^3)=\mathrm{Rm}(\mathbf{e}^1\wedge\mathbf{e}^4)=\mathrm{Rm}(\mathbf{e}^2\wedge\mathbf{e}^3)=\mathrm{Rm}(\mathbf{e}^2\wedge\mathbf{e}^4)=\mathrm{Rm}(\mathbf{e}^3\wedge\mathbf{e}^4)=0,\ \textit{so that}$$
$$\mathrm{Rm}(\omega_1^+)=\mathrm{Rm}(\omega_1^-)=\frac{1}{4}\omega_1^+ + \frac{1}{4}\omega_1^- = \left(\frac{1}{4},0,0,\frac{1}{4},0,0\right),\ \textit{and}\ \mathrm{Rm}(\omega_i^\pm)=(0,0,0,0,0,0),\ \forall i\neq 1,$$

as claimed. Recalling Proposition **(P.3.6)**, we then see that the matrix representations of $\mathcal{W}^+$ and $\mathcal{W}^-$ must satisfy:

$$\mathcal{W}^+ + \frac{1}{12}\,\mathrm{Id} = \mathcal{W}^- + \frac{1}{12}\,\mathrm{Id} = \begin{pmatrix}\frac{1}{4} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

from which it follows that

$$\mathcal{W}^\pm = \begin{pmatrix}\frac{1}{6} & 0 & 0\\ 0 & -\frac{1}{12} & 0\\ 0 & 0 & -\frac{1}{12}\end{pmatrix}.$$

We now claim that $\mathbb{S}^2\times\mathbb{R}^2$ satisfies

$$g(E\owedge E,\mathcal{W}^+)=\frac{1}{48}.$$

Indeed, we have by the definition of the Kulkarni-Nomizu product that:

$$\begin{aligned}2(E\owedge E)_{ijk\ell} &= E_{jk}E_{i\ell}-E_{ik}E_{j\ell}+E_{jk}E_{i\ell}-E_{ik}E_{j\ell}\\ &= -2(E_{ik}E_{j\ell}-E_{i\ell}E_{jk}).\end{aligned}$$

In particular, denoting by $\{\lambda_i\}_{1\leq i\leq 4}$ the eigenvalues of E we have that

$$\begin{aligned}\|(E\owedge E)\|^2 &= \sum_{1\leq i,j,k,\ell\leq 4}[(E\owedge E)_{ijk\ell}]^2\\ &= \sum_{1\leq i,j,k,\ell\leq 4}[\lambda_i\lambda_j(\delta_{jk}\delta_{i\ell}-\delta_{ik}\delta_{j\ell})]^2\\ &= \sum_{1\leq i,j,k,\ell\leq 4}\lambda_i^2\lambda_j^2(\delta_{jk}\delta_{i\ell}-2\delta_{jk}\delta_{i\ell}\delta_{ik}\delta_{j\ell}+\delta_{ik}\delta_{j\ell})\\ &= 2\left[\sum_{1\leq k\leq 4}\lambda_k^2\left(\sum_{1\leq \ell\leq 4}\lambda_\ell^2\right)-\sum_{1\leq i\leq 4}\lambda_i^4\right]\\ &= 2\left[\sum_{1\leq k\leq 4}\lambda_k^2\left(4\cdot\frac{1}{16}\right)-4\cdot\frac{1}{256}\right]\\ &= \frac{2}{16}-\frac{2}{64}=\frac{3}{32}.\end{aligned}$$

For a 2-form ω, we then see that:

$$\begin{aligned}(E\owedge E)(\omega)_{ij} &= -\frac{1}{2}\sum_{k,\ell}(E\owedge E)_{ijk\ell}\omega_{k\ell}\\ &= -\frac{1}{2}\sum_{k,\ell}\lambda_i\lambda_j(\delta_{jk}\delta_{i\ell}-\delta_{ik}\delta_{j\ell})\omega_{k\ell}\\ &= -\frac{1}{2}\lambda_i\lambda_j(\omega_{ji}-\omega_{ij})=\lambda_i\lambda_j\omega_{ij}.\end{aligned}$$

In particular, recalling (3.2) we have

$$\begin{aligned}[(E \owedge E)(\omega_1^+)]_{ij} &= \lambda_i\lambda_j(\delta_{ij}^{12} + \delta_{ij}^{34}) \\ &= \lambda_1\lambda_2\delta_{ij}^{12} + \lambda_3\lambda_4\delta_{ij}^{34} \\ &= \frac{1}{16}(\delta_{ij}^{12} + \delta_{ij}^{34}) \\ &= \frac{1}{16}(\omega_1^+)_{ij},\end{aligned}$$

so that

$$(E \owedge E)(\omega_1^+) = \frac{1}{16}\omega_1^+.$$

In an entirely analogous way, we see that

$$(E \owedge E)(\omega_2^+) = -\frac{1}{16}\omega_2^+,$$

and

$$(E \owedge E)(\omega_3^+) = -\frac{1}{16}\omega_3^+.$$

Now, since by definition $(E \owedge E)^{\pm}(\omega) = (E \owedge E)(\omega^{\pm})$, the matrix representation of the endomorphism $(E \owedge E)^+$ is given by

$$(E \owedge E)^+ = \begin{pmatrix} \frac{1}{16} & 0 & 0 \\ 0 & -\frac{1}{16} & 0 \\ 0 & 0 & -\frac{1}{16} \end{pmatrix}.$$

Consequently,

$$g((E \owedge E)^+, \mathcal{W}^+) = \frac{1}{16}\cdot\frac{1}{6} + \frac{1}{16}\cdot\frac{1}{12} + \frac{1}{16}\cdot\frac{1}{12} = \frac{1}{48},$$

as claimed.

In the proofs of the following results we will make crucial use of other facts already proved previously in the literature, and therefore it is convenient to state them now. In [10], Chen proved that every complete shrinking gradient Ricci soliton has nonnegative scalar curvature (that is, $\mathrm{Scal} \geq 0$). And in [7], Cao and Zhou proved the following

Lemma (L.3.1). *Let $(\mathcal{M}^n, g, f)$ be a complete, shrinking and noncompact gradient Ricci soliton satisfying (2.7) with $2\lambda = 1$. Then there exist $r_0 > 0$, $p_0 \in \mathcal{M}$ and real constants $c_1, c_2 > 0$ (which depend only on n and on the geometry of g on the unit ball centered at p_0) such that the potential function f satisfies the estimates*

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2,$$

whenever $r(x) \geq r_0$, where $r \doteq \mathrm{dist}(p_0, \bullet)$ denotes the Riemannian distance function to p_0.

Remark (O.80). Due to the existence of the Gaussian shrinking soliton (see Example (E.2.7)), it is not possible to improve the estimate above.

Moreover, in [16], Derdziński proved that every oriented Einstein manifold of dimension 4 $(\mathcal{M}^4, g)$ satisfies the following Weitzenböck formula:

$$\Delta\|W^{\pm}\|^2 = 2\cdot\|W^{\pm}\|^2 + \mathrm{Scal}\cdot\|W^{\pm}\|^2 - 36\cdot\det W^{\pm}.$$

This formula is a powerful tool in the theory of canonical metrics on manifolds of dimension 4, which can be used to obtain classification results as well as to rule out potential new examples.

Next, we recall the following Weitzenböck-type formula for gradient Ricci solitons, which also appears in [9] and [56] and plays an essential role in the proof of Theorem **(T.3.1)**.

Proposition (P.3.9). *Let $(\mathcal{M}^4, g, f)$ be a shrinking gradient Ricci soliton of dimension 4. Then we have:*

$$\Delta_f\left(\|W^{\pm}\|^2\right) = 2\cdot\|\nabla W^{\pm}\|^2 + 2\cdot\|W^{\pm}\|^2 - 36\cdot\det W^{\pm} - 2\cdot g\left((E\owedge E), W^{\pm}\right).$$

3.3 Main results

We will now present the proofs of Theorems **(T.3.1)** and **(T.3.2)**, which guarantee geometric classifications and controls on the Ricci curvature or the Riemannian curvature, provided that pointwise estimates on the self-dual or anti-self-dual parts of the Weyl tensor, or a certain control on the scalar curvature in terms of the potential function of the soliton, are satisfied.

Remark (O.81). As mentioned in [6], an important and novel aspect of Theorem **(T.3.1)** is the absence of hypotheses containing pointwise estimates on the Ricci tensor or assuming some vanishing of the Weyl tensor and its derivatives.

Moreover, as observed in [6], we can estimate the value $g\left((E\owedge E)^{\pm}, W^{\pm}\right)$ so as to replace the condition on the right-hand side of the inequalities of Theorem **(T.3.1)** by estimates on the Einstein tensor E. Indeed, as an immediate consequence of Proposition **(P.3.8)** and of the Cauchy-Schwarz inequality, we have

$$g\left((E\owedge E)^{\pm}, W^{\pm}\right) \leq \sqrt{\frac{3}{2}}\,\|E\|^2\,\|W^{\pm}\|. \tag{3.3}$$

Note that estimate **(3.3)** is not optimal. Indeed, as seen in Example **(E.3.5)**, the soliton $\mathbb{S}^2\times\mathbb{R}^2$ satisfies

$$g\left((E\owedge E)^{+}, W^{+}\right) = \frac{1}{48} = \frac{\sqrt{6}}{6}\,\|E\|^2\,\|W^{+}\|.$$

In [8], Xiaodong Cao, Ernani Ribeiro, and Hung Tran proved the novel (and optimal, due to Example **(E.3.5)**) estimate:

$$g\left((E\owedge E)^{+}, W^{+}\right) \leq \frac{\sqrt{6}}{6}\,\|E\|^2\,\|W^{+}\|.$$

Theorem (T.3.1). *Let $(\mathcal{M}^4, g, f)$ be a complete shrinking gradient Ricci soliton of dimension 4 that satisfies either*

$$\|W^{+}\|^2 - \sqrt{6}\cdot\|W^{+}\|^3 \geq g\left((E\owedge E)^{+}, W^{+}\right),$$

or

$$\|W^{-}\|^2 - \sqrt{6}\cdot\|W^{-}\|^3 \geq g\left((E\owedge E)^{-}, W^{-}\right).$$

Then $(\mathcal{M}^4, g, f)$ is either

(a) Einstein, or

(b) a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^4$, or of $\mathbb{S}^3 \times \mathbb{R}$, or of $\mathbb{S}^2 \times \mathbb{R}^2$.

Proof: First, we have by Proposition **(P.3.9)** that

$$\Delta_f \left(\|W^\pm\|^2 \right) = 2 \cdot \|\nabla W^\pm\|^2 + 2 \cdot \|W^\pm\|^2 - 36 \cdot \det W^\pm - 2 \cdot g\left((E \owedge E), W^\pm \right). \tag{3.4}$$

Now, by Lemma **(L.1.13)**, we also have that

$$\begin{aligned} \Delta_f \left(\|W^\pm\|^2 \right) &= 2 \cdot \|W^\pm\| \cdot \Delta_f \|W^\pm\| + 2 \cdot g\left(\nabla \|W^\pm\|, \nabla \|W^\pm\| \right) \\ &= 2 \cdot \|W^\pm\| \cdot \Delta_f \|W^\pm\| + \|\nabla \|W^\pm\|\|^2. \end{aligned}$$

It then follows that

$$\begin{aligned} 2 \cdot \|W^\pm\| \cdot \Delta_f \|W^\pm\| &= \Delta_f \left(\|W^\pm\|^2 \right) - 2\|\nabla \|W^\pm\|\|^2 \\ &= 2 \cdot \|\nabla W^\pm\|^2 - 2 \cdot \|\nabla \|W^\pm\|\|^2 + 2 \cdot \|W^\pm\|^2 - 36 \cdot \det W^\pm - 2 \cdot g\left((E \owedge E), W^\pm \right). \end{aligned}$$

By inequality **(1.9)**, we also have that $\|\nabla W^\pm\|^2 \geq \|\nabla \|W^\pm\|\|^2$, and therefore

$$2 \cdot \|W^\pm\| \cdot \Delta_f \|W^\pm\| \geq 2 \cdot \|W^\pm\|^2 - 36 \cdot \det W^\pm - 2 \cdot g\left((E \owedge E), W^\pm \right).$$

Now using the second part of Proposition **(P.3.7)**, we obtain

$$\|W^\pm\| \cdot \Delta_f \|W^\pm\| \geq \|W^\pm\|^2 - \sqrt{6} \cdot \|W^\pm\|^3 - g\left((E \owedge E)^\pm, W^\pm \right). \tag{3.5}$$

Recalling also that by hypothesis we have

$$\|W^\pm\|^2 - \sqrt{6} \cdot \|W^\pm\|^3 \geq g\left((E \owedge E)^\pm, W^\pm \right). \tag{3.6}$$

We see that

$$\|W^\pm\| \cdot \Delta_f \|W^\pm\| \geq 0,$$

that is, $\|W^\pm\| \cdot \Delta_f \|W^\pm\|$ is nonnegative. Next, we will prove that $\|W^\pm\|^2 \cdot e^{-f}$ is g-integrable, that is, that $\|W^\pm\| \in \mathcal{L}^2(e^{-f} \cdot \mathrm{dVol}_g)$. Indeed, it follows from the Cauchy-Schwarz inequality that

$$-\|(E \owedge E)^\pm\| \cdot \|W^\pm\| \leq g\left((E \owedge E)^\pm, W^\pm \right) \leq \|(E \owedge E)^\pm\| \cdot \|W^\pm\|.$$

Thus, using the first part of Proposition **(P.3.8)**, we see that

$$\begin{aligned} -g\left((E \owedge E)^\pm, W^\pm \right) &\leq \|(E \owedge E)^\pm\| \cdot \|W^\pm\| \\ &\leq \frac{\sqrt{3}}{\sqrt{2}} \cdot \|E\|^2 \cdot \|W^\pm\|^2. \end{aligned}$$

Now using equation (3.6), we obtain

$$\begin{aligned}\sqrt{6}\cdot\|W^{\pm}\|^3 &\leq \|W^{\pm}\|^2 - g\left((E \owedge E)^{\pm}, W^{\pm}\right)\\ &\leq \|W^{\pm}\|^2 + \frac{\sqrt{3}}{\sqrt{2}}\cdot\|E\|^2\cdot\|W^{\pm}\|\\ &\leq \|W^{\pm}\|^2 + \frac{\sqrt{3}}{\sqrt{2}}\cdot\|\mathrm{Ric}\|^2\cdot\|W^{\pm}\|.\end{aligned}$$

Thus, we have that

$$\sqrt{6}\cdot\|W^{\pm}\|^2 \leq \|W^{\pm}\| + \frac{\sqrt{3}}{\sqrt{2}}\cdot\|\mathrm{Ric}\|^2 \leq \frac{\sqrt{3}}{\sqrt{2}}\|W^{\pm}\|^2 + \frac{\sqrt{2}}{4\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{2}}\cdot\|\mathrm{Ric}\|^2, \tag{3.7}$$

or, equivalently,

$$\|W^{\pm}\|^2 \leq \frac{1}{6} + \|\mathrm{Ric}\|^2, \tag{3.8}$$

where in the last inequality in (3.7) we used that

$$\frac{\sqrt{3}}{\sqrt{2}}\cdot x^2 - x + \frac{\sqrt{2}}{4\sqrt{3}} \geq 0,\ \forall x\in\mathbb{R}.$$

Integrating equation (3.8) over $\mathcal{M}$, we see that

$$\int_{\mathcal{M}}\|W^{\pm}\|^2\cdot e^{-f}\,\mathrm{dVol}_g \leq \frac{1}{6}\cdot\int_{\mathcal{M}} e^{-f}\,\mathrm{dVol}_g + \int_{\mathcal{M}}\|\mathrm{Ric}\|^2\cdot e^{-f}\,\mathrm{dVol}_g\,. \tag{3.9}$$

Now, as seen in Corollary 1.1 of [7], the "weighted" volume of $\mathcal{M}^4$ is finite, that is,

$$\int_{\mathcal{M}} e^{-f}\,\mathrm{dVol}_g < \infty.$$

Moreover, Munteanu and Sesum (as seen in Theorem 1.1 in [37]) proved that

$$\int_{\mathcal{M}}\|\mathrm{Ric}\|^2 e^{-f}\,\mathrm{dVol}_g < \infty.$$

It then follows from equation (3.9) that $\|W^{\pm}\|^2\cdot e^{-f}$ is g-integrable. Let us now fix arbitrary $r>0, p\in\mathcal{M}$, and let $\varphi:\mathcal{M}\to\mathbb{R}$ be a smooth function satisfying:

$$\begin{aligned}&\varphi(q) = 1,\ \forall q\in B_p(r)\doteq \exp_p(B_r(0));\\ &\varphi(q) = 0,\ \forall q\notin B_p(2r),\ \text{and}\\ &\|\nabla\varphi\| \leq \frac{c}{r},\end{aligned}$$

where $c>0$ is a real constant. Note that we used the hypothesis of completeness of $\mathcal{M}$ to guarantee that the geodesic ball $\exp_p(B_r(0))$ is well defined for all $p\in\mathcal{M}$ and $r>0$. The existence of such a φ is guaranteed by Lemma (L.1.9). We will denote the volume form $e^{-f}\cdot\mathrm{dVol}_g$ by $\widetilde{\mathrm{dVol}_g}$. Integrating by

parts in the first equality below (as we saw in **(4.29)**), we then have that:

$$\begin{aligned}
0 &\geq -\int_{\mathcal{M}} \varphi^2 \cdot \|W^{\pm}\| \cdot \Delta_f \left(\|W^{\pm}\|\right) \widetilde{\mathrm{dVol}_g} \\
&= \int_{\mathcal{M}} g\left(\nabla\left(\varphi^2 \cdot \|W^{\pm}\|\right), \nabla\|W^{\pm}\|\right) \widetilde{\mathrm{dVol}_g} \\
&= \int_{\mathcal{M}} \varphi^2 \cdot \|\nabla\|W^{\pm}\|\|^2 \widetilde{\mathrm{dVol}_g} + 2\int_{\mathcal{M}} \varphi \cdot \|W^{\pm}\| \cdot g\left(\nabla\|W^{\pm}\|, \nabla\varphi\right) \widetilde{\mathrm{dVol}_g} \\
&= \int_{\mathcal{M}} \left\|\left(\varphi \cdot \|\nabla\|W^{\pm}\|\| + \|W^{\pm}\| \cdot \nabla\varphi\right)\right\|^2 \widetilde{\mathrm{dVol}_g} - \int_{\mathcal{M}} \|W^{\pm}\|^2 \cdot \|\nabla\varphi\|^2 \widetilde{\mathrm{dVol}_g},
\end{aligned}$$

where in the last equalities we used the bilinearity of g together with Lemma **(L.1.5)**, from which the following equalities follow

$$\nabla\left(\varphi^2 \cdot \|W^{\pm}\|\right) = \varphi^2 \cdot \nabla\left(\|W^{\pm}\|\right) + 2 \cdot \varphi \cdot \|W^{\pm}\| \cdot \nabla\varphi, \text{ as well as}$$
$$\left\|\nabla\left(\varphi \cdot \|W^{\pm}\|\right)\right\|^2 = \left\|\left(\varphi \cdot \nabla\left\|W^{\pm}\right\| + \|W^{\pm}\| \cdot \nabla\varphi\right)\right\|^2 = \varphi^2 \cdot \left\|\nabla\|W^{\pm}\|\right\|^2 + 2 \cdot \varphi \cdot \|W^{\pm}\| \cdot g\left(\nabla\|W^{\pm}\|, \nabla\varphi\right) + \|W^{\pm}\|^2 \cdot \|\nabla\varphi\|^2.$$

Consequently, we have

$$\int_{\mathcal{M}} \left\|\nabla\left(\varphi \cdot \|W^{\pm}\|\right)\right\|^2 \widetilde{\mathrm{dVol}_g} \leq \int_{\mathcal{M}} \|W^{\pm}\|^2 \cdot \|\nabla\varphi\|^2 \widetilde{\mathrm{dVol}_g}.$$

By the construction of φ, the following equalities are also obvious:

$$\int_{B_p(r)} \left\|\nabla\|W^{\pm}\|\right\|^2 \widetilde{\mathrm{dVol}_g} = \int_{\mathcal{M}} \left\|\nabla\left(\varphi \cdot \|W^{\pm}\|\right)\right\|^2 \widetilde{\mathrm{dVol}_g} - \int_{B_p(2r)\setminus B_p(r)} \left\|\nabla\left(\varphi \cdot \|W^{\pm}\|\right)\right\|^2 \widetilde{\mathrm{dVol}_g},$$
$$\int_{\mathcal{M}} \|W^{\pm}\|^2 \cdot \|\nabla\varphi\|^2 \widetilde{\mathrm{dVol}_g} = \int_{B_p(2r)\setminus B_p(r)} \|W^{\pm}\|^2 \cdot \|\nabla\varphi\|^2 \widetilde{\mathrm{dVol}_g}.$$

Hence, it follows that:

$$\begin{aligned}
0 \leq \int_{B_p(r)} \left\|\nabla\|W^{\pm}\|\right\|^2 \widetilde{\mathrm{dVol}_g} &\leq \int_{\mathcal{M}} \left\|\nabla\left(\varphi \cdot W^{\pm}\right)\right\|^2 \widetilde{\mathrm{dVol}_g} \\
&\leq \int_{B_p(2r)\setminus B_p(r)} \|W^{\pm}\|^2 \cdot \|\nabla\varphi\|^2 \widetilde{\mathrm{dVol}_g} \\
&\leq \frac{c^2}{r^2} \cdot \int_{\mathcal{M}} \|W^{\pm}\|^2 \widetilde{\mathrm{dVol}_g}.
\end{aligned}$$

Since the inequality above holds for every $r > 0$ and since we have already seen that $\|W^{\pm}\| \in \mathcal{L}^2\left(\widetilde{\mathrm{dVol}_g}\right)$, we conclude that

$$0 \leq \int_{\mathcal{M}} \left\|\nabla\|W^{\pm}\|\right\|^2 \widetilde{\mathrm{dVol}_g} \leq \left(\inf_{r>0} r^{-2}\right) \cdot c^2 \int_{\mathcal{M}} \|W^{\pm}\|^2 \widetilde{\mathrm{dVol}_g} = 0,$$

from which it follows that $\nabla\left(\|W^{\pm}\|\right) = 0$, so that $\|W^{\pm}\|$ is constant.

We then have two cases to analyze, namely, $\|W^{\pm}\| = 0$ (which obviously implies $\nabla W^{\pm} = 0$) and $\|W^{\pm}\| \neq 0$. If $\|W^{\pm}\| = 0$, then it follows from Theorem 1.2 in [11] that $\mathcal{M}^4$ is either Einstein or a finite quotient of either the Gaussian shrinking soliton $\mathbb{R}^4$ or the round cylinder $\mathbb{S}^3 \times \mathbb{R}$, as initially claimed.

On the other hand, if $\|W^{\pm}\|$ is identically equal to a nonzero constant, say c, then $\nabla W^{\pm} = 0$. Indeed, by equation (3.4), we have that

$$0 = \Delta_f \left(\|W^{\pm}\|^2 \right) = 2 \cdot \|\nabla W^{\pm}\|^2 + 2 \cdot c^2 - 36 \cdot \det W^{\pm} - 2 \cdot g\left((E \owedge E), W^{\pm} \right). \tag{3.10}$$

Recalling also equation (3.6), we obtain

$$c^2 - \sqrt{6} \cdot c^3 \geq g\left((E \owedge E)^{\pm}, W^{\pm} \right) \implies -2 \cdot g\left((E \owedge E)^{\pm}, W^{\pm} \right) \geq 2\sqrt{6} \cdot c^3 - 2 \cdot c^2.$$

Now, by Proposition (P.3.7), we have

$$36 \det W^{\pm} \leq 2\sqrt{6} \cdot c^3 \implies -36 \det W^{\pm} \geq -2\sqrt{6} \cdot c^3.$$

Therefore

$$2 \cdot c^2 - 36 \cdot \det W^{\pm} - 2 \cdot g\left((E \owedge E), W^{\pm} \right) \geq 0.$$

It then follows from equation (3.10) that

$$\nabla W^{\pm} = 0, \text{ and also } 2 \cdot c^2 = 36 \det W^{\pm} + \underbrace{2 \cdot g\left((E \owedge E), W^{\pm} \right)}_{\leq 2 \cdot c^2 - 2\sqrt{6} \cdot c^3},$$

from which we also conclude that

$$2 \cdot c^2 \leq 36 \det W^{\pm} + 2 \cdot c^2 - 2\sqrt{6} \cdot c^3 \implies \det W^{\pm} \geq \frac{\sqrt{6}}{18} \cdot c^3,$$

and therefore equality in Proposition (P.3.7) holds, so that $W^{\pm}$ has exactly two distinct eigenvalues. Moreover, as we saw in Proposition (P.4.14), we have $\mathrm{div} = \mathrm{tr} \circ \nabla$, from which it is clear that $\mathrm{div}\, W^{\pm} = 0$ whenever $\nabla W^{\pm} = 0$. Since we have already seen that this holds in either case, we can then apply Theorem 1.1 in [55] and conclude that $\mathcal{M}^4$ is either Einstein or a finite quotient of $\mathbb{S}^2 \times \mathbb{R}^2$.

■

Assuming now that Scal is suitably bounded by the potential function f, that is, that Scal satisfies

$$\mathrm{Scal} \leq A + \varepsilon f, \tag{3.11}$$

where $A > 0$ and $0 \leq \varepsilon < 1$ are constants, we will prove some curvature estimates for Rm and Ric. For this, it will be essential to recall a curvature estimate for Rm satisfied by every shrinking gradient Ricci soliton, proved by Munteanu and Wang in [38].

Proposition (P.3.10). *Let $(\mathcal{M}^4, g, f)$ be a complete and noncompact shrinking gradient Ricci soliton of dimension 4. Then there exists a positive universal constant C_0 such that:*

$$\|\mathrm{Rm}\| \leq C_0 \left(\|\mathrm{Ric}\| + \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|} \right),$$

whenever $\|\nabla f\| \neq 0$.

Proof: Let us fix an orthonormal frame $\{\mathbf{e}_i\}_{1\leq i\leq 4}$ with $e_4 = \dfrac{\nabla f}{\|\nabla f\|}$. By part (g) of Lemma **(L.2.4)**, we have:

$$\begin{aligned}|\mathrm{Rm}_{ij\ell 4}| &= \frac{1}{\|\nabla f\|}\cdot|(\nabla_i \mathrm{Ric})_{\ell j} - (\nabla_j \mathrm{Ric})_{\ell i}| \\ &\leq \frac{1}{\|\nabla f\|}\cdot(|(\nabla_i \mathrm{Ric})_{\ell j}| + |(\nabla_j \mathrm{Ric})_{\ell i}|) \\ &\leq \frac{1}{\|\nabla f\|}\cdot\left(\sum_{1\leq i,j,\ell\leq 4}|(\nabla_i \mathrm{Ric})_{\ell j}| + \sum_{1\leq i,j,\ell\leq 4}|(\nabla_j \mathrm{Ric})_{\ell i}|\right).\end{aligned}$$

Now, from the inequality:

$$\frac{\sum_{1\leq i\leq n}|a_i|}{n} \leq \frac{\sqrt{\sum_{1\leq i\leq n} a_i^2}}{\sqrt{n}}, \text{ for any } \{a_i\}_{1\leq i\leq n}\subset\mathbb{R} \text{ and } n\in\mathbb{N},$$

we obtain

$$\begin{aligned}|\mathrm{Rm}_{ij\ell 4}| &\leq \frac{2}{\|\nabla f\|}\cdot\left(\sqrt{\sum_{1\leq i,j,\ell\leq 4}|(\nabla_i \mathrm{Ric})_{\ell j}|^2} + \sqrt{\sum_{1\leq i,j,\ell\leq 4}|(\nabla_j \mathrm{Ric})_{\ell i}|^2}\right) \\ &= \frac{4}{\|\nabla f\|}\cdot\|\nabla \mathrm{Ric}\|.\end{aligned}$$

Note also that:

$$\|\mathrm{Rm}\| = \sqrt{\sum_{1\leq i,j,k,\ell\leq 4}(\mathrm{Rm}_{ijk\ell})^2} \leq \sum_{1\leq i,j,k,\ell\leq 4}|\mathrm{Rm}_{ijk\ell}|.$$

By Proposition **(P.3.2)**, we only need to worry about 20 terms of the last sum appearing in the expression above. All of them can be handled in a similar way. Note that, for example,

$$\begin{aligned}&\mathrm{Ric}_{11} = \mathrm{Rm}_{1212} + \mathrm{Rm}_{1313} + \mathrm{Rm}_{1414}, \qquad \mathrm{Ric}_{22} = \mathrm{Rm}_{1212} + \mathrm{Rm}_{2323} + \mathrm{Rm}_{2424}, \\ &\mathrm{Ric}_{33} = \mathrm{Rm}_{1313} + \mathrm{Rm}_{2323} + \mathrm{Rm}_{3434}, \qquad \mathrm{Ric}_{44} = \mathrm{Rm}_{1414} + \mathrm{Rm}_{2424} + \mathrm{Rm}_{3434}, \\ &\implies \mathrm{Rm}_{1212} = \frac{1}{2}(\mathrm{Ric}_{11} + \mathrm{Ric}_{22} - \mathrm{Ric}_{33} - \mathrm{Ric}_{44}) + \mathrm{Rm}_{3434}, \\ &\implies |\mathrm{Rm}_{1212}| \leq C_0\left(\|\mathrm{Ric}\| + \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|}\right),\end{aligned}$$

and we can deal with the other 19 components of Rm in an entirely analogous way, obtaining the desired estimate.

■

Remark (O.82). Since obviously the constants c_0, c_1 and c_2 in the following Theorem can be taken arbitrarily large, ε can be taken arbitrarily small.

Moreover, for each $K > 0$, since

$$D_K \doteq \{x\in\mathcal{M} \mid f(x)\leq K\} = f^{-1}((-\infty, K]),$$

it follows from the continuity of f that D_K is closed in $\mathcal{M}$. Now, if D_K were unbounded, then taking p_0 as in Lemma **(L.3.1)**, there would exist a sequence of points $(x_k)_{k\in\mathbb{N}}\subset D_k$ satisfying $\lim_{n\to\infty}\mathrm{dist}(p_0, x_k) = \infty$,

which is evidently absurd. The Hopf-Rinow Theorem then guarantees that D_K is compact for any $K > 0$.

Remark (O.83). By part **(d)** of Lemma **(L.2.4)**, we know that $\mathrm{Scal} + \|\nabla f\|^2 = f$, which obviously implies the estimate $\mathrm{Scal} \leq f$, satisfied by any shrinking gradient Ricci soliton. We therefore see that the hypothesis $\mathrm{Scal} \leq A + \varepsilon f$ of Theorem **(T.3.2)** is a very slight weakening of a property common to all shrinking gradient Ricci solitons. In this sense, it would then be interesting to investigate the possibility of removing this estimate as a hypothesis, which as of this writing constitutes an open problem.

Another interesting problem would be to investigate whether the estimate on Rm in the second part of Theorem **(T.3.2)** can be improved to $\|\mathrm{Rm}\| \leq c_0 + (c_2\varepsilon)\, f$, instead of $\|\mathrm{Rm}\| \leq c_0 + (c_2\varepsilon)\, f^2$.

Theorem (T.3.2). *Let $(\mathcal{M}^4, g, f)$ be a complete and noncompact shrinking gradient Ricci soliton of dimension 4 that satisfies*

$$\mathrm{Scal} \leq A + \varepsilon f,$$

where $A > 0$ and $0 \leq \varepsilon < 1$ are constants. Then there exist constants c_0, c_1 and c_2 such that the following curvature estimates hold on $\mathcal{M}$:

- $\|\mathrm{Ric}\| \leq c_0 + (c_1\varepsilon) f$;
- $\|\mathrm{Rm}\| \leq c_0 + (c_2\varepsilon) f^2$.

Proof: We will first prove the estimate on the Ricci tensor, closely following the same arguments given in [6] and [38]. As always, we will do all the computations in a geodesic frame centered around a fixed point $p \in \mathcal{M}$ which, to avoid overloading the notation, we will omit. Moreover, we will also use c or C to denote universal constants (i.e. depending only on the dimension of the manifold, in this case 4) that may change from one line to another. Thus, note that

$$\begin{aligned}
\Delta_f\left(\|\mathrm{Ric}\|^2\right) = \Delta_f\left(\sum_{1\leq i,j\leq 4}(\mathrm{Ric}_{ij})^2\right) &= \sum_{1\leq i,j\leq 4} 2\cdot \mathrm{Ric}_{ij}\Delta_f \mathrm{Ric}_{ij} + 2\sum_{1\leq i,j\leq 4} g\left(\nabla \mathrm{Ric}_{ij}, \nabla \mathrm{Ric}_{ij}\right) \\
&= 2\cdot \sum_{1\leq i,j\leq 4}\left[\mathrm{Ric}_{ij}\left(\mathrm{Ric}_{ij} - 2\sum_{1\leq k,\ell\leq 4}\mathrm{Rm}_{ikj\ell}\cdot \mathrm{Ric}_{k\ell}\right)\right] + 2\cdot\|\nabla \mathrm{Ric}\|^2 \\
&= 2\cdot\|\mathrm{Ric}\|^2 + 2\cdot\|\nabla\mathrm{Ric}\|^2 - 4\left(\sum_{1\leq i,j\leq 4}\mathrm{Ric}_{ij}\right)\cdot\left(\sum_{1\leq k,\ell\leq 4}\mathrm{Rm}_{ikj\ell}\cdot\mathrm{Ric}_{k\ell}\right),
\end{aligned}$$

where in the first equality we used Lemma **(L.1.13)** and in the second we used part **(f)** of Lemma **(L.2.4)**. Now, by Cauchy-Schwarz, we have that

$$\sum_{1\leq i,j,k,\ell\leq 4}\mathrm{Rm}_{ikj\ell}\cdot\mathrm{Ric}_{k\ell} \leq c\sqrt{\sum_{1\leq i,j,k,\ell\leq 4}(\mathrm{Rm}_{ikj\ell})^2}\cdot\sqrt{\sum_{1\leq k,\ell\leq 4}(\mathrm{Ric}_{k\ell})^2} = c\cdot\|\mathrm{Rm}\|\cdot\|\mathrm{Ric}\|, \text{ as well as}$$

$$\sum_{1\leq i,j\leq 4}\mathrm{Ric}_{ij} \leq c\cdot\|\mathrm{Ric}\|.$$

Therefore

$$\begin{aligned}
\Delta_f\left(\|\mathrm{Ric}\|^2\right) &\geq 2\cdot\|\nabla\mathrm{Ric}\|^2 + 2\cdot\|\mathrm{Ric}\|^2 - c\cdot\|\mathrm{Rm}\|\cdot\|\mathrm{Ric}\|^2 \\
&\geq 2\cdot\|\nabla\mathrm{Ric}\|^2 - c\cdot\|\mathrm{Ric}\|^3 - c\cdot\frac{\|\nabla\mathrm{Ric}\|}{\|\nabla f\|}\cdot\|\mathrm{Ric}\|^2,
\end{aligned} \tag{3.12}$$

where in the last inequality we discarded the term with $\|\mathrm{Ric}\|^2$ and used Proposition **(P.3.10)**. For any $0 < a < 1$, we also have:

$$\begin{aligned}
\Delta_f(\mathrm{Scal}^{-a}) &= \Delta(\mathrm{Scal}^{-a}) - \nabla_{\nabla f}(\mathrm{Scal}^{-a}) \\
&= \sum_{1\le i\le 4}\left[(\mathbf{e}_i\circ\mathbf{e}_i)(\mathrm{Scal}^{-a}) - \mathbf{e}_i(f)\cdot\mathbf{e}_i(\mathrm{Scal}^{-a})\right] \\
&= \sum_{1\le i\le 4}\left[\frac{a(a+1)}{\mathrm{Scal}^{a+2}}\cdot(\mathbf{e}_i(\mathrm{Scal}))^2 - a\cdot\mathrm{Scal}^{-a-1}\cdot(\mathbf{e}_i\circ\mathbf{e}_i)(\mathrm{Scal}) + a\cdot\mathbf{e}_i(f)\cdot\mathrm{Scal}^{-a-1}\cdot\mathbf{e}_i(\mathrm{Scal})\right] \\
&= \frac{a(a+1)}{\mathrm{Scal}^{a+2}}\cdot\|\nabla\mathrm{Scal}\|^2 - a\cdot\mathrm{Scal}^{-a-1}\cdot\Delta\mathrm{Scal} + a\cdot\mathrm{Scal}^{-a-1}\cdot g(\nabla f,\nabla\mathrm{Scal}) \\
&= \frac{a(a+1)}{\mathrm{Scal}^{a+2}}\cdot\|\nabla\mathrm{Scal}\|^2 - a\cdot\mathrm{Scal}^{-a-1}\Delta_f\mathrm{Scal} \\
&= -\frac{a}{\mathrm{Scal}^a} + 2a\cdot\frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^{a+1}} + \frac{a(a+1)}{\mathrm{Scal}^{a+2}}\cdot\|\nabla\mathrm{Scal}\|^2,
\end{aligned}$$

where in the last equality we used part **(c)** of Lemma **(L.2.4)**. This equality together with equation **(3.12)** and Lemma **(L.1.13)** guarantee that

$$\begin{aligned}
\Delta_f\left(\frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a}\right) &= \frac{\Delta_f\left(\|\mathrm{Ric}\|^2\right)}{\mathrm{Scal}^a} + \|\mathrm{Ric}\|^2\cdot\Delta_f\left(\frac{1}{\mathrm{Scal}^a}\right) + 2\cdot g\left(\nabla\left(\|\mathrm{Ric}\|^2\right),\nabla\left(\frac{1}{\mathrm{Scal}^a}\right)\right) \\
&\ge \frac{1}{\mathrm{Scal}^a}\left[2\cdot\|\nabla\mathrm{Ric}\|^2 - c\cdot\|\mathrm{Ric}\|^3 - c\cdot\frac{\|\nabla\mathrm{Ric}\|}{\|\nabla f\|}\cdot\|\mathrm{Ric}\|^2\right] \\
&\quad + \|\mathrm{Ric}\|^2\cdot\left[-\frac{a}{\mathrm{Scal}^a} + 2a\cdot\frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^{a+1}} + \frac{a(a+1)}{\mathrm{Scal}^{a+2}}\cdot\|\nabla\mathrm{Scal}\|^2\right]\cdot\Delta_f\left(\frac{1}{\mathrm{Scal}^a}\right) \\
&\quad + 2\cdot g\left(\nabla\left(\|\mathrm{Ric}\|^2\right),\nabla\left(\frac{1}{\mathrm{Scal}^a}\right)\right).
\end{aligned} \tag{3.13}$$

Now, since

$$\nabla\left(\|\mathrm{Ric}\|^2\right) = 2\cdot\|\mathrm{Ric}\|\cdot\nabla\left(\|\mathrm{Ric}\|\right),\ \text{and}\ \nabla\left(\mathrm{Scal}^{-a}\right) = -a\cdot\mathrm{Scal}^{-a-1}\cdot\nabla\mathrm{Scal},$$

it follows that

$$2\cdot g\left(\nabla\left(\|\mathrm{Ric}\|^2\right),\nabla\left(\frac{1}{\mathrm{Scal}^a}\right)\right) = -4a\cdot\mathrm{Scal}^{-a-1}\cdot\|\mathrm{Ric}\|\cdot g\left(\nabla\mathrm{Scal},\nabla\left(\|\mathrm{Ric}\|\right)\right).$$

But by the Cauchy-Schwarz and Kato inequalities, we have

$$g(\nabla\mathrm{Scal},\nabla\|\mathrm{Ric}\|) \le \|\nabla\mathrm{Scal}\|\cdot\|\nabla\|\mathrm{Ric}\|\| \le \|\nabla\mathrm{Scal}\|\cdot\|\nabla\mathrm{Ric}\|.$$

Hence,

$$2\cdot g\left(\nabla\left(\|\mathrm{Ric}\|^2\right),\nabla\left(\frac{1}{\mathrm{Scal}^a}\right)\right) \ge -4a\cdot\mathrm{Scal}^{-a-1}\cdot\|\mathrm{Ric}\|\cdot\|\nabla\mathrm{Scal}\|\cdot\|\nabla\mathrm{Ric}\|.$$

But from

$$\left[\sqrt{a+1}\cdot\|\mathrm{Ric}\|\cdot\|\nabla\mathrm{Scal}\|\cdot\mathrm{Scal}^{-1} - \frac{2}{\sqrt{a+1}}\right]^2 \ge 0,$$

it follows that

$$-4\cdot \mathrm{Scal}^{-1}\cdot \|\mathrm{Ric}\|\cdot \|\nabla \mathrm{Scal}\|\cdot \|\nabla \mathrm{Ric}\| \geq -(a+1)\cdot \|\mathrm{Ric}\|^2\cdot \|\nabla \mathrm{Scal}\|^2\cdot \mathrm{Scal}^{-2} - \frac{4}{a+1}\cdot \|\nabla \mathrm{Ric}\|^2 \geq 0,$$

and therefore that

$$\begin{aligned}-4a\cdot \mathrm{Scal}^{-a-1}\cdot \|\mathrm{Ric}\|\cdot \|\nabla \mathrm{Scal}\|\cdot \|\nabla \mathrm{Ric}\| \geq -a(a+1)\cdot \|\mathrm{Ric}\|^2\cdot \|\nabla \mathrm{Scal}\|^2\cdot \mathrm{Scal}^{-a-2}\\ - \frac{4a}{a+1}\cdot \|\nabla \mathrm{Ric}\|^2\cdot \mathrm{Scal}^{-a}.\end{aligned}$$

We then obtain

$$2\cdot g\left(\nabla\left(\|\mathrm{Ric}\|^2\right), \nabla\left(\frac{1}{\mathrm{Scal}^a}\right)\right) \geq -a(a+1)\cdot \frac{\|\mathrm{Ric}\|^2\cdot \|\nabla \mathrm{Scal}\|^2}{\mathrm{Scal}^{a+2}} - \frac{4a}{a+1}\cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\mathrm{Scal}^a},$$

Therefore, equation (3.13) can be written as

$$\begin{aligned}\Delta_f\left(\frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a}\right) &\geq \frac{1}{\mathrm{Scal}^a}\left[2\cdot \|\nabla \mathrm{Ric}\|^2 - c\cdot \|\mathrm{Ric}\|^3 - c\cdot \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|}\cdot \|\mathrm{Ric}\|^2\right]\\ &\quad + \|\mathrm{Ric}\|^2\cdot \left[-\frac{a}{\mathrm{Scal}^a} + 2a\cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^{a+1}} + \frac{a(a+1)}{\mathrm{Scal}^{a+2}}\cdot \|\nabla \mathrm{Scal}\|^2\right]\cdot \Delta_f\left(\frac{1}{\mathrm{Scal}^a}\right)\\ &\quad - a(a+1)\cdot \frac{\|\mathrm{Ric}\|^2\cdot \|\nabla \mathrm{Scal}\|^2}{\mathrm{Scal}^{a+2}} - \frac{4a}{a+1}\cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\mathrm{Scal}^a}\\ &\geq \frac{2\cdot \|\nabla \mathrm{Ric}\|^2}{\mathrm{Scal}^a} - \frac{4a}{a+1}\cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\mathrm{Scal}^a} - c\cdot \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|}\cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a} - c\cdot \frac{\|\mathrm{Ric}\|^3}{\mathrm{Scal}^a} - a\cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a}\\ &\quad + 2a\cdot \frac{\|\mathrm{Ric}\|^4}{\mathrm{Scal}^{a+1}}\\ &= \frac{2(1-a)}{(1+a)}\cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\mathrm{Scal}^a} - c\cdot \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|}\cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a} - c\cdot \frac{\|\mathrm{Ric}\|^3}{\mathrm{Scal}^a} - a\cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a} + 2a\cdot \frac{\|\mathrm{Ric}\|^4}{\mathrm{Scal}^{a+1}}.\end{aligned} \tag{3.14}$$

Now, from

$$\left(4\cdot (1-a)\cdot \|\nabla \mathrm{Ric}\|\cdot \|\nabla f\| - (1+a)\cdot c\cdot \|\mathrm{Ric}\|^2\right)^2 \geq 0,$$

we have

$$16\cdot (1-a)^2\cdot \|\nabla \mathrm{Ric}\|^2\cdot \|\nabla f\|^2 + (1+a)^2\cdot c^2\cdot \|\mathrm{Ric}\|^4 - 8c\cdot (1-a)\cdot (1+a)\cdot \|\nabla \mathrm{Ric}\|\cdot \|\mathrm{Ric}\|^2\cdot \|\nabla f\| \geq 0,$$

whence

$$\frac{2\cdot (1-a)}{1+a}\cdot \|\nabla \mathrm{Ric}\|^2\cdot \|\nabla f\|^2 + \frac{(1+a)}{8\cdot (1-a)}\cdot c^2\cdot \|\mathrm{Ric}\|^4 - c\cdot \|\nabla \mathrm{Ric}\|\cdot \|\mathrm{Ric}\|^2\cdot \|\nabla f\| \geq 0,$$

or, equivalently,

$$\frac{2\cdot (1-a)}{1+a}\cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\mathrm{Scal}^a} - c\cdot \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|}\cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a} \geq -\frac{(1+a)}{8\cdot (1-a)}\cdot \frac{c^2}{\|\nabla f\|^2}\cdot \frac{\|\mathrm{Ric}\|^4}{\mathrm{Scal}^a}.$$

And therefore, now replacing c by $\tilde{c} = -\frac{1+a}{8} \cdot c^2$, we obtain

$$\begin{aligned}\Delta_f\left(\frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a}\right) &\geq \left(2a - \frac{\tilde{c}}{(1-a)} \cdot \frac{\mathrm{Scal}}{\|\nabla f\|^2}\right) \cdot \frac{\|\mathrm{Ric}\|^4}{\mathrm{Scal}^{a+1}} - c \cdot \frac{\|\mathrm{Ric}\|^3}{\mathrm{Scal}^a} - a \cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a} \\ &\geq \left(2a - \frac{C}{(1-a)} \cdot \frac{\mathrm{Scal}}{\|\nabla f\|^2}\right) \cdot \frac{\|\mathrm{Ric}\|^4}{\mathrm{Scal}^{a+1}} - C \cdot \frac{\|\mathrm{Ric}\|^3}{\mathrm{Scal}^a} - a \cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a},\end{aligned} \tag{3.15}$$

where $C \doteq \max\{c, \tilde{c}\}$. For simplicity we will commit the abuse of notation of denoting C by c. But as we already saw in part **(d)** of Lemma **(L.2.4)**, we have $\mathrm{Scal} + \|\nabla f\|^2 = f$, so that our hypothesis on Scal implies that

$$\|\nabla f\|^2 = f - \mathrm{Scal} \geq f - A - \varepsilon f = (1-\varepsilon)f - A. \tag{3.16}$$

It then follows that

$$\frac{\mathrm{Scal}}{\|\nabla f\|^2} \leq \frac{\mathrm{Scal}}{(1-\varepsilon)f - A} \leq \frac{\varepsilon f + A}{(1-\varepsilon)f - A}.$$

Using this inequality in inequality **(3.15)**, we have

$$\Delta_f\left(\frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a}\right) \geq \left(2a - \frac{c}{1-a} \cdot \frac{\varepsilon f + A}{(1-\varepsilon)f - A}\right) \cdot \frac{\|\mathrm{Ric}\|^4}{\mathrm{Scal}^{a+1}} - c \cdot \frac{\|\mathrm{Ric}\|^3}{\mathrm{Scal}^a} - a \cdot \frac{\|\mathrm{Ric}\|^2}{\mathrm{Scal}^a}. \tag{3.17}$$

Now, we then need to consider the term

$$\Gamma = 2a - \frac{c}{1-a} \cdot \frac{\varepsilon f + A}{(1-\varepsilon)f - A}.$$

Taking $a = \frac{1}{2}$, we see that

$$\Gamma = 1 - \frac{2c \cdot (\varepsilon f + A)}{(1-\varepsilon)f - A}.$$

Now, defining $C \doteq \frac{1 - \varepsilon - 4c \cdot \varepsilon}{1 + 4c}$ and choosing $r_0 > 0$ large enough in Lemma **(L.3.1)** so that

$$r_0 > \max\left\{\frac{A}{C}, \frac{1+4c}{4c} \cdot (1 + A)\right\}, \tag{3.18}$$

we then see that

$$A \leq \left(\frac{1 - \varepsilon - 4c \cdot \varepsilon}{1 + 4c}\right) f,$$

on $\mathcal{M} \setminus D_{r_0}$, where $D_{r_0} \doteq \{x \in \mathcal{M} \mid f(x) \leq r_0\}$. On $\mathcal{M} \setminus D_{r_0}$, we then have

$$\begin{aligned}2c \cdot (\varepsilon f + A) &\leq 2c\left(\varepsilon f + \frac{1 - \varepsilon - 4c \cdot \varepsilon}{1 + 4c} f\right) \\ &= \frac{2c}{1 + 4c} f.\end{aligned}$$

and also

$$\frac{1}{(1-\varepsilon)f - A} \leq \frac{1 + 4c}{(1 + 4c)(1-\varepsilon)f - (1 - \varepsilon - 4c \cdot \varepsilon)f}$$

$$= \frac{1+4c}{4cf}.$$

Hence,

$$\frac{2c(\varepsilon f + A)}{(1-\varepsilon)f - A} \leq \frac{2c \cdot f}{1+4c} \cdot \frac{1+4c}{4c \cdot f} = \frac{1}{2}$$

on $\mathcal{M} \setminus D_{r_0}$. Recalling remark **(O.82)**, we may also assume without loss of generality that $\varepsilon < (1+4c)^{-1}$, so that

$$\begin{aligned}(1-\varepsilon)f - A &> \left(1 - \frac{1}{1+4c}\right) f - A \\ &= \frac{4c \cdot f}{1+4c} - A \\ &> \frac{4c \cdot r_0}{1+4c} - A \\ &> \frac{4c}{1+4c} \cdot \frac{1+4c}{4c} \cdot (1+A) - A \\ &= 1,\end{aligned}$$

on $\mathcal{M} \setminus D_{r_0}$. We have then proved that the asymptotic behavior of f predicted by **(L.3.1)** guarantees the existence of an $r_0 > 0$ depending only on A and ε such that

$$\Gamma \geq \frac{1}{2}, \text{ and } (1-\varepsilon f) - A \geq 1 \text{ on } \mathcal{M} \setminus D_{r_0}.$$

Therefore, defining $u \doteq \frac{\|\mathrm{Ric}\|^2}{\sqrt{\mathrm{Scal}}}$ and recalling that we took $a = \frac{1}{2}$, equation **(3.17)** can be written as

$$\begin{aligned}\Delta_f u &\geq \Gamma \cdot \frac{\|\mathrm{Ric}\|^4}{\mathrm{Scal}^{\frac{3}{2}}} - c \cdot \frac{\|\mathrm{Ric}\|^3}{\sqrt{\mathrm{Scal}}} - \frac{\|\mathrm{Ric}\|^2}{2 \cdot \sqrt{\mathrm{Scal}}} \\ &\geq \frac{1}{2} \cdot u^2 \cdot \mathrm{Scal}^{-\frac{1}{2}} - c \cdot u^{\frac{3}{2}} \cdot \mathrm{Scal}^{\frac{1}{4}} - \frac{1}{2}u \\ &\geq \frac{1}{2} \cdot u^2 \cdot \mathrm{Scal}^{-\frac{1}{2}} - c \cdot u^{\frac{3}{2}} \cdot \mathrm{Scal}^{\frac{1}{4}} - u,\end{aligned} \tag{3.19}$$

on $\mathcal{M} \setminus D_{r_0}$. Consider now a smooth function $\eta : [0, \infty) \to [0, 1]$ satisfying

$$\eta(x) = \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{1}{2}. \\ 1, & \text{if } 1 \leq t \leq 2. \\ 0, & \text{if } t > 3; \end{cases}$$

and let us define $\varphi : \mathcal{M} \to \mathbb{R}$ by $\varphi(x) = \eta\left(\frac{f(x)}{\rho^2}\right)$, where $\rho > 2 \cdot r_0$ is a constant.

Remark (O.84). Note that $\varphi(x) = 1$ whenever $\frac{f(x)}{\rho^2} \in [1, 2]$, and $\varphi(x) = 0$ whenever $\frac{f(x)}{\rho^2} > 3$ or $\frac{f(x)}{\rho^2} \leq \frac{1}{2}$. Therefore, in the argument that follows, we may without loss of generality work only with

the points $x \in \mathcal{M}$ satisfying

$$\frac{\rho^2}{2} \leq f(x) \leq 3 \cdot \rho^2.$$

Now, since

$$(\nabla\varphi)_x = \frac{1}{\rho^2} \cdot \eta'\left(\frac{f(x)}{\rho^2}\right) \cdot (\nabla f)_x, \ \forall x \in \mathcal{M},$$

and also (by part **(d)** of Lemma **(L.2.4)**):

$$\|\nabla f\|^2 = f - \mathrm{Scal} \leq f,$$

it follows that

$$\|\nabla\varphi\| \leq |\eta'| \cdot \frac{\|\nabla f\|}{\rho^2} \leq |\eta'| \cdot \frac{\sqrt{f}}{\rho^2} \leq |\eta'| \cdot \frac{\sqrt{3}}{\rho}.$$

Note also that for every $x \in \mathcal{M}$ we have

$$\begin{aligned}(\Delta_f\varphi)(x) &= \sum_{1\leq i\leq 4} [(\mathbf{e}_i \circ \mathbf{e}_i)(\varphi) - \mathbf{e}_i(f) \cdot \mathbf{e}_i(\varphi)] \\ &= \sum_{1\leq i\leq 4} \left[\frac{1}{\rho^4} \cdot \eta''\left(\frac{f(x)}{\rho^2}\right) \cdot (\mathbf{e}_i(f))^2 + \frac{1}{\rho^2} \cdot \eta'\left(\frac{f(x)}{\rho^2}\right) \cdot (\mathbf{e}_i \circ \mathbf{e}_i)(f) - \frac{1}{\rho^2} \cdot (\mathbf{e}_i(f))^2 \cdot \eta'\left(\frac{f(x)}{\rho^2}\right)\right] \\ &= \frac{1}{\rho^4} \cdot \eta''\left(\frac{f(x)}{\rho^2}\right) \cdot \|\nabla f\|^2 + \frac{1}{\rho^2} \cdot \eta'\left(\frac{f(x)}{\rho^2}\right) \cdot \Delta_f f.\end{aligned}$$

By construction, η and all its derivatives have compact support, so there exists a constant $C > 0$ such that

$$C > \max\left\{\frac{\eta''}{2 \cdot \rho^4}, \frac{\eta'}{2 \cdot \rho^2}\right\}.$$

Also using parts **(a)** and **(d)** of Lemma **(L.2.4)**, we see that

$$\Delta_f f = \Delta f - \nabla_{\nabla f} f = \Delta f - \|\nabla f\|^2 = 2 - \mathrm{Scal} - (f - \mathrm{Scal}) = 2 - f.$$

It then follows that:

$$\Delta_f\varphi = \eta'' \cdot \frac{\|\nabla f\|^2}{\rho^4} + \eta' \cdot \frac{1}{\rho^2} \cdot \Delta_f f \leq \eta'' \cdot \frac{\|\nabla f\|^2}{\rho^4} + \eta' \cdot \frac{2 - f}{\rho^2} \leq \frac{C}{2} \cdot 2 = C.$$

We now define $G = \varphi^2 \cdot u = \varphi^2 \cdot \dfrac{\|\mathrm{Ric}\|^2}{\sqrt{\mathrm{Scal}}}$, and from now on we will denote Scal by R. By Lemmas **(L.1.5)** and **(L.1.13)**, we have

$$\nabla G = \nabla(\varphi^2 \cdot u) = (\nabla u) \cdot \varphi^2 + u \cdot \nabla(\varphi^2), \text{ and } \Delta_f(\varphi^2) = 2 \cdot \|\nabla\varphi\|^2 + 2\varphi \cdot \Delta_f\varphi.$$

Therefore, using equation **(3.19)** and Lemma **(L.1.13)** again we see that

$$\begin{aligned}\varphi^2 \cdot \Delta_f G &= \varphi^2 \cdot \Delta_f(u \cdot \varphi^2) = \varphi^2 \cdot [u \cdot \Delta_f(\varphi^2) + \varphi^2 \cdot \Delta_f u + 2 \cdot g(\nabla u, \nabla[\varphi^2])] \\ &= \varphi^4 \cdot \Delta_f u + \varphi^2 u \cdot (2 \cdot \|\nabla\varphi\|^2 + 2\varphi \cdot \Delta_f\varphi) + 2 \cdot g((\nabla u) \cdot \varphi^2, \nabla[\varphi^2]) \\ &\geq \varphi^4 \left[\frac{1}{2}u^2 R^{-\frac{1}{2}} - cu^{\frac{3}{2}}R^{\frac{1}{4}} - cu\right] + \varphi^2 u \cdot (2 \cdot \|\nabla\varphi\|^2 + 2\varphi \cdot \Delta_f\varphi) + 2 \cdot g(\nabla G - u \cdot \nabla(\varphi^2), \nabla[\varphi^2])\end{aligned}$$

$$
\begin{aligned}
&= \varphi^4\left[\frac{1}{2}u^2R^{-\frac{1}{2}} - cu^{\frac{3}{2}}R^{\frac{1}{4}} - cu\right] + \varphi^2 u\cdot(2\cdot\|\nabla\varphi\|^2 + 2\varphi\cdot\Delta_f\varphi)\\
&+ 2\cdot g(\nabla G, \nabla[\varphi^2]) - 2u\cdot\left\|\nabla\left(\varphi^2\right)\right\|^2\\
&= \frac{1}{2}\cdot\varphi^4u^2R^{-\frac{1}{2}} - \varphi\cdot c\cdot\varphi^3u^{\frac{3}{2}}R^{\frac{1}{4}} - \varphi^2\cdot c\cdot\varphi^2u + 2\cdot g(\nabla G, \nabla[\varphi^2])\\
&+ \varphi^2u\cdot(2\cdot\|\nabla\varphi\|^2 + 2\varphi\cdot\Delta_f\varphi) - 8\varphi^2u\cdot\|\nabla\varphi\|^2.
\end{aligned}
$$

Now using that (by construction) φ and all its derivatives have compact support, we will take constants $c_1, c_2 > 0$ such that

$$2\cdot\|\nabla\varphi\|^2 + 2\varphi\cdot\Delta_f\varphi \geq -c_1, \text{ and } -8\cdot\|\nabla\varphi\|^2 \geq -c_2.$$

We then obtain (now also using that $0 \leq \varphi \leq 1$):

$$
\begin{aligned}
\varphi^2\cdot\Delta_f G &\geq \frac{1}{2}G^2R^{-\frac{1}{2}} - c\cdot G^{\frac{3}{2}}\cdot R^{\frac{1}{4}} - (c + c_1 + c_2)G + 2\cdot g(\nabla G, \nabla[\varphi^2])\\
&\geq \frac{1}{2}G^2R^{-\frac{1}{2}} - c_3\cdot G^{\frac{3}{2}}\cdot R^{\frac{1}{4}} - c_3G + 2\cdot g(\nabla G, \nabla[\varphi^2]),
\end{aligned}
$$

where $c_3 = c + c_1 + c_2$. For simplicity we will commit the abuse of notation of denoting c_3 by c. Now, since $D_{3\rho^2}$ is compact (as we saw in remark **(O.82)**), there exists $q \in D_{3\rho^2}$ such that $G(q) \geq G(x)\ \forall x \in D_{3\rho^2}$. At q, we then have (according to Lemma **(L.1.11)**):

$$\varphi^2\cdot(\Delta_f G)_q = \varphi^2[(\Delta G)(q) - g((\nabla f)_q, (\nabla G)_q)] = \varphi^2\cdot(\Delta G)(q) \leq 0,$$

and therefore

$$0 \geq \frac{1}{2}G^2(q)R^{\frac{1}{2}}(q) - c\cdot G^{\frac{3}{2}}(q) - cG(q).$$

Defining $v \doteq G^{\frac{1}{2}}$, it then follows that

$$0 \geq \frac{1}{2}v^4R^{-\frac{1}{2}} - c\cdot v^3R^{\frac{1}{4}} - v^2,$$

at q. Or, equivalently,

$$0 \geq \frac{1}{2}R^{-\frac{1}{2}}v^2 - cR^{\frac{1}{4}}v - c,$$

at q. We then see that v must satisfy (at q):

$$\frac{cR^{\frac{1}{4}} - \sqrt{c^2R^{\frac{1}{2}} + 2R^{-\frac{1}{2}}c}}{R^{-\frac{1}{2}}} \leq v \leq \frac{cR^{\frac{1}{4}} + \sqrt{c^2R^{\frac{1}{2}} + 2R^{-\frac{1}{2}}c}}{R^{-\frac{1}{2}}}.$$

Hence, at q we have

$$v \leq cR^{\frac{3}{4}} + \sqrt{c^2R^{\frac{3}{2}} + 2R^{\frac{1}{2}}} \leq cR^{\frac{3}{4}} + cR^{\frac{3}{4}} + \sqrt{2}\cdot R^{\frac{1}{4}} \leq 3c\cdot R^{\frac{3}{4}} = CR^{\frac{3}{4}},$$

where we assumed without loss of generality that $c \geq \sqrt{2}\cdot R^{-\frac{1}{2}}(q)$. Again we will commit the abuse of denoting C by c. Therefore,

$$v(q) \leq cR^{\frac{3}{4}}(q),$$

and consequently,

$$G(q) \leq cR^{\frac{3}{2}}(q).$$

By the way the point q was chosen and by our hypothesis on R, we also have that

$$G(x) \leq G(q) \leq cR^{\frac{3}{2}} \leq c \cdot (\varepsilon f(q) + A)^{\frac{3}{2}}, \ \forall x \in D_{3\rho^2}.$$

By definition, on $D_{2\rho^2} \setminus D_{\rho^2}$ we have $\rho^2 < f(x) \leq 2\rho^2$. Since by construction $q \in D_{3\rho^2}$, it follows that there exists $C > 0$ (it suffices to take $C \geq 3$) such that $Cf(x) \geq f(q), \ \forall x \in D_{2\rho^2} \setminus D_{\rho^2}$.

It then follows that on $D_{2\rho^2} \setminus D_{\rho^2}$, $\|\mathrm{Ric}\|$ satisfies:

$$\begin{aligned} \|\mathrm{Ric}\|^2(x) = G(x)R^{\frac{1}{2}}(x) &\leq G(q)R^{\frac{1}{2}}(x) \\ &\leq c \cdot (\varepsilon f(q) + A)^{\frac{3}{2}} \cdot (\varepsilon f(x) + A)^{\frac{1}{2}} \\ &\leq c \cdot (\varepsilon C f(x) + A)^{\frac{3}{2}} \cdot (\varepsilon f(x) + A)^{\frac{1}{2}}. \end{aligned}$$

Now take a constant $C' > 0$ such that

$$C' \geq \left(\frac{2C\varepsilon\rho^2 + A}{2\varepsilon\rho^2 + A} \right)^{\frac{3}{2}}.$$

Such a C' obviously satisfies:

$$C' \geq \left(\frac{C\varepsilon f(x) + A}{\varepsilon f(x) + A} \right)^{\frac{3}{2}}, \ \forall x \in D_{2\rho^2} \setminus D_{\rho^2} \iff (C\varepsilon f(x) + A)^{\frac{3}{2}} \leq C'(\varepsilon f(x) + A)^{\frac{3}{2}}, \ \forall x \in D_{2\rho^2} \setminus D_{\rho^2}.$$

And, therefore, on $D_{2\rho^2} \setminus D_{\rho^2}$, $\|\mathrm{Ric}\|$ satisfies:

$$\|\mathrm{Ric}\|^2(x) \leq C_1 \cdot (\varepsilon f(x) + A)^2,$$

where $C_1 \doteq c \cdot C'$. Then defining $c_0 \doteq A\sqrt{C_1}$ and $c_1 \doteq \sqrt{C'}$, we obtain:

$$\|\mathrm{Ric}\|(x) \leq c_0 + (c_1\varepsilon) \cdot f(x), \ \forall x \in D_{2\rho^2} \setminus D_{\rho^2}.$$

Since $\rho > 0$ can be taken arbitrarily large, we then have (now denoting $4r_0^2$ by r_0):

$$\|\mathrm{Ric}\|(x) \leq c_0 + (c_1\varepsilon) \cdot f(x), \ \forall x \in \bigcup_{\rho > 2r_0} f^{-1}((\rho^2, 2\rho^2]) = \mathcal{M} \setminus D_{r_0}.$$

Since D_{r_0} is compact, there exist constants $M, \beta \in \mathbb{R}$ such that $\|\mathrm{Ric}\|(x) \leq M$ and $f(x) \geq \beta$ whenever $x \in D_{r_0}$. Taking $c_0 > 0$ large enough, we then have $\|\mathrm{Ric}\| \leq c_0 + (c_1\varepsilon)f$ on D_{r_0} as well, so that the estimate

$$\|\mathrm{Ric}\| \leq c_0 + (c_1\varepsilon)f$$

is in fact valid on all of $\mathcal{M}$, as desired.

We will now deal with the second estimate of the Theorem, that is, the estimate on the curvature tensor Rm. Again we will use the same approach seen in [6]. First, note that (using **(2.13)** and **(2.14)**),

the coordinate representation (again, in a geodesic frame) of (2.15) is given by:

$$(\Delta_f \mathrm{Rm})_{ijk\ell} = \mathrm{Rm}_{ijk\ell} - 2 \cdot \sum_{1\leq s,t\leq 4} (\mathrm{Rm}_{\ell kst}\mathrm{Rm}_{ijst} - \mathrm{Rm}_{jst\ell}\mathrm{Rm}_{sikt} + \mathrm{Rm}_{ist\ell}\mathrm{Rm}_{sjkt}.) \tag{3.20}$$

Now, note that (by Lemma (L.1.13))

$$\Delta_f(\|\mathrm{Rm}\|^2) = 2 \cdot \|\mathrm{Rm}\| \cdot \Delta_f(\|\mathrm{Rm}\|) + 2 \cdot \|\nabla\|\mathrm{Rm}\|\|^2,$$

and therefore

$$\|\mathrm{Rm}\| \cdot \Delta_f(\|\mathrm{Rm}\|) = \Delta_f(\|\mathrm{Rm}\|^2) - \|\nabla\|\mathrm{Rm}\|\|^2.$$

But since

$$\begin{aligned}\Delta_f(\|\mathrm{Rm}\|^2) = \Delta_f\left(\sum_{1\leq i,j,k,\ell\leq 4} (\mathrm{Rm}_{ijk\ell})^2\right) &= \sum_{i,j,k,\ell}\left[\mathrm{Rm}_{ijk\ell}\cdot\Delta_f\mathrm{Rm}_{ijk\ell} + \|\nabla\mathrm{Rm}_{ijk\ell}\|^2\right] \\ &= \left(\sum_{i,j,k,\ell}\mathrm{Rm}_{ijk\ell}\cdot\Delta_f\mathrm{Rm}_{ijk\ell}\right) + \|\nabla\mathrm{Rm}\|^2,\end{aligned}$$

and by Kato's inequality we have

$$\|\nabla\mathrm{Rm}\|^2 - \|\nabla\|\mathrm{Rm}\|\|^2 \geq 0,$$

it then follows from equation (3.20) that

$$\begin{aligned}\|\mathrm{Rm}\| \cdot \Delta_f(\|\mathrm{Rm}\|) &\geq \sum_{i,j,k,\ell}\mathrm{Rm}_{ijk\ell}\cdot\Delta_f\mathrm{Rm}_{ijk\ell} \\ &= \|\mathrm{Rm}\|^2 - 2\cdot\sum_{i,j,k,\ell,s,t}\mathrm{Rm}_{ijk\ell}(\mathrm{Rm}_{\ell kst}\mathrm{Rm}_{ijst} - \mathrm{Rm}_{jst\ell}\mathrm{Rm}_{sikt} + \mathrm{Rm}_{ist\ell}\mathrm{Rm}_{sjkt}).\end{aligned}$$

We can deal with all the terms in the sums appearing in the second summand of the expression above in the obvious way (using the Cauchy-Schwarz inequality), so that we obtain

$$\Delta_f(\|\mathrm{Rm}\|) \geq \|\mathrm{Rm}\| - c\cdot\|\mathrm{Rm}\|^2 \geq -c\cdot\|\mathrm{Rm}\|^2. \tag{3.21}$$

On the other hand, it follows from equation (3.12) that

$$\begin{aligned}\Delta_f(\|\mathrm{Ric}\|^2) &\geq 2\cdot\|\nabla\mathrm{Ric}\|^2 + 2\cdot\|\mathrm{Ric}\|^2 - 4\cdot\|\mathrm{Rm}\|\cdot\|\mathrm{Ric}\|^2 \\ &\geq 2\cdot\|\nabla\mathrm{Ric}\|^2 + 2\cdot\|\mathrm{Ric}\|^2 - C\cdot\|\mathrm{Ric}\|^3 - C\cdot\frac{\|\nabla\mathrm{Ric}\|}{\|\nabla f\|}\cdot\|\mathrm{Ric}\|^2.\end{aligned}$$

Thus, using equation (3.21), we obtain:

$$\begin{aligned}\Delta_f\left(\|\mathrm{Rm}\| + \|\mathrm{Ric}\|^2\right) &\geq \underbrace{-c}_{=1-(c+1)}\cdot\|\mathrm{Rm}\|^2 + 2\cdot\|\nabla\mathrm{Ric}\|^2 + 2\cdot\|\mathrm{Ric}\|^2 - C\cdot\|\mathrm{Ric}\|^3 - C\cdot\frac{\|\nabla\mathrm{Ric}\|}{\|\nabla f\|}\cdot\|\mathrm{Ric}\|^2 \\ &= \|\mathrm{Rm}\|^2 - (c+1)\cdot\|\mathrm{Rm}\|^2 + 2\cdot\|\nabla\mathrm{Ric}\|^2 + 2\cdot\|\mathrm{Ric}\|^2\end{aligned}$$

$$- C \cdot \|\mathrm{Ric}\|^3 - C \cdot \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|} \cdot \|\mathrm{Ric}\|^2.$$

Now, by Proposition **(P.3.10)**, there exists $C_0 > 0$ such that:

$$-(c+1) \cdot \|\mathrm{Rm}\|^2 \geq -C_0 \cdot \|\mathrm{Ric}\|^2 - C_0 \cdot \frac{\|\nabla \mathrm{Ric}\| \cdot \|\mathrm{Ric}\|}{\|\nabla f\|} - C_0 \cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\|\nabla f\|^2}.$$

Using the fact that

$$-2ab \geq -(a^2 + b^2), \ \forall a, b \in \mathbb{R},$$

it then follows that there exists $c_0 > 0$ such that

$$-(c+1) \cdot \|\mathrm{Rm}\|^2 \geq -c_0 \cdot \|\mathrm{Ric}\|^2 - c_0 \cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\|\nabla f\|^2}.$$

Hence, (now denoting c_0 by c)

$$\begin{aligned} \Delta_f \left(\|\mathrm{Rm}\| + \|\mathrm{Ric}\|^2 \right) \geq \|\mathrm{Rm}\|^2 - c \cdot \|\mathrm{Ric}\|^2 - c \cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\|\nabla f\|^2} \\ + 2 \cdot \|\nabla \mathrm{Ric}\|^2 - C \cdot \|\mathrm{Ric}\|^3 - C \cdot \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|} \cdot \|\mathrm{Ric}\|^2. \end{aligned} \tag{3.22}$$

Now, from

$$\left(\frac{C}{2} \cdot \|\mathrm{Ric}\|^2 - \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|} \right)^2 \geq 0,$$

we obtain

$$C \cdot \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|} \cdot \|\mathrm{Ric}\|^2 \leq \frac{\|\nabla \mathrm{Ric}\|^2}{\|\nabla f\|^2} + \frac{C^2}{4} \cdot \|\mathrm{Ric}\|^4.$$

A fortiori,

$$-C \cdot \frac{\|\nabla \mathrm{Ric}\|}{\|\nabla f\|} \cdot \|\mathrm{Ric}\|^2 \geq -2 \cdot \|\nabla \mathrm{Ric}\|^2 - \frac{C^2}{4} \cdot \|\mathrm{Ric}\|^4.$$

Thus (now already assuming without loss of generality that $c \geq C$ and $C^2 \geq 4c$), equation **(3.22)** can be written as

$$\begin{aligned} \Delta_f \left(\|\mathrm{Rm}\| + \|\mathrm{Ric}\|^2 \right) \geq \|\mathrm{Rm}\|^2 - c \cdot \|\mathrm{Ric}\|^2 - c \cdot \|\mathrm{Ric}\|^3 - c \cdot \|\mathrm{Ric}\|^4 \\ + 2 \cdot \|\nabla \mathrm{Ric}\|^2 - c \cdot \frac{\|\nabla \mathrm{Ric}\|^2}{\|\nabla f\|^2} - \frac{\|\nabla \mathrm{Ric}\|^2}{\|\nabla f\|^2}. \end{aligned} \tag{3.23}$$

We may also assume without loss of generality that r_0 is large enough so that $\|\nabla f\|^2 \geq (1-\varepsilon) f - A \geq \frac{c+1}{2}$. In this way, equation **(3.23)** can be written as

$$\Delta_f \left(\|\mathrm{Rm}\| + \|\mathrm{Ric}\|^2 \right) \geq \|\mathrm{Rm}\|^2 - c \cdot \|\mathrm{Ric}\|^2 - c \cdot \|\mathrm{Ric}\|^3 - c \cdot \|\mathrm{Ric}\|^4,$$

on $D_{2\rho^2} \setminus D_{\rho^2}$. The compactness of $D_{2\rho^2}$ also guarantees the existence of $c_0, A > 0$ such that $-c_0 \|\mathrm{Ric}\|^2 \geq -A\|\mathrm{Ric}\|^3$ on $D_{2\rho^2}$. Committing the usual abuse of not making the renaming of constants explicit, we

have

$$\Delta_f\left(\|\mathrm{Rm}\|+\|\mathrm{Ric}\|^2\right)\geq\|\mathrm{Rm}\|^2-c\cdot\|\mathrm{Ric}\|^3-c\cdot\|\mathrm{Ric}\|^4,$$

on $D_{2\rho^2}\setminus D_{\rho^2}$. Again, since $\rho>2\cdot r_0$ can be taken arbitrarily large, we conclude that

$$\Delta_f\left(\|\mathrm{Rm}\|+\|\mathrm{Ric}\|^2\right)\geq\|\mathrm{Rm}\|^2-c\cdot\|\mathrm{Ric}\|^3-c\cdot\|\mathrm{Ric}\|^4$$

holds on $\mathcal{M}\setminus D_{r_0}$. Now, from

$$a^2=\frac{1}{2}\left[(a+b^2)^2+(a-b^2)^2-2b^4\right]\geq\frac{1}{2}(a+b^2)^2-2b^4,\ \forall a,b\in\mathbb{R},$$

we conclude that

$$\Delta_f\left(\|\mathrm{Rm}\|+\|\mathrm{Ric}\|^2\right)\geq\frac{1}{2}\cdot\left(\|\mathrm{Rm}\|+\|\mathrm{Ric}\|^2\right)^2-c\cdot\|\mathrm{Ric}\|^4-c\cdot\|\mathrm{Ric}\|^3,$$

on $\mathcal{M}\setminus D_{r_0}$. Moreover, using the estimate $\|\mathrm{Ric}\|\leq c_0+(c_1\varepsilon)f$ established in the first part of the proof, we see that

$$\Delta_f\left(\|\mathrm{Rm}\|+\|\mathrm{Ric}\|^2\right)\geq\frac{1}{2}\cdot\left(\|\mathrm{Rm}\|+\|\mathrm{Ric}\|^2\right)^2-\beta(f),\tag{3.24}$$

on $\mathcal{M}\setminus D_{r_0}$, where $\beta(f)\doteq c\cdot\left[(c_0+(c_1\varepsilon)f)^4+(c_0+(c_1\varepsilon)f)^3\right]$. Just as we did in the first part of the proof, let us now consider the same functions η and $\varphi=\eta\left(\frac{f(x)}{\rho^2}\right)$, which have compact supports contained in $\left[2^{-1},2\right]$ and $D_{3\rho^2}$, respectively. We will also define $Q\doteq\varphi^2\cdot\psi$, with $\psi\doteq\|\mathrm{Rm}\|+\|\mathrm{Ric}\|^2$. By Lemma **(L.1.13)**, we see that

$$\Delta_f Q=\psi\cdot\Delta_f(\varphi^2)+\varphi^2\cdot(\Delta_f\psi)+2\cdot g\left(\nabla[\varphi^2],\nabla\psi\right).$$

Now, recalling that (by Lemma **(L.1.5)**)

$$g\left(\nabla[\varphi^2],\nabla Q\right)=g\left(\nabla\varphi^2,\psi\cdot\nabla[\varphi^2]+\varphi^2\cdot\nabla\psi\right)=\psi\cdot\|\nabla(\varphi^2)\|^2+\varphi^2\cdot g\left(\nabla[\varphi^2],\nabla\psi\right),$$

we obtain

$$\begin{aligned}\varphi^2\cdot\Delta_f Q&=(\varphi^2\psi)\cdot\Delta_f(\varphi^2)+\varphi^4\cdot\Delta_f\psi+2\varphi^2\cdot g\left(\nabla[\varphi^2],\nabla\psi\right)\\&=(\varphi^2\psi)\cdot\Delta_f(\varphi^2)+\varphi^4\cdot\Delta_f\psi+2\cdot g\left(\nabla(\varphi^2),\nabla Q\right)-2\psi\cdot\|\nabla(\varphi^2)\|^2.\end{aligned}$$

Therefore, using that $\Delta_f(\varphi^2)=2\cdot\|\nabla\varphi\|^2+2\cdot\varphi\cdot\Delta_f\varphi$ and recalling equation **(3.24)**, we obtain

$$\varphi^2\cdot\Delta_f Q=Q\cdot(2\cdot\|\nabla\varphi\|^2+2\cdot\varphi\cdot\Delta_f\varphi)+\varphi^4\cdot\underbrace{\Delta_f\psi}_{\geq\frac{1}{2}\psi^2-\beta}+2\cdot g\left(\nabla(\varphi^2),\nabla Q\right)-\underbrace{2\psi\cdot\|\nabla(\varphi^2)\|^2}_{-8Q\cdot\|\nabla\varphi\|^2}.$$

As we argued before, the fact that φ and η and all their derivatives have compact support guarantees the existence of a constant $c>0$ such that

$$2\cdot\|\nabla\varphi\|^2+2\cdot\varphi\cdot\Delta_f\varphi\geq-c,\ \text{and}\ -8\cdot\|\nabla\varphi\|^2\geq-c.$$

Hence, also using that $0 \leq \varphi \leq 1$, we obtain

$$\varphi^2 \cdot \Delta_f Q \geq \frac{1}{2}Q^2 - cQ + 2g\left(\nabla[\varphi^2], \nabla Q\right) - \beta.$$

Using Lemma **(L.1.11)**, we then obtain that at a maximum point of Q in $D_{3\rho^2}$, say $y \in \mathcal{M}$, we have

$$0 \geq \left(\frac{1}{2}Q^2 - cQ - \beta\right)(y).$$

It follows that

$$c - \sqrt{c^2 + 2\beta} \leq Q(y) \leq c + \sqrt{c^2 + 2\beta}.$$

Using repeatedly the inequality

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, \ \forall a, b \in \mathbb{R}_{\geq 0},$$

we then obtain

$$Q(y) \leq (c + c_0 \cdot \sqrt{f} + c_1 \cdot f + c_2 \cdot f^2 + c_3 \cdot f^{\frac{3}{2}})(y).$$

Using the fact that $y \in D_{3\rho^2}$ is a maximum point and that ρ can be arbitrarily large, we conclude that

$$Q = \varphi^2 \cdot (\|\mathrm{Rm}\| + \|\mathrm{Ric}\|^2) \leq c + (c_2\varepsilon)f^2,$$

on $\mathcal{M} \setminus D_{r_0}$. Since φ has compact support (in particular there exist constants $s, t \in \mathbb{R}$ such that $\varphi([0,\infty)) \subset [s,t]$), we conclude that

$$\|\mathrm{Rm}\| \leq \|\mathrm{Rm}\| + \|\mathrm{Ric}\|^2 \leq c + (c_2\varepsilon)f^2,$$

on $\mathcal{M} \setminus D_{r_0}$. As we argued in the first part of the proof, the constant c can be chosen so that the desired estimate holds on all of $\mathcal{M}$. This completes the proof of the Theorem.

■

Appendix

Some important computations

On the extension of the metric to tensors: There is a subtlety implicit in definition (D.1.21). What we are actually saying is that to compute the value of the function $g(P,Q)$ at a point $q \in \mathcal{M}$, we take a chart φ around a neighborhood $U_q \ni q$ (which is where the local components of the definition above come from, which are computed using the frame and coframe induced by φ), and then we compute the expression

$$\left(\sum_{\substack{1\leq i_1,\dots,i_k\leq n\\ 1\leq j_1,\dots,j_\ell\leq n\\ 1\leq r_1,\dots,r_k\leq n\\ 1\leq s_1,\dots,s_\ell\leq n}} g_{i_1r_1}g_{i_2r_2}\cdots g_{i_kr_k}g^{j_1s_1}g^{j_2s_2}\cdots g^{j_\ell s_\ell}P^{i_1\dots i_k}_{j_1\dots j_\ell}Q^{r_1\dots r_k}_{s_1\dots s_\ell} \right)(q). \tag{4.25}$$

A priori, it could happen that, by taking another chart ψ around x, which induces another frame $\left\{\dfrac{\partial}{\partial \widetilde{x}^i}\right\}_{1\leq i\leq n}$ and another coframe $\{\mathrm{d}\widetilde{x}^i\}_{1\leq i\leq n}$, we would have

$$\left(\sum_{\substack{1\leq i_1,\dots,i_k\leq n\\ 1\leq j_1,\dots,j_\ell\leq n\\ 1\leq r_1,\dots,r_k\leq n\\ 1\leq s_1,\dots,s_\ell\leq n}} g_{i_1r_1}g_{i_2r_2}\cdots g_{i_kr_k}g^{j_1s_1}g^{j_2s_2}\cdots g^{j_\ell s_\ell}P^{i_1\dots i_k}_{j_1\dots j_\ell}Q^{r_1\dots r_k}_{s_1\dots s_\ell} \right)(q)$$

$$\neq \left(\sum_{\substack{1\leq i_1,\dots,i_k\leq n\\ 1\leq j_1,\dots,j_\ell\leq n\\ 1\leq r_1,\dots,r_k\leq n\\ 1\leq s_1,\dots,s_\ell\leq n}} g_{\widetilde{i}_1\widetilde{r}_1}g_{\widetilde{i}_2\widetilde{r}_2}\cdots g_{\widetilde{i}_k\widetilde{r}_k}g^{\widetilde{j}_1\widetilde{s}_1}g^{\widetilde{j}_2\widetilde{s}_2}\cdots g^{\widetilde{j}_\ell\widetilde{s}_\ell}P^{\widetilde{i}_1\dots\widetilde{i}_k}_{\widetilde{j}_1\dots\widetilde{j}_\ell}Q^{\widetilde{r}_1\dots\widetilde{r}_k}_{\widetilde{s}_1\dots\widetilde{s}_\ell} \right)(q).$$

We will now verify that this does not happen. To avoid extreme notational overload, we will use Einstein notation in the computations below (that is, we will adopt the convention that when an index appears both up and down, we are summing over that index). Let us fix a new chart $\psi : V \subset \mathcal{M}^n \to \mathbb{R}^n$. We will denote by $\widetilde{T}^{j_1\dots j_a}_{i_1\dots i_b}$ the components of a tensor T in the new coordinate system ψ, and we will also use the notation

$$\left.\frac{\partial}{\partial x^{j_k}}\right|_p = \partial_k,$$

and

$$\left.\mathrm{d}x^{j_k}\right|_p = \partial^k.$$

We then want to establish the following equality:

$$g_{j_1r_1}\dots g_{j_ar_a}g^{i_1m_1}\dots g^{i_bm_b}P^{j_1\dots j_a}_{i_1\dots i_b}Q^{r_1\dots r_a}_{m_1\dots m_b} = \widetilde{g}_{\mu_{j_1}\mu_{r_1}}\dots\widetilde{g}_{\mu_{j_a}\mu_{r_a}}\widetilde{g}^{\theta_{i_1}\theta_{m_1}}\dots\widetilde{g}^{\theta_{i_b}\theta_{m_b}}\widetilde{P}^{\mu_{j_1}\dots\mu_{j_a}}_{\theta_{i_1}\dots\theta_{i_b}}\widetilde{Q}^{\mu_{r_1}\dots\mu_{r_a}}_{\theta_{m_1}\dots\theta_{m_b}}.$$

We will call the left-hand side of the expression above A and the right-hand side B. Let us first observe that if $\{\mathbf{e}_i\}_{1\le i\le n}$ is a basis of a vector space $\mathbb{V}$ and $\{\mathbf{e}^i\}_{1\le i\le n}$ is its associated dual basis, then every $v \in \mathbb{V}$ is written as

$$v = \mathbf{e}^i(v)\,\mathbf{e}_i,$$

and every $f \in \mathbb{V}^*$ is written as

$$f = f(\mathbf{e}_i)\,\mathbf{e}^i.$$

In particular, for each $1 \le k \le n$, we have

$$\partial_{j_k} = \widetilde{\partial}^{\mu_{j_k}}(\partial_{j_k})\widetilde{\partial}_{\mu_{j_k}}.$$

Analogously, we have

$$\partial^{i_k} = \partial^{i_k}(\widetilde{\partial}_{\theta_{i_k}})\widetilde{\partial}^{\theta_{i_k}},$$

Therefore, defining the expressions A_1, A_2, A_3 and A_4 by:

$$\begin{aligned}
A_1 &\doteq g_{j_1r_1}\dots g_{j_ar_a},\\
A_2 &\doteq g^{i_1m_1}\dots g^{i_bm_b},\\
A_3 &\doteq P^{j_1\dots j_a}_{i_1\dots i_b},\\
A_4 &\doteq Q^{r_1\dots r_a}_{m_1\dots m_b};
\end{aligned}$$

we see that

$$\begin{aligned}
A_1 &= \widetilde{\partial}^{\mu_{j_1}}(\partial_{j_1})\widetilde{\partial}^{\mu_{r_1}}(\partial_{j_1})\dots\widetilde{\partial}^{\mu_{j_a}}(\partial_{j_a})\widetilde{\partial}^{\mu_{r_a}}(\partial_{r_a})\widetilde{g}_{\mu_{j_1}\mu_{r_1}}\dots\widetilde{g}_{\mu_{j_a}\mu_{r_a}},\\
A_2 &= \partial^{i_1}(\widetilde{\partial}_{\theta_{i_1}})\partial^{m_1}(\widetilde{\partial}_{\theta_{m_1}})\dots\partial^{i_b}(\widetilde{\partial}_{\theta_{i_b}})\partial^{m_b}(\widetilde{\partial}_{\theta_{m_b}})\widetilde{g}^{\theta_{i_1}\theta_{m_1}}\dots\widetilde{g}^{\theta_{i_b}\theta_{m_b}},\\
A_3 &= \partial^{j_1}(\widetilde{\partial}_{\beta_{j_1}})\dots\partial^{j_a}(\widetilde{\partial}_{\beta_{j_a}})\widetilde{\partial}^{\lambda_{i_1}}(\partial_{i_1})\dots\widetilde{\partial}^{\lambda_{i_b}}(\partial_{i_b})\widetilde{P}^{\beta_{j_1}\dots\beta_{j_a}}_{\lambda_{i_1}\dots\lambda_{i_b}},\\
A_4 &= \partial^{r_1}(\widetilde{\partial}_{\xi_{r_1}})\widetilde{\partial}^{\xi_{r_1}}\dots\partial^{r_a}(\widetilde{\partial}_{\xi_{r_a}})\widetilde{\partial}^{\xi_{r_a}}\widetilde{\partial}^{\eta_{m_1}}(\partial_{m_1})\dots\widetilde{\partial}^{\eta_{m_b}}(\partial_{m_b})\widetilde{Q}^{\xi_{r_1}\dots\xi_{r_a}}_{\eta_{m_1}\dots\eta_{m_b}}.
\end{aligned}$$

Grouping the terms conveniently, it follows that

$$
\begin{aligned}
A &= \left(\prod_{1\le k\le a} \widetilde{\partial}^{\mu_{j_k}}(\partial_{j_k})\partial^{j_k}(\widetilde{\partial}_{\beta_{j_k}})\right)\cdot\left(\prod_{1\le k\le a} \widetilde{\partial}^{\mu_{r_k}}(\partial_{r_k})\partial^{r_k}(\widetilde{\partial}_{\xi_{r_k}})\right)\\
&\cdot\left(\prod_{1\le k\le b} \widetilde{\partial}^{\lambda_{i_k}}(\partial_{i_k})\partial^{i_k}(\widetilde{\partial}_{\theta_{i_k}})\right)\cdot\left(\prod_{1\le k\le b} \widetilde{\partial}^{\eta_{m_k}}(\partial_{m_k})\partial^{m_k}(\widetilde{\partial}_{\theta_{m_k}})\right)\cdot \widetilde{g}_{\mu_{j_1}\mu_{r_1}}\dots\widetilde{g}_{\mu_{j_a}\mu_{r_a}}\widetilde{g}^{\theta_{i_1}\theta_{m_1}}\dots\widetilde{g}^{\theta_{i_b}\theta_{m_b}}\\
&\cdot \widetilde{P}^{\beta_{j_1}\dots\beta_{j_a}}_{\lambda_{i_1}\dots\lambda_{i_b}}\widetilde{Q}^{\xi_{r_1}\dots\xi_{r_a}}_{\eta_{m_1}\dots\eta_{m_b}}\\
&= \left(\prod_{1\le k\le a} \widetilde{\partial}^{\mu_{j_k}}(\partial^{j_k}(\widetilde{\partial}_{\beta_{j_k}})\partial_{j_k})\right)\cdot\left(\prod_{1\le k\le a} \widetilde{\partial}^{\mu_{r_k}}(\partial^{r_k}(\widetilde{\partial}_{\xi_{r_k}})\partial_{r_k})\right)\\
&\cdot\left(\prod_{1\le k\le b} \widetilde{\partial}^{\lambda_{i_k}}(\partial^{i_k}(\widetilde{\partial}_{\theta_{i_k}})\partial_{i_k})\right)\cdot\left(\prod_{1\le k\le b} \widetilde{\partial}^{\eta_{m_k}}(\partial^{m_k}(\widetilde{\partial}_{\theta_{m_k}})\partial_{m_k})\right)\widetilde{g}_{\mu_{j_1}\mu_{r_1}}\dots\widetilde{g}_{\mu_{j_a}\mu_{r_a}}\widetilde{g}^{\theta_{i_1}\theta_{m_1}}\dots\widetilde{g}^{\theta_{i_b}\theta_{m_b}}\text{Í}\\
&\widetilde{P}^{\beta_{j_1}\dots\beta_{j_a}}_{\lambda_{i_1}\dots\lambda_{i_b}}\widetilde{Q}^{\xi_{r_1}\dots\xi_{r_a}}_{\eta_{m_1}\dots\eta_{m_b}}\\
&= \left(\prod_{1\le k\le a} \widetilde{\partial}^{\mu_{j_k}}(\widetilde{\partial}_{\beta_{j_k}})\right)\cdot\left(\prod_{1\le k\le a} \widetilde{\partial}^{\mu_{r_k}}(\widetilde{\partial}_{\xi_{r_k}})\right)\cdot\left(\prod_{1\le k\le b} \widetilde{\partial}^{\lambda_{i_k}}(\widetilde{\partial}_{\theta_{i_k}})\right)\cdot\left(\prod_{1\le k\le b} \widetilde{\partial}^{\eta_{m_k}}(\widetilde{\partial}_{\theta_{m_k}})\right)\\
&\cdot \widetilde{g}_{\mu_{j_1}\mu_{r_1}}\dots\widetilde{g}_{\mu_{j_a}\mu_{r_a}}\cdot\widetilde{g}^{\theta_{i_1}\theta_{m_1}}\dots\widetilde{g}^{\theta_{i_b}\theta_{m_b}}\widetilde{P}^{\beta_{j_1}\dots\beta_{j_a}}_{\lambda_{i_1}\dots\lambda_{i_b}}\widetilde{Q}^{\xi_{r_1}\dots\xi_{r_a}}_{\eta_{m_1}\dots\eta_{m_b}}\\
&= \left(\prod_{1\le k\le a} \delta^{\mu_{j_k}}_{\beta_{j_k}}\right)\cdot\left(\prod_{1\le k\le a} \delta^{\mu_{r_k}}_{\xi_{r_k}}\right)\cdot\left(\prod_{1\le k\le b} \delta^{\lambda_{i_k}}_{\theta_{i_k}}\right)\cdot\left(\prod_{1\le k\le b} \delta^{\eta_{m_k}}_{\theta_{m_k}}\right)\\
&\cdot\cdot\widetilde{g}_{\mu_{j_1}\mu_{r_1}}\dots\widetilde{g}_{\mu_{j_a}\mu_{r_a}}\cdot\widetilde{g}^{\theta_{i_1}\theta_{m_1}}\dots\widetilde{g}^{\theta_{i_b}\theta_{m_b}}\widetilde{P}^{\beta_{j_1}\dots\beta_{j_a}}_{\lambda_{i_1}\dots\lambda_{i_b}}\widetilde{Q}^{\xi_{r_1}\dots\xi_{r_a}}_{\eta_{m_1}\dots\eta_{m_b}}\\
&= \widetilde{g}_{\mu_{j_1}\mu_{r_1}}\dots\widetilde{g}_{\mu_{j_a}\mu_{r_a}}\widetilde{g}^{\theta_{i_1}\theta_{m_1}}\dots\widetilde{g}^{\theta_{i_b}\theta_{m_b}}\widetilde{P}^{\mu_{j_1}\dots\mu_{j_a}}_{\theta_{i_1}\dots\theta_{i_b}}\widetilde{Q}^{\mu_{r_1}\dots\mu_{r_a}}_{\theta_{m_1}\dots\theta_{m_b}}\\
&= B.
\end{aligned}
$$

Proof of Identity (1.10): It is a direct consequence of the definitions of ∇^2 and of Rm. First, note that we may assume without loss of generality that $\nabla_w v_i = 0$ for every $w \in T_p\mathcal{M}$ (see remark **(O.3)**) and $1 \le i \le n$. We then have

$$
\begin{aligned}
(\nabla^2_{x,y}T)(v_1,\dots,v_k) &= [\nabla_x(\nabla_y T)](v_1,\dots,v_k) - (\nabla_{\nabla_x y}T)(v_1,\dots,v_k)\\
&= \nabla_x\left[(\nabla_y T)(v_1,\dots,v_k)\right] - \sum_{1\le i\le k}(\nabla_y T)(v_1,\dots,\nabla_x v_i,\dots,v_k)\\
&\quad - \nabla_{\nabla_x y}(T(v_1,\dots,v_k)) + \sum_{1\le i\le k} T(v_1,\dots,\nabla_{\nabla_x y}v_i,\dots,v_k)\\
&= \nabla_x\left[\nabla_y(T(v_1,\dots,v_k)) - \sum_{1\le i\le k} T(v_1,\dots,\nabla_y v_i,\dots,v_k)\right]\\
&\quad - \sum_{1\le i\le k}\left[\nabla_y(T(v_1,\dots,\nabla_x v_i,\dots,v_k)) - \sum_{\substack{1\le j\le k\\ j\ne i}} T(v_1,\dots,\nabla_y v_j,\dots,\nabla_x v_i,\dots,v_k)\right]\\
&\quad + \sum_{1\le i\le k} T(v_1,\dots,\nabla_y\nabla_x v_i,\dots,v_k)\\
&\quad - \nabla_{\nabla_x y}(T(v_1,\dots,v_k)) + \sum_{1\le i\le k} T(v_1,\dots,\nabla_{\nabla_x y}v_i,\dots,v_k).
\end{aligned}
$$

Hence,

$$\begin{aligned}
(\mathrm{Rm}_{x,y}T)(v_1,\dots,v_k) &= (\nabla^2_{x,y}T - \nabla^2_{y,x}T)(v_1,\dots,v_k)\\
&= (\nabla_x\nabla_y)(T(v_1,\dots,v_k)) - (\nabla_y\nabla_x)(T(v_1,\dots,v_k))\\
&- \sum_{1\le i\le k} \nabla_x(T(v_1,\dots,\nabla_y v_i,\dots,v_k)) + \sum_{1\le i\le k} \nabla_y(T(v_1,\dots,\nabla_x v_i,\dots,v_k))\\
&- \sum_{1\le i\le k} \nabla_y(T(v_1,\dots,\nabla_x v_i,\dots,v_k)) + \sum_{1\le i\le k} \nabla_x(T(v_1,\dots,\nabla_y v_i,\dots,v_k))\\
&+ \sum_{\substack{1\le i\le k\\ 1\le j\le k\\ j\neq i}} T(v_1,\dots,\nabla_y v_j,\dots,\nabla_x v_i,\dots,v_k) - \sum_{\substack{1\le i\le k\\ 1\le j\le k\\ j\neq i}} T(v_1,\dots,\nabla_x v_j,\dots,\nabla_y v_i,\dots,v_k)\\
&+ \sum_{1\le i\le k} T(v_1,\dots,\nabla_y\nabla_x v_i,\dots,v_k) - \sum_{1\le i\le k} T(v_1,\dots,\nabla_x\nabla_y v_i,\dots,v_k)\\
&- \nabla_{\nabla_x y}(T(v_1,\dots,v_k)) + \nabla_{\nabla_y x}(T(v_1,\dots,v_k))\\
&+ \sum_{1\le i\le k} T(v_1,\dots,\nabla_{\nabla_x y} v_i,\dots,v_k) - \sum_{1\le i\le k} T(v_1,\dots,\nabla_{\nabla_y x} v_i,\dots,v_k).
\end{aligned}$$

We will denote the terms in the expression above as follows:

$$\begin{aligned}
A_1 &\doteq (\nabla_x\nabla_y)(T(v_1,\dots,v_k)), & A_2 &\doteq (\nabla_y\nabla_x)(T(v_1,\dots,v_k)),\\
B_1 &\doteq \sum_{1\le i\le k} \nabla_x(T(v_1,\dots,\nabla_y v_i,\dots,v_k)), & C_1 &\doteq \sum_{1\le i\le k} \nabla_y(T(v_1,\dots,\nabla_x v_i,\dots,v_k)),\\
D_1 &\doteq \sum_{\substack{1\le i\le k\\ 1\le j\le k\\ j\neq i}} T(v_1,\dots,\nabla_y v_j,\dots,\nabla_x v_i,\dots,v_k), & D_2 &\doteq \sum_{\substack{1\le i\le k\\ 1\le j\le k\\ j\neq i}} T(v_1,\dots,\nabla_x v_j,\dots,\nabla_y v_i,\dots,v_k),\\
E_1 &\doteq \sum_{1\le i\le k} T(v_1,\dots,\nabla_y\nabla_x v_i,\dots,v_k), & E_2 &\doteq \sum_{1\le i\le k} T(v_1,\dots,\nabla_x\nabla_y v_i,\dots,v_k),\\
A_3 &\doteq \nabla_{\nabla_x y}(T(v_1,\dots,v_k)), & A_4 &\doteq \nabla_{\nabla_y x}(T(v_1,\dots,v_k)),\\
F_1 &\doteq \sum_{1\le i\le k} T(v_1,\dots,\nabla_{\nabla_x y} v_i,\dots,v_k), & F_2 &\doteq \sum_{1\le i\le k} T(v_1,\dots,\nabla_{\nabla_y x} v_i,\dots,v_k).
\end{aligned}$$

We then have

$$\begin{aligned}
(\mathrm{Rm}_{x,y}T)(v_1,\dots,v_k) &= A_1 - A_2 - A_3 + A_4 - B_1 + B_1 + C_1 - C_1 + D_1 - D_2 + E_1 - E_2 + F_1 - F_2\\
&= A_1 - A_2 - A_3 + A_4 + D_1 - D_2 + E_1 - E_2 + F_1 - F_2.
\end{aligned}$$

Note now that

$$A_1 - A_2 - A_3 + A_4 = [\nabla^2(T(v_1,\dots,v_k))](x,y) - [\nabla^2(T(v_1,\dots,v_k))](y,x) = 0,$$

since the Hessian is symmetric, as we saw in remark **(O.45)**. Since $\nabla_w v_i = 0$ for every $w \in T_p\mathcal{M}$ and T is a tensor, we also have

$$D_1 = D_2 = F_1 = F_2 = 0.$$

We then conclude that

$$(\mathrm{Rm}_{x,y}T)(v_1,\dots,v_k) = E_1 - E_2 = -\sum_{1\le i\le k} T(v_1,\dots,(\nabla_x\nabla_y - \nabla_y\nabla_x)v_i,\dots,v_k)$$
$$= -\sum_{1\le i\le k} T(v_1,\dots,\mathrm{Rm}_{x,y}(v_i),\dots,v_k).$$

■

Remark (O.85). In the proof above, the fact

$$\nabla_w v_i = 0,\ \forall w \in T_p\mathcal{M},$$

does not imply that B_1 or C_1 vanish at p. Even if T is a tensor and therefore

$$T(v_1,\dots,\nabla_y v_i,\dots,v_k) = T(v_1,\dots,\nabla_x v_i,\dots,v_k) = 0,$$

holds, the fact that a real function vanishes at a point does not allow us to conclude anything about the derivative of such a function at that point, and therefore we do not know whether B_1 or C_1 vanish.

Differential forms

Definition (D.4.7). We will say that a tensor $\omega \in \mathscr{T}^0_k(T\mathcal{M}^n)$ is alternating when

$$\omega(X_1,\dots,X_i,\dots,X_j,\dots,X_k) = -\omega(X_1,\dots,X_j,\dots,X_i,\dots,X_k),$$

for all $\{X_i\}_{1\le i\le k} \subset \Gamma(T\mathcal{M})$ and $1 \le i,j \le k$.

Definition (D.4.8). A differential form ω of degree k on $\mathcal{M}^n$ is an alternating tensor $\omega \in \mathscr{T}^0_k(T\mathcal{M}^n)$.

Remark (O.86). The bundle of all differential forms of degree k on $\mathcal{M}$ will be denoted by $\Lambda^k(\mathcal{M})$. Again, to avoid notational overload, we will commit the abuse of regarding elements of $\Gamma\left(\Lambda^k(\mathcal{M})\right)$ as elements of $\Lambda^k(\mathcal{M})$.

Definition (D.4.9). An orientation form on $\mathcal{M}^n$ is a differential form ω of degree n such that $T^*_p\mathcal{M} \ni \omega(p) \neq 0$ for every $p \in \mathcal{M}$.

The proofs of the next three propositions can be found in [31].

Proposition (P.4.11). *Let $(\mathcal{M}^n, g)$ be an oriented Riemannian manifold. Then there exists a unique orientation form on $\mathcal{M}$, which we will denote by dVol_g, called the* ***Riemannian volume form****, which satisfies*

$$\mathrm{dVol}_g(E_1,\dots,E_n) = 1,$$

for any oriented local orthonormal frame $\{E_i\}_{1\le i\le n}$ of $\mathcal{M}$.

Proposition (P.4.12). *Let $(\mathcal{M}, g)$ be an oriented Riemannian manifold and $f \in \mathscr{C}^\infty(\mathcal{M})$ a smooth function with compact support satisfying $f \geq 0$. Then*

$$\int_{\mathcal{M}} f \, \mathrm{dVol}_g \geq 0,$$

and equality holds if, and only if, $f \equiv 0$.

Theorem (I.4.1) (Stokes' Theorem). *Let $\mathcal{M}^n$ be an oriented differentiable manifold with boundary, and suppose that $\omega \in \Lambda^{n-1}(\mathcal{M})$ has compact support. Then*

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega. \tag{4.26}$$

Remark (O.87). The Riemannian volume form dVol_g on $\mathcal{M}$ determines a natural measure μ on $\mathcal{M}$ that makes $(\mathcal{M}, \mu)$ a measure space, so that it makes sense to talk about $\mathcal{L}^2(\mathrm{dVol}_g)$, the space of all real functions u on $\mathcal{M}$ satisfying

$$\int_{\mathcal{M}} u^2 \, \mathrm{dVol}_g < \infty.$$

A common and harmless abuse of notation is to denote μ by dVol_g. Moreover, $\mathcal{L}^2(\mathrm{dVol}_g)$ admits an inner product, given by

$$\langle u, v \rangle_{\mathcal{L}^2(\mathrm{dVol}_g)} \doteq \int_{\mathcal{M}} u \cdot v \, \mathrm{dVol}_g \, .$$

Definition (D.4.10). Let $v \in \mathbb{V}$ be a vector of a finite-dimensional real vector space. For each $v \in \mathbb{V}$, we define a map $i_v : \Lambda^k(\mathbb{V}^*) \to \Lambda^{k-1}(\mathbb{V}^*)$, called ***interior multiplication by*** $\mathbf{v}$, by

$$[i_v\omega](\omega_1, \ldots, \omega_{k-1}) \doteq \omega(v, \omega_1, \ldots, \omega_{k-1}), \text{ for each } \omega \in \Lambda^k(\mathbb{V}^*).$$

Remark (O.88). By convention, we interpret $i_v\omega$ as being zero when ω is a real number. We will also use the notation

$$v \lrcorner \omega = i_v\omega.$$

Proposition (P.4.13). *For each field $X \in \Gamma(T\mathcal{M})$, there exists a unique real function, which we will denote by $\operatorname{div} X$, such that*

$$(\operatorname{div} X) \, \mathrm{dVol}_g = \mathrm{d}\left(X \lrcorner \mathrm{dVol}_g\right). \tag{4.27}$$

The map $\operatorname{div} : \Gamma(T\mathcal{M}) \to \mathscr{C}^\infty(\mathcal{M})$ is then well defined.

Proof: See [31].

■

Using the tensoriality of all the entities present in equation (4.27) together with a geodesic frame centered around an arbitrarily fixed point $p \in \mathcal{M}$, a direct computation shows the following

Proposition (P.4.14). *Let $(\mathcal{M}^n, g)$ be an oriented Riemannian manifold. For any $X \in \Gamma(T\mathcal{M})$, we have*

$$\operatorname{div}(X) = \operatorname{tr}(\nabla X),$$

Equivalently,

$$\operatorname{div} = \operatorname{tr} \circ \nabla.$$

Remark (O.89). By the previous proposition (and by the fact that the map tr does not depend on bases), we conclude that if $\{E_i\}_{1\leq i\leq n}$ is any local orthonormal frame in a neighborhood of an arbitrarily fixed $p \in \mathcal{M}$, then

$$[\mathrm{div}(X)](p) = \sum_{1\leq i\leq n} g\left([\nabla X](E_i), E_i\right) = \sum_{1\leq i\leq n} g\left(\nabla_{E_i(p)} X, E_i(p)\right).$$

Equivalently, the representation in local coordinates of $\mathrm{div}(X)$ is given by

$$\mathrm{tr}(\nabla X) = \sum_{1\leq i\leq n} \mathrm{d}x^i\left(\nabla_{\partial_i} X\right).$$

Lemma (L.4.2). Let $f \in \mathscr{C}^\infty(\mathcal{M}^n)$ and $X \in \Gamma(T\mathcal{M})$. Then

$$\mathrm{div}(f \cdot X) = X(f) + f \cdot \mathrm{div}(X).$$

Proof: Immediate consequence of the previous remark. Indeed,

$$\begin{aligned}\mathrm{div}(f \cdot X) = \sum_{1\leq i\leq n} g\left(\nabla_{\mathbf{e}_i}(fX), \mathbf{e}_i\right) &= \sum_{1\leq i\leq n} g\left(\mathbf{e}_i(f)X + f \cdot \nabla_{\mathbf{e}_i}(X), \mathbf{e}_i\right) \\ &= \left(\sum_{1\leq i\leq n} X(\mathbf{e}_i)\cdot \mathbf{e}_i(f)\right) + f \cdot \mathrm{div}(X) \\ &= X(f) + f \cdot \mathrm{div}(X).\end{aligned}$$

■

Lemma (I.4.2) (Integration by parts on Riemannian manifolds). *Let $u, v \in \mathscr{C}^\infty(\mathcal{M})$ be smooth real functions and suppose that u has compact support. Then*

$$\int_{\mathcal{M}} g\left(\nabla u, \nabla v\right) \mathrm{dVol}_g = -\int_{\mathcal{M}} u \cdot \Delta v \; \mathrm{dVol}_g. \tag{4.28}$$

Proof: By Stokes' theorem (see **(4.26)**) and by definition **(4.27)**, we have

$$\int_{\mathcal{M}} \mathrm{div}(u \cdot \nabla v)\, \mathrm{dVol}_g = 0.$$

On the other hand, by lemma **(L.4.2)** and by definition **(D.1.30)**, we have

$$\mathrm{div}(u \cdot \nabla v) = u \cdot \Delta v + g(\nabla u, \nabla v).$$

It then follows that

$$0 = \int_{\mathcal{M}} u \cdot \Delta v \, \mathrm{dVol}_g + \int_{\mathcal{M}} g\left(\nabla u, \nabla v\right) \mathrm{dVol}_g,$$

as claimed.

■

Corollary (C.4.3). *The operator $u \mapsto \Delta u$ is self-adjoint with respect to the inner product on $\mathcal{L}^2(\mathrm{dVol}_g)$ induced by dVol_g, id est,*

$$\int_{\mathcal{M}} f \cdot \Delta h \, \mathrm{dVol}_g = \int_{\mathcal{M}} h \cdot \Delta f \, \mathrm{dVol}_g,$$

for all $f, h \in \mathcal{L}^2(\mathrm{dVol}_g)$.

Proof: Immediate consequence of the previous lemma.

■

Lemma (L.4.3). *Let $u, v \in \mathscr{C}^\infty(\mathcal{M})$ be smooth real functions and suppose that u has compact support. Then*

$$\int_{\mathcal{M}} g\left(\nabla u, \nabla v\right) \cdot e^{-f} \, \mathrm{dVol}_g = -\int_{\mathcal{M}} u \cdot e^{-f} \Delta_f v \, \mathrm{dVol}_g \,. \tag{4.29}$$

Proof: Immediate consequence of the previous lemmas.

■

Corollary (C.4.4). *The operator $u \mapsto \Delta_f u$ is self-adjoint with respect to the inner product on $\mathcal{L}^2(e^{-f}\,\mathrm{dVol}_g)$ induced by $e^{-f}\,\mathrm{dVol}_g$, id est,*

$$\int_{\mathcal{M}} u \cdot \Delta_f v \cdot e^{-f} \, \mathrm{dVol}_g = \int_{\mathcal{M}} v \cdot \Delta_f u \cdot e^{-f} \, \mathrm{dVol}_g,$$

for all $u, v \in \mathcal{L}^2(e^{-f}\,\mathrm{dVol}_g)$.

Proof: Immediate consequence of the previous lemma.

■

The definition of div provided by proposition **(P.4.14)** allows us to naturally extend div to arbitrary tensors, as we see in the following

Definition (D.4.11). Let

$$\mathscr{T}^1_k(\mathcal{M}) \ni T : \underbrace{\Gamma(T\mathcal{M}) \times \ldots \times \Gamma(T\mathcal{M})}_{k \text{ times}} \to \Gamma(T\mathcal{M})$$

be a tensor of type $(1, k)$. The divergence of T, which we will denote by $\operatorname{div} T$, is the tensor of type $(0, k)$ determined by

$$\begin{aligned}(\operatorname{div} T)(Y_1, \ldots, Y_k) &= \operatorname{tr}\left([\nabla T](\bullet, Y_1, \ldots, Y_k)\right) \\ &= \operatorname{tr}\left(X \mapsto (\nabla_X T)(Y_1, \ldots, Y_k)\right).\end{aligned}$$

Remark (O.90). In arbitrary local coordinates, if $T \in \mathscr{T}^1_k(T\mathcal{M})$, we then have

$$\begin{aligned}(\operatorname{div} T)(Y_1, \ldots, Y_k) &= \sum_{1 \leq a, b \leq n} g^{ab} \left\langle (\nabla_{\partial_a} T)(Y_1, \ldots, Y_k), \partial_b \right\rangle \\ &= \sum_{1 \leq b \leq n} g^{ab} (\nabla_{\partial_a} T)(Y_1, \ldots, Y_k, \partial_b) \\ &= \sum_{1 \leq b \leq n} (\nabla^{\partial_b} T)(Y_1, \ldots, Y_k, \partial_b).\end{aligned}$$

where in the second equality we used an argument entirely analogous to the one seen in example **(E.1.9)**. In particular, we see that if T is a 2-tensor, we have

$$(\operatorname{div} T)(\bullet) = \sum_{1\leq b\leq n} (\nabla^b T)(\bullet, \partial_b).$$

And if $T = \mathrm{Rm}$, we have (now using the Einstein notation convention):

$$[\operatorname{div}(\mathrm{Rm})]_{ijk} = (\nabla^\ell \mathrm{Rm})_{ijk\ell}.$$

Remark (O.91). It is also common to denote $\operatorname{div} T$ by δT.

Remark (O.92). Definition **(D.4.11)** also extends in the obvious way to purely covariant tensors once we make the identification seen in remark **(O.17)**. In this case, if $T \in \mathscr{T}^0_k(T\mathcal{M})$, $\operatorname{div}(T) \in \mathscr{T}^0_{k-1}(T\mathcal{M})$ is the tensor determined by

$$\operatorname{div}(T)(X_1, \ldots, X_{k-1}) = \operatorname{tr}_g((X, Y) \mapsto (\nabla_X T)(X_1, \ldots, X_{k-1}, Y)).$$

In arbitrary local coordinates, we have

$$(\operatorname{div} T)(Y_1, \ldots, Y_{k-1}) = \sum_{1\leq i,j\leq n} g^{ij} (\nabla_{\partial_i} T)(Y_1, \ldots, Y_{k-1}, \partial_j).$$

Remark (O.93). In local components, we then have (using the Einstein convention):

$$(\delta T)_{i_1\ldots i_{k-1}} = g^{ij} T_{i_1\ldots i_{k-1} i;j} \text{ , if } T \text{ is of type } (0, k).$$
$$(\delta T)_{i_1\ldots i_k} = T_{i_1\ldots i_k}{}^{j}{}_{;j} \text{ , if } T \text{ is of type } (1, k).$$

Another commonly present operator is given by the following proposition, proved in [20]

Proposition (P.4.15). Let $(\mathcal{M}^n, g)$ be a Riemannian manifold. There exists a unique operator, called the Hodge star operator, $\star : \Lambda^k(\mathcal{M}) \to \Lambda^{n-k}(\mathcal{M})$, such that

$$\alpha \wedge (\star\beta) = g(\alpha, \beta)\, \mathrm{dVol}_g,$$

for all $\alpha, \beta \in \Lambda^k(\mathcal{M})$.

The method of moving frames

In general, the vast majority of computations in Riemannian Geometry boil down to proving tensorial equalities. This is usually done in one of the following three ways:

- *the coordinate-free way*: in this case, we make explicit the action of each tensor on arbitrary fields on the manifold in question, verifying that the results coincide. See, for example, the proofs of **(2.11)** and of item **(b)** in lemma **(L.2.4)**.

- *using local coordinates*: in this case, we fix appropriate local coordinates that make the computations easier (for example, normal coordinates) and take advantage of the multilinearity of tensors, which guarantees that the equality only needs to be verified on a basis (as we observed in remark (O.11)).

- *using the method of moving frames*: in this case, we fix a local orthonormal frame (which induces a local orthonormal coframe) and compute the connection 1-forms and curvature 2-forms, respectively, which determine the connection and curvature of the manifold.

Depending on the situation, one of the three methods above can be significantly easier to apply than the other two. In this subsection, based on the reference [53] (a good alternative reference is [18]), we will devote ourselves to exploring the last method mentioned above as far as suffices for our applications.

Let $(\mathcal{M}^n, g)$ be a Riemannian manifold and fix $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ a local g-orthonormal frame defined on an open set $U \subset \mathcal{M}$, with $\{\mathbf{e}^1, \dots, \mathbf{e}^n\}$ the associated coframe (determined by $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$). Since any field $X \in \Gamma(T\mathcal{M})$ is written locally as

$$\Gamma(TU) \ni X|_U = \sum_{1 \leq j \leq n} X^j \, \mathbf{e}_j, \text{ where } \{X^j\}_{1 \leq j \leq n} \subset \mathscr{C}^\infty(\mathcal{M}),$$

then given $s \in \Gamma(TU)$ we can determine (using the linearity of ∇ in the first entry and the Leibniz rule) $\nabla_X s$ from knowledge of the fields $\{\nabla_X \mathbf{e}_j\}_{1 \leq j \leq n}$. Being a field on U, $\nabla_X \mathbf{e}_j$ is a linear combination of the $\mathbf{e}_i$'s with coefficients ω^i_j depending on X, *id est:*

$$\nabla_X \mathbf{e}_j = \sum_{1 \leq i \leq n} \omega^i_j(X) \, \mathbf{e}_i.$$

The $\mathscr{C}^\infty(\mathcal{M})$-linearity of $\nabla_X \mathbf{e}_j$ in X guarantees the $\mathscr{C}^\infty(\mathcal{M})$-linearity of ω^i_j in X, so that $\omega^i_j \in \Gamma(T^*U)$ is a 1-form on U. The 1-forms ω^i_j on U are called the ***connection forms***, and the matrix $\omega = (\omega^i_j)_{1 \leq i,j \leq n}$ is called the ***connection matrix***, of the connection ∇ with respect to the frame $\{\mathbf{e}_i\}_{1 \leq i \leq n} \subset \Gamma(TU)$.

Analogously, given $X, Y \in \Gamma(TU)$, the field $\mathrm{Rm}(X,Y)\mathbf{e}_j$ is a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_n$, *id est:*

$$\mathrm{Rm}(X,Y)\mathbf{e}_j = \sum_{1 \leq i \leq n} \Omega^i_j(X,Y) \, \mathbf{e}_i.$$

Since

$$\mathrm{Rm}(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

is alternating and $\mathscr{C}^\infty(\mathcal{M})$-bilinear, Ω^i_j is also alternating and $\mathscr{C}^\infty(\mathcal{M})$-bilinear. By the properties of Rm, Ω^i_j is therefore a 2-form on U. The 2-forms Ω^i_j are called the ***curvature forms***, and the matrix $\Omega = (\Omega^i_j)$ is called the curvature matrix, of the connection ∇ with respect to the frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ on U.

Remark (O.94). Note that Ω^i_j is determined by Rm, since

$$\begin{aligned} \Omega^i_j &= \frac{1}{2} \sum_{1 \leq k,\ell \leq n} \Omega^i_j(\mathbf{e}_k, \mathbf{e}_\ell) \, \mathbf{e}^k \wedge \mathbf{e}^\ell \\ &= -\frac{1}{2} \sum_{1 \leq k,\ell \leq n} \mathrm{Rm}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_\ell) \, \mathbf{e}^k \wedge \mathbf{e}^\ell, \end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}\Omega^i_j(\mathbf{e}_k,\mathbf{e}_\ell) &= g\left(\sum_{1\leq s\leq n}\Omega^s_j(\mathbf{e}_k,\mathbf{e}_\ell)\mathbf{e}_s,\mathbf{e}_i\right)\\ &= g\left(\mathrm{Rm}(\mathbf{e}_k,\mathbf{e}_\ell)\mathbf{e}_j,\mathbf{e}_i\right)\\ &= -\mathrm{Rm}(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_\ell).\end{aligned}$$

Direct computations show the following results:

Theorem (T.4.3). *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold and $U \subset \mathcal{M}$ an open set on which an orthonormal frame $\mathbf{e}_1,\ldots,\mathbf{e}_n$ is defined, with associated coframe $\mathbf{e}^1,\ldots,\mathbf{e}^n$. Then there exists an antisymmetric matrix $\left(\omega^i_j\right)_{1\leq i,j\leq n}$ of 1-forms such that*

$$\mathrm{d}\mathbf{e}^i + \sum_{1\leq j\leq n}\omega^i_j\wedge\mathbf{e}^j = 0, \text{ for every } 1\leq i\leq n.$$

Moreover, if $\left(\alpha^i_j\right)_{1\leq i,j\leq n}$ is any other antisymmetric matrix of 1-forms satisfying

$$\mathrm{d}\mathbf{e}^i + \sum_{1\leq j\leq n}\alpha^i_j\wedge\mathbf{e}^j = 0, \text{ for every } 1\leq i\leq n, \tag{4.30}$$

then

$$\left(\alpha^i_j\right)_{1\leq i,j\leq n} = \left(\omega^i_j\right)_{1\leq i,j\leq n}.$$

Theorem (T.4.4). *The curvature forms Ω^i_j are related to the connection forms ω^i_j by the following equation, commonly called* ***Cartan's second structure equation****:*

$$\Omega^i_j = \mathrm{d}\,\omega^i_j + \sum_{1\leq k\leq n}\omega^i_k\wedge\omega^k_j, \text{ for all } \{i,j\}\subset\{1,\ldots,n\}.$$

Proof: See [53] or [18].

■

Warped products

This section is devoted to computing the geometric entities of an arbitrary multiply warped product. Warped products (commonly translated in the Brazilian literature as "produtos torcidos") are one of the most important and fruitful concepts of Riemannian geometry, also having great relevance in physics (appearing as models of the Cosmos itself, as in the Friedmann–Lemaître–Robertson–Walker cosmological model (see, for example, [22] or [40]), or of black holes).

Our approach is based on [43] and on [17], the main difference being that in some computations we use the method of moving frames. In [43], the geometric entities of a simple warped product $\mathcal{B}\times_h\mathcal{N}$ are computed, while in [17] the authors state (without proof) several propositions that determine the

geometric entities of an arbitrary multiply warped product. Using the method of moving frames, we generalize the computations done in [43] and prove propositions 2.2, 2.3, 2.4, 2.5 and 2.6 stated in [17].

Definition (D.4.12). Let $(\mathcal{B}, g_B)$ and $(\mathcal{M}_i, g_{\mathcal{M}_i})$ be Riemannian manifolds and $b_i : \mathcal{B} \to (0, \infty)$ smooth functions for each $i \in \{1, 2, \cdots, k\}$. The multiple warped product of these manifolds is the product manifold $\mathcal{N} = \mathcal{B} \times \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_k$ endowed with the metric $g = g_{\mathcal{B}} \oplus b_1^2 g_{\mathcal{M}_1} \oplus b_2^2 g_{\mathcal{M}_2} \oplus \cdots \oplus b_k^2 g_{\mathcal{M}_k}$, defined by

$$g = \sigma_0^*(g_{\mathcal{B}}) \oplus (b_1 \circ \pi)^2 \sigma_1^* (g_{\mathcal{M}_1}) \oplus \cdots \oplus (b_k \circ \pi)^2 \sigma_k^* (g_{\mathcal{M}_k}),$$

where $\sigma_0 : \mathcal{N} \to \mathcal{B}$ is the map defined by

$$\begin{aligned} \sigma_0 : \mathcal{N} &\to \mathcal{B} \\ (p, q_1, \cdots, q_k) &\mapsto p \in \mathcal{B}, \end{aligned}$$

and for $i \geq 1$, $\sigma_i : \mathcal{N} \to \mathcal{M}_i$ is the map defined by

$$\begin{aligned} \sigma_i : \mathcal{N} &\to \mathcal{M}_i \\ (p, q_1, \cdots, q_i, \cdots, q_k) &\mapsto q_i \in \mathcal{M}_i. \end{aligned}$$

Remark (O.95).

- It is common to denote the Riemannian manifold $(\mathcal{N} = \mathcal{B} \times \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_k, g = g_{\mathcal{B}} \oplus b_1^2 g_{\mathcal{M}_1} \oplus b_2^2 g_{\mathcal{M}_2} \oplus \cdots \oplus b_k^2 g_{\mathcal{M}_k})$ simply by $\mathcal{N} = \mathcal{B} \times_{h_1} \mathcal{M}_1^{d_1} \times \cdots \times_{h_k} \mathcal{M}_k^{d_k}$.
- When $k = 1$, we have the usual simple warped product.
- If all $b_i \equiv 1$, we have a trivial Riemannian product.

Definition (D.4.13). If $\varphi : \mathcal{M} \to \mathcal{N}$ is a smooth map, we will say that two fields $X \in \Gamma(T\mathcal{M})$ and $Y \in \Gamma(T\mathcal{N})$ are φ-related (a fact we will denote by $X \overset{\varphi}{\sim} Y$) when $Y \circ \varphi = \mathrm{d}\varphi \circ X$.

Lemma (L.4.4). The tangent space $T_p\mathcal{M}$ at a point $p = (p_0, p_1, \cdots, p_k) \in \mathcal{M} = \mathcal{B} \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_k$ is canonically isomorphic (in the sense that there exists an isomorphism that does not depend on choices of bases) to the product $T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k$.

Proof: Consider the maps

$$\begin{aligned} \iota_0 : \mathcal{B} &\to \mathcal{M} \\ \mathcal{B} \ni q_0 &\mapsto (q_0, p_1, \ldots, p_k) \in \mathcal{M}, \end{aligned}$$

and, for $\alpha \geq 1$,

$$\begin{aligned} \iota_\alpha : \mathcal{N}_\alpha &\to \mathcal{M} \\ \mathcal{N}_\alpha \ni q_\alpha &\mapsto (p_0, p_1, \ldots, p_{\alpha-1}, q_\alpha, p_{\alpha+1}, \ldots, p_k). \end{aligned}$$

As we will see now, the map ψ defined by

$$\psi : T_{(p_0, p_1, \cdots, p_k)} (\mathcal{B} \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_k) \to T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k$$

$$v \mapsto \left(\mathrm{d}\left(\sigma_0\right)_p(v), \mathrm{d}\left(\sigma_1\right)_p(v), \mathrm{d}\left(\sigma_2\right)_p(v), \cdots, \mathrm{d}\left(\sigma_k\right)_p(v)\right),$$

is an isomorphism, with inverse given by

$$\begin{aligned}\varphi : T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k &\to T_{(p_0,p_1,\cdots,p_k)}\left(\mathcal{B} \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_k\right)\\ (v_0, v_1, \ldots, v_k) &\mapsto \sum_{0\leq \ell \leq k} \mathrm{d}\left(\iota_\ell\right)_{p_\ell}(v_\ell).\end{aligned}$$

Indeed, first note that

$$\sigma_i \circ \iota_\ell = \begin{cases}\text{the identity map } \mathrm{Id}_{\mathcal{N}_i} : \mathcal{N}_i \to \mathcal{N}_i, \text{ if } i = \ell;\\ \text{the constant map } \mathcal{N}_\ell \ni q_\ell \mapsto p_i \in \mathcal{N}_i, \text{ if } i \neq \ell.\end{cases}$$

And therefore, it follows from the chain rule that

$$\begin{aligned}\mathrm{d}\left(\sigma_i \circ \iota_\ell\right)_{p_\ell} &= \mathrm{d}\left(\sigma_i\right)_p \circ \mathrm{d}\left(\iota_\ell\right)_{p_\ell}\\ &= \begin{cases}\text{the identity map } \mathrm{Id}_{T_{p_i}\mathcal{N}_i} : T_{p_i}\mathcal{N}_i \to T_{p_i}\mathcal{N}_i, \text{ if } i = \ell;\\ \text{the identically zero map } T_{p_\ell}\mathcal{N}_\ell \ni v_\ell \mapsto 0 \in T_{p_\ell}\mathcal{N}_\ell, \text{ if } i \neq \ell.\end{cases}\end{aligned} \tag{4.31}$$

Therefore, for every $(v_0, v_1, \ldots, v_k) \in T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k$, we have

$$\begin{aligned}(\psi \circ \varphi)(v_0, v_1, \ldots, v_k) &= \left(\mathrm{d}\left(\sigma_0\right)_p\left(\sum_{0\leq \ell \leq k} \mathrm{d}\left(\iota_\ell\right)_{p_\ell}(v_\ell)\right), \ldots, \mathrm{d}\left(\sigma_k\right)_p\left(\sum_{0\leq \ell \leq k} \mathrm{d}\left(\iota_\ell\right)_{p_\ell}(v_\ell)\right)\right)\\ &= \left(\sum_{0\leq \ell \leq k} \mathrm{d}\left(\sigma_0\right)_p\left(\mathrm{d}\left(\iota_\ell\right)_{p_\ell}(v_\ell)\right), \ldots, \sum_{0\leq \ell \leq k} \mathrm{d}\left(\sigma_k\right)_p\left(\mathrm{d}\left(\iota_\ell\right)_{p_\ell}(v_\ell)\right)\right)\\ &= (v_0, v_1, \ldots, v_k),\end{aligned}$$

where in the last equality we used (4.31). It follows that ψ is surjective. Since $T_{(p_0,p_1,\cdots,p_k)}\left(\mathcal{B} \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_k\right)$ and $T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k$ have the same dimension, it follows that ψ is an isomorphism. Since a bijective map has a unique right inverse, it also follows that φ is the inverse of ψ.

■

Remark (O.96). Note that lemma (L.4.4) does not use at any point any Riemannian structure of $\mathcal{M}$, thus characterizing the tangent space of a product of differentiable manifolds.

Remark (O.97). For each $i \in \{0, 1, \cdots, k\}$, the set of fields on $\mathcal{M}$ that arise as lifts of fields on factors of the product will be denoted by

$$\mathcal{L}(\mathcal{N}_i) = \{X \in \Gamma(T\mathcal{M}) \mid \mathrm{d}\sigma_j(X) = 0, \text{ for every } j \neq i.\}$$

Given $X_i \in \Gamma(T\mathcal{N}_i)$, it is easy to see that there exists a unique field $\mathrm{Lev}(X_i) \in \mathcal{L}(\mathcal{N}_i)$ satisfying $\left(\mathrm{d}\sigma_i \circ \mathrm{Lev}\right)(X_i) = X_i$ (indeed, $\mathrm{Lev}(X_i) = \psi^{-1}\left((0, \cdots, X_i, 0, \cdots, 0)\right)$). The field $\mathrm{Lev}(X_i)$ is called the lift of the field X_i. The abuse of notation of identifying a field or a 1-form with its lift is therefore harmless,

and we will commit it several times from now on. Note that $\mathrm{Lev}(X_i)$ is the unique field σ_i-related to X_i and σ_j-related to $0 \in \Gamma(T\mathcal{N}_j)$ for all $j \neq i$.

Remark (O.98). It is now convenient to recall the Gauss equation. We say that a map between Riemannian manifolds $f : (\mathcal{M}^n, g_{\mathcal{M}}) \to \left(\widetilde{\mathcal{M}}^{n+p}, \widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}\right)$ is an isometric immersion when

$$f^*\left(\widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}\right) = g_{\mathcal{M}},$$

or, equivalently,

$$g_{\mathcal{M}}(p)(v,w) = \widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}\left(f(p)\right)\left(\mathrm{d}f_p(v), \mathrm{d}f_p(w)\right),$$

for all $p \in \mathcal{M}$ and $v, w \in T_p\mathcal{M}$. In this case, committing the usual abuse of identifying fields with their extensions, we have

$$\widetilde{\mathrm{Rm}}(X,Y,Z,W) = \mathrm{Rm}(X,Y,Z,W) - \widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}\left(\mathbf{II}(X,W), \mathbf{II}(Y,Z)\right) + \widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}\left(\mathbf{II}(X,Z), \mathbf{II}(Y,W)\right), \tag{4.32}$$

for all $X, Y, Z, W \in \Gamma(T\mathcal{M})$. In this regard, it is important to emphasize that, once we fix

$$p = (p_0, p_1, \ldots, p_k) \in \mathcal{M} = \mathcal{B} \times \mathcal{N}_1 \times \ldots \times \mathcal{N}_k,$$

the inclusion

$$\begin{aligned} \iota_\alpha : (\mathcal{N}_\alpha, g_{\mathcal{N}_\alpha}) &\hookrightarrow \mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1 \times \ldots \times_{h_k} \mathcal{N}_k \\ q_\alpha &\mapsto (p_0, \ldots, p_{\alpha-1}, q_\alpha, p_{\alpha+1}, \ldots, p_k), \end{aligned}$$

is not an isometric immersion. Indeed, arbitrarily fixing $q_\alpha \in \mathcal{N}_\alpha$ and denoting

$$P = (p_0, \ldots, p_{\alpha-1}, q_\alpha, p_{\alpha+1}, \ldots, p_k),$$

we have

$$\begin{aligned} \left[(\iota_\alpha)^*(g_{\mathcal{M}})\right](q_\alpha)(v,w) &= g_{\mathcal{M}}(P)\left(\mathrm{d}(\iota_\alpha)_{q_\alpha}(v), \mathrm{d}(\iota_\alpha)_{q_\alpha}(w)\right) \\ &= h_\alpha^2(p_0)\left[(\sigma_\alpha)^*(g_{\mathcal{N}_\alpha})\right]_P\left(\mathrm{d}(\iota_\alpha)_{q_\alpha}(v), \mathrm{d}(\iota_\alpha)_{q_\alpha}(w)\right) \\ &= h_\alpha^2(p_0)\,(g_{\mathcal{N}_\alpha})_{q_\alpha}\left(\mathrm{d}(\sigma_\alpha)_P\left(\mathrm{d}(\iota_\alpha)_{q_\alpha}(v)\right), \mathrm{d}(\sigma_\alpha)_{q_\alpha}\left(\mathrm{d}(\iota_\alpha)_{q_\alpha}(w)\right)\right) \\ &= h_\alpha^2(p_0)(q_\alpha)\,(g_{\mathcal{N}_\alpha})_{q_\alpha}(v,w), \end{aligned} \tag{4.33}$$

where in the last equality we used equation (4.31). However, it is clear from the computations done in (4.33) that with the metric $(h_\alpha(p_0))^2\, g_{\mathcal{N}_\alpha}$, the inclusion

$$\begin{aligned} \iota_\alpha : \left(\mathcal{N}_\alpha, (h_\alpha(p_0))^2\, g_{\mathcal{N}_\alpha}\right) &\hookrightarrow \mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1 \times \ldots \times_{h_k} \mathcal{N}_k \\ q_\alpha &\mapsto (p_0, \ldots, p_{\alpha-1}, q_\alpha, p_{\alpha+1}, \ldots, p_k), \end{aligned}$$

is an isometric immersion, so that in this context we can use equation (4.32) (as we will do later on).

Remark (O.99). Although equation (4.32) is usually used to determine the curvature of submanifolds, in some situations we can use it in the opposite direction: that is, via knowledge of the curvature of specific submanifolds, we can determine the curvature of the total manifold. An interesting exercise

in Riemannian Geometry (which we will comment on in more detail in example (E.4.6)) that uses this technique involves computing the curvature of the warped product $\mathcal{I} \times_\varphi \mathbb{Q}^n(\kappa)$, where $\mathcal{I} \subset \mathbb{R}$ is an interval, $\varphi : \mathcal{I} \to (0, \infty)$ is a smooth function, and we define

$$\mathbb{Q}^n(\kappa) \doteq \begin{cases} \mathbb{R}^n, \text{ if } \kappa = 0; \\ \mathbb{S}^n, \text{ if } \kappa = 1; \\ \mathbb{H}^n, \text{ if } \kappa = -1. \end{cases}$$

Lemma (L.4.5). If $\varphi : \mathcal{M} \to \mathcal{N}$ is a smooth map, then

$$\mathrm{d}\varphi([X, Y]) = [\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)], \ \forall X, Y \in \Gamma(T\mathcal{M}).$$

Proof: It follows from the definition of the Lie bracket and from the definition of the differential of a map. Indeed, for any $p \in \mathcal{M}$ and $g \in \mathscr{C}^\infty(\mathcal{N})$ we have

$$\begin{aligned} \left(\mathrm{d}\varphi_p\left([X, Y](p)\right)\right)(g) &= [X, Y](p)(g \circ \varphi) \\ &= X_p\left(Y(g \circ \varphi)\right) - Y_p\left(X(g \circ \varphi)\right) \\ &= X_p\left(\mathrm{d}\varphi(Y)(g) \circ \varphi\right) - Y_p\left(\mathrm{d}\varphi(X)(g) \circ \varphi\right) \\ &= \mathrm{d}\varphi_p(X_p)\left(\mathrm{d}\varphi(Y)(g)\right) - \mathrm{d}\varphi_p(Y_p)\left(\mathrm{d}\varphi(X)(g)\right) \\ &= [\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)]_p(g). \end{aligned}$$

■

Lemma (L.4.6). If $\varphi : \mathcal{M} \to \mathcal{N}$ is a smooth map and $X_1, X_2 \in \Gamma(T\mathcal{M}), Y_1, Y_2 \in \Gamma(T\mathcal{N})$ are fields satisfying $X_i \overset{\varphi}{\sim} Y_i$ for each $1 \leq i \leq 2$, then $[X_1, X_2] \overset{\varphi}{\sim} [Y_1, Y_2]$.

Proof: Immediate consequence of the previous lemma. Indeed, given any $p \in \mathcal{M}$ we have

$$\begin{aligned} \mathrm{d}\varphi_p\left([X_1, X_2](p)\right) &= \left[\mathrm{d}\varphi_p\left(X_1(p)\right), \mathrm{d}\varphi_p\left(X_2(p)\right)\right] \\ &= [Y_1, Y_2](\varphi(p)). \end{aligned}$$

■

Naturally, knowledge of the connections and curvatures of the factors of such a product determines the connection and curvature of the product itself. To determine these, we will use the method of moving frames. First, we will establish some notation that is important for the proofs of the following lemmas (also stated in [17], with a sign error in item (8) of Proposition 2.4, corrected in item (8) of lemma (L.4.10)).

Let $\mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiple warped product with metric

$$g = g_{\mathcal{B}} \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k},$$

and connection $\nabla^g = \nabla$. We will denote $\mathcal{N}_0^r \doteq \mathcal{B}$ and $d_i \doteq \dim(\mathcal{N}_i)$ for each $0 \leq i \leq k$. For each $0 \leq i \leq n$, we will fix local $g_{\mathcal{N}_i}$-orthonormal frames, which we will denote by $\{\mathbf{e}_{(i,j)}\}_{1 \leq j \leq d_i}$. Their

associated coframes will be denoted by $\{\Theta^{(i,j)}\}_{1\leq i,j\leq d_i}$ (and are determined by $\Theta^{(i,j)}(\mathbf{e}_{(i,s)}) = \delta^j_s$). The connection 1-forms and curvature 2-forms associated with these frames will be denoted by $\{\omega_q^{(i,j)}\}_{1\leq j,q\leq d_i}$ and $\{\Omega_q^{(i,j)}\}_{1\leq j,q\leq d_i}$, respectively. Thus, note that

$$\left\{\mathbf{e}_{(0,1)},\dots,\mathbf{e}_{(0,r)},\frac{1}{h_1}\,\mathbf{e}_{(1,1)},\dots,\frac{1}{h_1}\,\mathbf{e}_{(1,d_1)},\dots,\frac{1}{h_k}\,\mathbf{e}_{(k,1)},\dots,\frac{1}{h_k}\,\mathbf{e}_{(k,d_k)}\right\}$$

is a local g-orthonormal frame. We will use the notation

$$\widetilde{\mathbf{e}}_{(i,j)} = \begin{cases}\mathbf{e}_{(0,j)}, & \text{if } i=0;\\ \dfrac{1}{h_i}\,\mathbf{e}_{(i,j)}, & \text{if } i\neq 0;\end{cases}$$

so that

$$\{\widetilde{\mathbf{e}}_{(i,j)}\}_{0\leq i\leq k,1\leq j\leq d_i}$$

is a local g-orthonormal frame, its associated local coframe being given by

$$\{\widetilde{\Theta}^{(i,j))}\}_{0\leq i\leq k,1\leq j\leq d_i},$$

where

$$\widetilde{\Theta}^{(i,j)} = \begin{cases}\Theta^{(0,j)}, & \text{if } i=0;\\ h_i\Theta^{(i,j)}, & \text{if } i\neq 0;\end{cases}$$

so that on an open set $U\subset\mathcal{M}$, every field $s\in\Gamma(TU)$ is written as

$$s|_U = \sum_{\substack{0\leq i\leq k\\ 1\leq j\leq d_i}} \widetilde{s}^{(i,j)}\,\widetilde{\mathbf{e}}_{(i,j)} = \sum_{\substack{0\leq i\leq k\\ 1\leq j\leq d_i}} \widetilde{\Theta}^{(i,j)}(s)\,\widetilde{\mathbf{e}}_{(i,j)}.$$

Finally, the connection 1-forms and curvature 2-forms of $(\mathcal{M},g)$ with respect to this frame will be denoted by $\{\widetilde{\omega}^{(i,j)}_{(p,m)}\}_{\substack{0\leq i,p\leq k\\ 1\leq j,m\leq d_i}}$ and $\{\widetilde{\Omega}^{(i,j)}_{(p,m)}\}_{\substack{0\leq i,p\leq k\\ 1\leq j,m\leq d_i}}$, so that

$$\nabla_s\left(\widetilde{\mathbf{e}}_{(p,m)}\right) = \sum_{\substack{0\leq i\leq k\\ 1\leq j\leq d_i}} \widetilde{\omega}^{(i,j)}_{(p,m)}(s)\,\widetilde{\mathbf{e}}_{(i,j)},$$

and

$$\mathrm{Rm}(s_1,s_2)\widetilde{\mathbf{e}}_{(p,m)} = \sum_{\substack{0\leq i\leq k\\ 1\leq j\leq d_i}} \widetilde{\Omega}^{(i,j)}_{(p,m)}(s_1,s_2)\,\widetilde{\mathbf{e}}_{(i,j)}.$$

Lemma (L.4.7). Following the notation above, let $\mathcal{M} = \mathcal{B}^r\times_{h_1}\mathcal{N}_1^{d_1}\times\cdots\times_{h_k}\mathcal{N}_k^{d_k}$ be a multiple warped product with metric $g = g_{\mathcal{B}}\oplus h_1^2 g_{\mathcal{N}_1}\oplus\cdots\oplus h_k^2 g_{\mathcal{N}_k}$ and connection $\nabla^g=\nabla$. The connection 1-forms $\{\widetilde{\omega}^{(i,j)}_{(p,m)}\}_{\substack{0\leq i,p\leq k\\ 1\leq j,m\leq d_i}}$ are given by

(a)

$$\widetilde{\omega}^{(0,j)}_{(0,\ell)} = \omega^{(0,j)}_{\ell},\ \text{for all } 1\leq j,\ell\leq r.$$

(b)

$$\widetilde{\omega}^{(0,j)}_{(\beta,\ell)} = -\frac{\mathbf{e}_{(0,j)}(h_\beta)}{h_\beta}\,\widetilde{\Theta}^{(\beta,\ell)},\ \text{for any } 1\leq j\leq r \text{ and } \beta,\ell\geq 1.$$

(c)

$$\widetilde{\omega}^{(\gamma,j)}_{(0,a)} = \frac{\mathbf{e}_{(0,a)}(h_\gamma)}{h_\gamma}\,\widetilde{\Theta}^{(\gamma,j)},\ \text{if } 1\leq\gamma\leq k \text{ and } 1\leq a,j\leq r.$$

(d)

$$\widetilde{\omega}^{(\gamma,q)}_{(\beta,p)} = \delta_{\gamma\beta}\,\omega^{(\gamma,q)}_{p},\ \text{if } 1\leq\gamma,\beta\leq k \text{ and } 1\leq p,q\leq d_\gamma.$$

Proof: The idea of the proof is classical: motivated by theorem **(T.4.3)**, we will define an antisymmetric matrix $\left(\alpha^{(i,j)}_{(p,m)}\right)_{\substack{0\leq i,p\leq k\\ 1\leq j,m\leq d_i}}$ that satisfies

$$\begin{cases} \mathrm{d}\widetilde{\Theta}^{(0,j)} + \displaystyle\sum_{1\leq\ell\leq r}\alpha^{(0,j)}_{(0,\ell)}\wedge\widetilde{\Theta}^{(0,\ell)} + \sum_{\substack{1\leq\gamma\leq k\\ 1\leq\beta\leq d_\gamma}}\alpha^{(0,j)}_{(\gamma,\beta)}\wedge\widetilde{\Theta}^{(\gamma,\beta)} = 0;\\ \mathrm{d}\widetilde{\Theta}^{(\beta,\gamma)} + \displaystyle\sum_{1\leq\ell\leq r}\alpha^{(\beta,\gamma)}_{(0,\ell)}\wedge\widetilde{\Theta}^{(0,\ell)} + \sum_{\substack{1\leq\xi\leq k\\ 1\leq\eta\leq d_\xi}}\alpha^{(\beta,\gamma)}_{(\xi,\eta)}\wedge\widetilde{\Theta}^{(\xi,\eta)} = 0; \end{cases} \tag{4.34}$$

for all $1\leq j\leq r$, $1\leq\beta\leq k$, $1\leq\gamma\leq d_\beta$. From the uniqueness stated in theorem **(T.4.3)**, it will follow that $\left(\alpha^{(i,j)}_{(p,m)}\right)_{\substack{0\leq i,p\leq k\\ 1\leq j,m\leq d_i}}$ is the matrix of 1-forms of the Levi-Civita connection of $\mathcal{M}$. Since

$$\begin{aligned} \mathrm{d}\widetilde{\Theta}^{(0,j)} &= \mathrm{d}\Theta^{(0,j)}\\ &= -\sum_{1\leq\ell\leq r}\omega^{(0,j)}_{\ell}\wedge\widetilde{\Theta}^{(0,\ell)}, \end{aligned} \tag{4.35}$$

and also,

$$\begin{aligned} \mathrm{d}\widetilde{\Theta}^{(\gamma,j)} &= \mathrm{d}\left(h_\gamma\Theta^{(\gamma,j)}\right)\\ &= \mathrm{d}h_\gamma\wedge\Theta^{(\gamma,j)} + h_\gamma\,\mathrm{d}\Theta^{(\gamma,j)}\\ &= \left(\sum_{1\leq a\leq r}\mathbf{e}_{(0,a)}\left(h_\gamma\right)\,\Theta^{(0,a)}\right)\wedge\Theta^{(\gamma,j)} - h_\gamma\sum_{1\leq\ell\leq d_\gamma}\omega^{(\gamma,j)}_{\ell}\wedge\Theta^{(\gamma,\ell)}\\ &= -\sum_{1\leq a\leq r}\left(\mathbf{e}_{(0,a)}\left(h_\gamma\right)\,\Theta^{(\gamma,j)}\right)\wedge\widetilde{\Theta}^{(0,a)} - \sum_{1\leq\ell\leq d_\gamma}\omega^{(\gamma,j)}_{\ell}\wedge\widetilde{\Theta}^{(\gamma,\ell)}, \end{aligned} \tag{4.36}$$

we will define

$$\begin{aligned} \alpha^{(0,j)}_{(0,\ell)} &= \omega^{(0,j)}_{\ell};\\ \alpha^{(0,j)}_{(\gamma,\beta)} &= -\frac{\mathbf{e}_{(0,j)}(h_\gamma)}{h_\gamma}\,\widetilde{\Theta}^{(\gamma,\beta)};\\ \alpha^{(\beta,\gamma)}_{(0,\ell)} &= \frac{\mathbf{e}_{(0,\ell)}(h_\beta)}{h_\beta}\,\widetilde{\Theta}^{(\beta,\gamma)};\\ \alpha^{(\beta,\gamma)}_{(\xi,\eta)} &= 0,\ \text{for any } \xi\geq 1 \text{ with } \xi\neq\beta. \end{aligned}$$

It is easy to see that the matrix α thus defined is antisymmetric. Moreover, it follows from equations **(4.35)** and **(4.36)** that α thus defined satisfies **(4.34)**, from which the desired result follows.

■

Lemma (L.4.8). Let $\mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiple warped product with metric $g = g_{\mathcal{B}} \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$ and connection $\nabla^g = \nabla$. Suppose that $X, Y \in \mathcal{L}(\mathcal{B})$, $V \in \mathcal{L}(\mathcal{N}_\alpha)$ and $W \in \mathcal{L}(\mathcal{N}_\eta)$. Then:

(1) $\nabla_X Y = \nabla^{\mathcal{B}}_X Y$;

(2) $\nabla_X V = \nabla_V X = \dfrac{X(h_\alpha)}{h_\alpha} V$;

(3) $\nabla_V W = \begin{cases} 0, \text{ if } \alpha \neq \eta; \\ \nabla^{\mathcal{N}_\eta}_V W - \dfrac{g(V,W)}{h_\eta} \operatorname{grad}_{\mathcal{B}} h_\eta, \text{ if } \alpha = \eta. \end{cases}$

Proof: We have

$$\begin{aligned} \nabla_X Y &= \nabla_X \left(\sum_{1 \leq a \leq r} \widetilde{Y}^{(0,a)} \, \widetilde{\mathbf{e}}_{(0,a)} \right) \\ &= \sum_{1 \leq a \leq r} \left\{ X\left(\widetilde{Y}^{(0,a)}\right) \widetilde{\mathbf{e}}_{(0,a)} + \widetilde{Y}^{(0,a)} \nabla_X \left(\widetilde{\mathbf{e}}_{(0,a)} \right) \right\} \\ &= \sum_{1 \leq a \leq r} \left\{ X\left(Y^{(0,a)}\right) \mathbf{e}_{(0,a)} + Y^{(0,a)} \nabla_X \left(\widetilde{\mathbf{e}}_{(0,a)} \right) \right\}. \end{aligned}$$

Now, note that

$$\begin{aligned} \nabla_X \left(\widetilde{\mathbf{e}}_{(0,a)} \right) &= \sum_{\substack{0 \leq i \leq k \\ 1 \leq j \leq d_i}} \widetilde{\omega}^{(i,j)}_{(0,a)}(X) \, \widetilde{\mathbf{e}}_{(i,j)} \\ &= \sum_{1 \leq j \leq r} \widetilde{\omega}^{(0,j)}_{(0,a)}(X) \, \widetilde{\mathbf{e}}_{(0,j)} + \sum_{\substack{1 \leq \gamma \leq k \\ 1 \leq j \leq d_\gamma}} \widetilde{\omega}^{(\gamma,j)}_{(0,a)}(X) \, \widetilde{\mathbf{e}}_{(\gamma,j)} \\ &= \sum_{1 \leq j \leq r} \omega^{(0,j)}_{a}(X) \, \mathbf{e}_{(0,j)} + \sum_{\substack{1 \leq \gamma \leq k \\ 1 \leq j \leq d_\gamma}} \mathbf{e}_{(0,a)}(h_\gamma) \underbrace{\Theta^{(\gamma,j)}(X)}_{=0, \text{ since } X \in \mathcal{L}(\mathcal{B})} \widetilde{\mathbf{e}}_{(\gamma,j)} \\ &= \nabla^{\mathcal{B}}_X \left(\mathbf{e}_{(0,a)} \right), \end{aligned}$$

therefore

$$\begin{aligned} \nabla_X Y &= \sum_{1 \leq a \leq r} \left\{ X\left(Y^{(0,a)}\right) \mathbf{e}_{(0,a)} + Y^{(0,a)} \nabla_X \left(\mathbf{e}_{(0,a)} \right) \right\} \\ &= \nabla^{\mathcal{B}}_X \left(\sum_{1 \leq a \leq r} Y^{(0,a)} \, \mathbf{e}_{(0,a)} \right) \\ &= \nabla^{\mathcal{B}}_X Y, \end{aligned}$$

where in the third equality we used items **(a)** and **(c)** of lemma **(L.4.7)**. Since $X\left(V^{(\alpha,\beta)}\right) = 0$, we also have

$$
\begin{aligned}
\nabla_X V &= \nabla_X \left(\sum_{1\leq\beta\leq d_\alpha} \widetilde{V}^{(\alpha,\beta)}\, \widetilde{\mathbf{e}}_{(\alpha,\beta)} \right) \\
&= \sum_{1\leq\beta\leq d_\alpha} \left\{ X\left(\widetilde{V}^{(\alpha,\beta)}\right) \widetilde{\mathbf{e}}_{(\alpha,\beta)} + \widetilde{V}^{(\alpha,\beta)}\, \nabla_X \left(\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right) \right\} \\
&= \sum_{1\leq\beta\leq d_\alpha} \left\{ \frac{X(h_\alpha)}{h_\alpha} V^{(\alpha,\beta)}\, \mathbf{e}_{(\alpha,\beta)} + \widetilde{V}^{(\alpha,\beta)} \left(\sum_{1\leq\ell\leq r} \underbrace{\widetilde{\omega}^{(0,\ell)}_{(\alpha,\beta)}(X)}_{=0,\text{ by (b) of (L.4.7)}}\, \widetilde{\mathbf{e}}_{(0,\ell)} + \sum_{\substack{1\leq\gamma\leq k \\ 1\leq s\leq d_\gamma}} \widetilde{\omega}^{(\gamma,s)}_{(\alpha,\beta)}(X)\, \widetilde{\mathbf{e}}_{(\gamma,s)} \right) \right\} \\
&= \frac{X(h_\alpha)}{h_\alpha} V + \sum_{1\leq s,\beta\leq d_\alpha} \underbrace{\omega^{(\alpha,s)}_\beta(X)}_{=0,\text{since } X\in\mathcal{L}(\mathcal{B})}\, \widetilde{\mathbf{e}}_{(\alpha,s)} \\
&= \frac{X(h_\alpha)}{h_\alpha} V,
\end{aligned}
$$

where in the second-to-last equality we used item **(d)** of lemma **(L.4.7)**.

Finally, again using lemma **(L.4.7)**, we have

$$
\begin{aligned}
\nabla_V W &= \nabla_V \left(\sum_{1\leq\beta\leq d_\eta} \widetilde{W}^{(\eta,\beta)}\, \widetilde{\mathbf{e}}_{(\eta,\beta)} \right) \\
&= \sum_{1\leq\beta\leq d_\eta} V\left(\widetilde{W}^{(\eta,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\beta)} + \sum_{1\leq\beta\leq d_\eta} \widetilde{W}^{(\eta,\beta)}\, \nabla_V \left(\widetilde{\mathbf{e}}_{(\eta,\beta)}\right) \\
&= \sum_{1\leq\beta\leq d_\eta} V\left(\widetilde{W}^{(\eta,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\beta)} + \sum_{1\leq\beta\leq d_\eta} \widetilde{W}^{(\eta,\beta)} \left\{ \sum_{1\leq a\leq r} \widetilde{\omega}^{(0,a)}_{(\eta,\beta)}(V)\, \widetilde{\mathbf{e}}_{(0,a)} + \sum_{\substack{1\leq\xi\leq k \\ 1\leq\gamma\leq d_\xi}} \widetilde{\omega}^{(\xi,\gamma)}_{(\eta,\beta)}(V)\, \widetilde{\mathbf{e}}_{(\xi,\gamma)} \right\} \\
&= \sum_{1\leq\beta\leq d_\eta} V\left(\widetilde{W}^{(\eta,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\beta)} + \sum_{1\leq\beta\leq d_\eta} \widetilde{W}^{(\eta,\beta)} \left\{ -\sum_{1\leq a\leq r} \frac{\mathbf{e}_{(0,a)}(h_\eta)}{h_\eta}\, \widetilde{V}^{(\eta,\beta)}\, \widetilde{\mathbf{e}}_{(0,a)} + \sum_{1\leq\gamma\leq d_\eta} \omega^{(\eta,\gamma)}_\beta(V)\, \widetilde{\mathbf{e}}_{(\eta,\gamma)} \right\} \\
&= \sum_{1\leq\beta\leq d_\eta} \left\{ V\left(\widetilde{W}^{(\eta,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\beta)} + \widetilde{W}^{(\eta,\beta)}\, \nabla_V \left(\widetilde{\mathbf{e}}_{(\eta,\beta)}\right) \right\} - \sum_{1\leq a\leq r} \frac{\mathbf{e}_{(0,a)}(h_\eta)}{h_\eta}\, g(V,W)\, \mathbf{e}_{(0,a)} \\
&= \nabla^{\mathcal{N}_\eta}_{\mathrm{d}\sigma_\eta(V)} W - \delta_{\alpha\eta} \frac{g(V,W)}{h_\eta} \mathrm{grad}_{\mathcal{B}} h_\eta \\
&= \delta_{\alpha\eta} \left(\nabla^{\mathcal{N}_\eta}_V W - \frac{g(V,W)}{h_\eta} \mathrm{grad}_{\mathcal{B}} h_\eta \right).
\end{aligned}
$$

■

Remark (O.100). The "natural" progression after having computed the connection 1-forms in lemma **(L.4.7)** would be to use theorem **(T.4.4)** to compute the curvature 2-forms. We will not do this because, to determine Rm, this path is much more complicated compared to simply using the definition of Rm in the computations, supported by lemma **(L.4.7)**. We will do this in the proof of lemma **(L.4.10)**.

Lemma (L.4.9). Let $\mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiple warped product with metric $g = g_{\mathcal{B}} \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$. Let us denote $\mathrm{grad}^{\mathcal{M}} = \mathrm{grad}$ and $\Delta^{\mathcal{M}} = \Delta$. If $\phi \in \mathscr{C}^\infty(\mathcal{B})$ and $\psi \in \mathscr{C}^\infty(\mathcal{N}_i)$, then

(1) $\operatorname{grad}(\phi \circ \pi) = \operatorname{grad}_{\mathcal{B}} \phi$;

(2) $\operatorname{grad}(\psi \circ \sigma_i) = \dfrac{\operatorname{grad}_{\mathcal{N}_i} \psi}{h_i^2}$;

(3) $\Delta(\phi \circ \pi) = \Delta_{\mathcal{B}} \phi + \displaystyle\sum_{1 \leq i \leq k} d_i \frac{g_{\mathcal{B}}\left(\operatorname{grad}_{\mathcal{B}} \phi, \operatorname{grad}_{\mathcal{B}} h_i\right)}{h_i}$;

(4) $\Delta(\psi \circ \sigma_i) = \dfrac{\Delta_{\mathcal{N}_i} \psi}{h_i^2}$.

Proof: Fix $p = (p_0, q_1, \ldots, q_k) \in \mathcal{M}$. Since $\phi \circ \pi$ is constant on $\{p_0\} \times \mathcal{N}_1 \times \ldots \times \mathcal{N}_k$ for every $p_0 \in \mathcal{B}$, we have

$$\begin{aligned}
\operatorname{grad}\left(\phi \circ \pi\right)(p) &= \sum_{1 \leq i \leq r} \mathbf{e}_{(0,i)}(p)\left(\phi \circ \pi\right) \mathbf{e}_{(0,i)}(p) \\
&= \sum_{1 \leq i \leq r} \mathrm{d}\left(\phi \circ \pi\right)_p \left(\mathbf{e}_{(0,i)}(p)\right) \mathbf{e}_{(0,i)}(p) \\
&= \sum_{1 \leq i \leq r} \mathrm{d}\phi_{p_0} \left(\mathbf{e}_{(0,i)}(p_0)\right) \mathbf{e}_{(0,i)}(p) \\
&= \sum_{1 \leq i \leq r} \mathbf{e}_{(0,i)}(p_0)\left(\phi\right) \mathbf{e}_{(0,i)}(p) \\
&= \left(\operatorname{grad}_{\mathcal{B}} \phi\right)(p).
\end{aligned}$$

Analogously,

$$\begin{aligned}
\operatorname{grad}\left(\psi \circ \sigma_i\right)(p) &= \sum_{1 \leq j \leq d_i} \tilde{\mathbf{e}}_{(i,j)}(p)(\psi)\, \tilde{\mathbf{e}}_{(i,j)}(p) \\
&= h_i^{-2} \left(\operatorname{grad}_{\mathcal{N}_i} \psi\right)(p).
\end{aligned}$$

From the definition of the Laplacian, from item **(2)** of lemma **(L.4.8)** and from item **(1)** of the current lemma, it also follows that

$$\begin{aligned}
\Delta\left(\phi \circ \pi\right) &= \sum_{1 \leq i \leq r} g\left(\nabla_{\mathbf{e}_{(0,i)}} \nabla^{\mathcal{B}} \phi, \mathbf{e}_{(0,i)}\right) + \sum_{\substack{1 \leq \alpha \leq k \\ 1 \leq \beta \leq d_\alpha}} g\left(\nabla_{\tilde{\mathbf{e}}_{(\alpha,\beta)}} \nabla^{\mathcal{B}} \phi, \tilde{\mathbf{e}}_{(\alpha,\beta)}\right) \\
&= \Delta_{\mathcal{B}} \phi + \sum_{\substack{1 \leq \alpha \leq k \\ 1 \leq \beta \leq d_\alpha}} g\left(\frac{\left(\nabla^{\mathcal{B}} \phi\right)(h_\alpha)}{h_\alpha} \tilde{\mathbf{e}}_{(\alpha,\beta)}, \tilde{\mathbf{e}}_{(\alpha,\beta)}\right) \\
&= \Delta_{\mathcal{B}} \phi + \sum_{1 \leq \alpha \leq k} d_\alpha \frac{\left(\nabla^{\mathcal{B}} \phi\right)(h_\alpha)}{h_\alpha} \\
&= \Delta_{\mathcal{B}} \phi + \sum_{1 \leq \alpha \leq k} d_\alpha \frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}} \phi, \nabla^{\mathcal{B}} h_\alpha\right)}{h_\alpha}.
\end{aligned}$$

Finally,

$$\Delta\left(\psi \circ \sigma_i\right) = \sum_{1 \leq j \leq r} g\left(\nabla_{\mathbf{e}_{(0,j)}} \nabla \psi, \mathbf{e}_{(0,j)}\right) + \sum_{\substack{1 \leq \alpha \leq k \\ 1 \leq \beta \leq d_\alpha}} g\left(\nabla_{\tilde{\mathbf{e}}_{(\alpha,\beta)}} \nabla \psi, \tilde{\mathbf{e}}_{(\alpha,\beta)}\right)$$

$$
\begin{aligned}
&= \sum_{1\le j\le r} g\left(\nabla_{\mathbf{e}_{(0,j)}}\left(\frac{\nabla^{\mathcal{N}_i}\psi}{h_i^2}\right), \mathbf{e}_{(0,j)}\right) + \sum_{\substack{1\le\alpha\le k\\ 1\le\beta\le d_\alpha}} g\left(\nabla_{\widetilde{\mathbf{e}}_{(\alpha,\beta)}}\left(\frac{\nabla^{\mathcal{N}_i}\psi}{h_i^2}\right), \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\\
&= \sum_{1\le j\le r} g\left(-\frac{2\,\mathbf{e}_{(0,j)}(h_i)}{h_i^3}\nabla^{\mathcal{N}_i}\psi + \frac{1}{h_i^2}\nabla_{\mathbf{e}_{(0,j)}}\nabla^{\mathcal{N}_i}\psi, \mathbf{e}_{(0,j)}\right)\\
&+ \sum_{\substack{1\le\alpha\le k\\ 1\le\beta\le d_\alpha}} h_i^{-4}\, g\left(\nabla_{\mathbf{e}_{(\alpha,\beta)}}\nabla^{\mathcal{N}_i}\psi, \mathbf{e}_{(\alpha,\beta)}\right)\\
&= \sum_{\substack{1\le\alpha\le k\\ 1\le\beta\le d_\alpha}} h_i^{-2}\, g^{\mathcal{N}_i}\left(\nabla_{\mathbf{e}_{(\alpha,\beta)}}\nabla^{\mathcal{N}_i}\psi, \mathbf{e}_{(\alpha,\beta)}\right)\\
&= h_i^{-2}\,\Delta_{\mathcal{N}_i}\psi.
\end{aligned}
$$

■

Lemma (L.4.10). Let $\mathcal{M} = \mathcal{B}\times_{h_1}\mathcal{N}_1^{d_1}\times\cdots\times_{h_k}\mathcal{N}_k^{d_k}$ be a multiple warped product with metric $g = g_{\mathcal{B}}\oplus h_1^2 g_{\mathcal{N}_1}\oplus\cdots\oplus h_k^2 g_{\mathcal{N}_k}$. Let us denote $\mathrm{Rm}^{\mathcal{M}} = \mathrm{Rm}$ and take $X, Y, Z \in \mathcal{L}(\mathcal{B})$, $V \in \mathcal{L}(\mathcal{N}_\alpha)$, $W \in \mathcal{L}(\mathcal{N}_\eta)$ and $U \in \mathcal{L}(\mathcal{N}_\beta)$. Then

(1) $\mathrm{Rm}(X,Y)Z = \mathrm{Rm}_B(X,Y)Z$;

(2) $\mathrm{Rm}(V,X)Y = -\dfrac{(\nabla^2_{\mathcal{B}}h_\alpha)(X,Y)}{h_\alpha}V$;

(3) $\mathrm{Rm}(X,V)W = \mathrm{Rm}(V,W)X = \mathrm{Rm}(V,X)W = 0$ whenever $\alpha\notin\{\eta\}$;

(4) $\mathrm{Rm}(X,Y)V = 0$;

(5) $\mathrm{Rm}(V,W)X = 0$, whenever $\alpha = \eta$;

(6) $\mathrm{Rm}(V,W)U = 0$, whenever $\alpha\in\{\eta\}\neq\{\beta\}$;

(7) $\mathrm{Rm}(U,V)W = -g(V,W)\,\dfrac{g_{\mathcal{B}}(\mathrm{grad}_{\mathcal{B}}h_\alpha, \mathrm{grad}_{\mathcal{B}}h_\beta)}{h_\alpha h_\beta}\,U$ whenever $\alpha\in\{\eta\}\neq\{\beta\}$;

(8) $\mathrm{Rm}(X,V)W = -\dfrac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X(\mathrm{grad}_{\mathcal{B}}h_\alpha)$, whenever $\alpha\in\{\eta\}$;

(9) $\mathrm{Rm}(V,W)U = \mathrm{Rm}_{\mathcal{N}_\alpha}(V,W)U + \dfrac{\|\mathrm{grad}_{\mathcal{B}}h_\alpha\|^2}{(h_\alpha^2)}\cdot(g(V,U)W - g(W,U)V)$, whenever $\{\alpha,\eta\} = \{\beta\}$.

Proof: Item **(1)** is a direct consequence of item **(1)** of lemma **(L.4.8)**. To prove item **(2)**, first note that $[X,V] = 0$ as an immediate consequence of lemma **(L.4.6)**, so that

$$
\begin{aligned}
\mathrm{Rm}(V,X)Y &= \nabla_V\nabla_X Y - \nabla_X\nabla_V Y\\
&= \nabla_V\nabla^{\mathcal{B}}_X Y - \nabla_X\left(\frac{Y(h_\alpha)}{h_\alpha}V\right)\\
&= \frac{\left(\nabla^{\mathcal{B}}_X Y\right)(h_\alpha)}{h_\alpha}V - X\left(\frac{Y(h_\alpha)}{h_\alpha}\right)V - \frac{Y(h_\alpha)}{h_\alpha}\frac{X(h_\alpha)}{h_\alpha}V\\
&= \frac{\nabla_{\nabla^{\mathcal{B}}_X Y}(h_\alpha)}{h_\alpha}V - \frac{X(Y(h_\alpha))}{h_\alpha}V + \frac{X(h_\alpha)}{h_\alpha^2}Y(h_\alpha)\,V - \frac{X(h_\alpha)\,Y(h_\alpha)}{h_\alpha^2}V
\end{aligned}
$$

$$= -\frac{\left(\mathrm{Hess}_g\left(h_\alpha\right)\right)(X,Y)}{h_\alpha} V.$$

Item (3) follows as a direct consequence of item (3) of lemma (L.4.8). Now, it follows from item (2) of lemma (L.4.8) that

$$\begin{aligned}
\nabla_X\nabla_Y V - \nabla_Y\nabla_X V &= \nabla_X\left(\frac{Y\left(h_\alpha\right)}{h_\alpha} V\right) - \nabla_Y\left(\frac{X\left(h_\alpha\right)}{h_\alpha} V\right) \\
&= X\left(\frac{Y\left(h_\alpha\right)}{h_\alpha}\right) V + \frac{Y\left(h_\alpha\right)}{h_\alpha}\nabla_X V - Y\left(\frac{X\left(h_\alpha\right)}{h_\alpha}\right) V - \frac{X\left(h_\alpha\right)}{h_\alpha}\nabla_Y V \\
&= \frac{X\left(Y\left(h_\alpha\right)\right)}{h_\alpha} V - \frac{X\left(h_\alpha\right)}{\left(h_\alpha\right)^2} Y\left(h_\alpha\right) V + \frac{Y\left(h_\alpha\right)}{h_\alpha}\frac{X\left(h_\alpha\right)}{h_\alpha} V \\
&- \frac{Y\left(X\left(h_\alpha\right)\right)}{h_\alpha} V + \frac{X\left(h_\alpha\right)}{\left(h_\alpha\right)^2} Y\left(h_\alpha\right) V - \frac{X\left(h_\alpha\right)}{h_\alpha}\frac{Y\left(h_\alpha\right)}{h_\alpha} V \\
&= \frac{[X,Y]\left(h_\alpha\right)}{h_\alpha} V \\
&= \nabla_{[X,Y]} V,
\end{aligned}$$

and therefore $\mathrm{Rm}(X,Y)V = 0$, as claimed in item (4). Item (5) is a simple consequence of item (2) of lemma (L.4.8). Indeed,

$$\begin{aligned}
\mathrm{Rm}(V,W)X &= \nabla_V\nabla_W X - \nabla_W\nabla_V X - \nabla_{[V,W]} X \\
&= \nabla_V\left(\frac{X\left(h_\alpha\right)}{h_\alpha} W\right) - \nabla_W\left(\frac{X\left(h_\alpha\right)}{h_\alpha} V\right) - \frac{X\left(h_\alpha\right)}{h_\alpha}[V,W] \\
&= \frac{X\left(h_\alpha\right)}{h_\alpha}\left(\nabla_V W - \nabla_W V - [V,W]\right) \\
&= 0.
\end{aligned}$$

Item (6) is a direct consequence of items (3) and (1) of lemma (L.4.8). To prove item (7), note that again as a consequence of lemma (L.4.6), we have $[U,V] = 0$, so that using lemma (L.4.8), we obtain:

$$\begin{aligned}
\mathrm{Rm}(U,V)W &= \nabla_U\nabla_V W - \nabla_V \underbrace{\nabla_U W}_{=0} \\
&= \nabla_U\left(\nabla_V^{\mathcal{N}_\alpha} W - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}} h_\alpha\right) \\
&= -\frac{g(V,W)}{h_\alpha}\nabla_U\nabla^{\mathcal{B}} h_\alpha \\
&= -\frac{g(V,W)}{h_\alpha}\frac{\left(\nabla^{\mathcal{B}} h_\alpha\right)\left(h_\beta\right)}{h_\beta} U \\
&= -g(V,W)\frac{g_{\mathcal{B}}(\mathrm{grad}_{\mathcal{B}} h_\alpha, \mathrm{grad}_{\mathcal{B}} h_\beta)}{h_\alpha h_\beta} U.
\end{aligned}$$

We will now prove item (8). First, we have

$$\nabla_X\nabla_V W = \nabla_X\left(\nabla_V^{\mathcal{N}_\alpha} W - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}} h_\alpha\right)$$

$$
\begin{aligned}
&= \frac{X(h_\alpha)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - X\left(\frac{g(V,W)}{h_\alpha}\right)\nabla^{\mathcal{B}}h_\alpha - \frac{g(V,W)}{h_\alpha}\nabla_X\nabla^{\mathcal{B}}h_\alpha \\
&= \frac{X(h_\alpha)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - \frac{\left(g\left(\nabla_X V, W\right) + g\left(V, \nabla_X W\right)\right)}{h_\alpha}\nabla^{\mathcal{B}}h_\alpha + \frac{X(h_\alpha)}{(h_\alpha)^2}\, g(V,W)\,\nabla^{\mathcal{B}}h_\alpha \\
&- \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X\nabla^{\mathcal{B}}h_\alpha \\
&= \frac{X(h_\alpha)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - 2\frac{X(h_\alpha)}{(h_\alpha)^2}\, g(V,W)\,\nabla^{\mathcal{B}}h_\alpha + \frac{X(h_\alpha)}{(h_\alpha)^2}\, g(V,W)\,\nabla^{\mathcal{B}}h_\alpha - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X\nabla^{\mathcal{B}}h_\alpha \\
&= \frac{X(h_\alpha)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - \frac{X(h_\alpha)}{(h_\alpha)^2}\, g(V,W)\,\nabla^{\mathcal{B}}h_\alpha - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X\nabla^{\mathcal{B}}h_\alpha.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\nabla_V\nabla_X W &= \nabla_V\left(\frac{X(h_\alpha)}{h_\alpha}W\right) \\
&= \frac{X(h_\alpha)}{h_\alpha}\left(\nabla^{\mathcal{N}_\alpha}_V W - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}h_\alpha\right) \\
&= \frac{X(h_\alpha)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - \frac{X(h_\alpha)}{(h_\alpha)^2}\, g(V,W)\,\nabla^{\mathcal{B}}h_\alpha.
\end{aligned}
$$

Since $[X,V]=0$, it then follows that

$$
\mathrm{Rm}(X,V)W = -\frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X(\mathrm{grad}_{\mathcal{B}}h_\alpha).
$$

Finally, to prove **(9)**, first note that $\mathrm{Rm}(V,W)U \in \mathcal{L}(\mathcal{N}_\alpha)$, since given $X \in \mathcal{L}(\mathcal{B})$ and $L \in \mathcal{L}(\mathcal{N}_\xi)$ with $\xi \neq \alpha$, we have

$$
\begin{aligned}
&g\left(\mathrm{Rm}(V,W)U, X\right) = -g\left(\mathrm{Rm}(V,W)X, U\right) = 0, \text{ by item } \textbf{(5)}; \\
&g\left(\mathrm{Rm}(V,W)U, L\right) = -g\left(\mathrm{Rm}(V,W)L, U\right) = 0, \text{ by item } \textbf{(6)}.
\end{aligned}
$$

It then follows from the Gauss equation and from item **(3)** of lemma **(L.4.8)** that, given any $C \in \mathcal{L}(\mathcal{N}_\alpha)$, we have

$$
\begin{aligned}
g\left(\mathrm{Rm}(V,W)U, C\right) &= (h_\alpha)^2\ \mathrm{Rm}^{\mathcal{N}_\alpha}(V,W,U,C) - g\left(\mathbf{II}(V,C), \mathbf{II}(W,U)\right) + g\left(\mathbf{II}(V,U), \mathbf{II}(W,C)\right) \\
&= (h_\alpha)^2\ \mathrm{Rm}^{\mathcal{N}_\alpha}(V,W,U,C) - \frac{g(V,C)\,g(W,U)}{(h_\alpha)^2}\left\|\nabla^{\mathcal{B}}h_\alpha\right\|^2 + \frac{g(V,U)\,g(W,C)}{(h_\alpha)^2}\left\|\nabla^{\mathcal{B}}h_\alpha\right\|^2.
\end{aligned}
$$

Equivalently, since $\mathrm{Rm}(V,W)U \in \mathcal{L}(\mathcal{N}_\alpha)$,

$$
\flat^g\left(\mathrm{Rm}(V,W)U\right) = \flat^g\left(\mathrm{Rm}^{\mathcal{N}_\alpha}(V,W)U + \frac{\left\|\nabla^{\mathcal{B}}h_\alpha\right\|^2}{(h_\alpha)^2}\cdot\left\{g(V,U)\,W - g(W,U)\,V\right\}\right).
$$

Since $\flat^g$ is an isomorphism on each fiber of $T\mathcal{M}$, it follows, as we wanted to show, that

$$
\mathrm{Rm}(V,W)U = \mathrm{Rm}^{\mathcal{N}_\alpha}(V,W)U + \frac{\left\|\nabla^{\mathcal{B}}h_\alpha\right\|^2}{(h_\alpha)^2}\cdot\left\{g(V,U)\,W - g(W,U)\,V\right\}.
$$

■

Lemma (L.4.11). Let $\mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiple warped product with metric $g = g_{\mathcal{B}} \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$. Suppose that $X, Y, Z \in \mathcal{L}(\mathcal{B})$, $V \in \mathcal{L}(\mathcal{N}_\alpha)$ and $W \in \mathcal{L}(\mathcal{N}_\eta)$. Then:

(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}_{\mathcal{B}}(X,Y) - \sum_{1\leq\mu\leq k} \frac{d_\mu}{h_\mu} \nabla^2_{\mathcal{B}}(h_\mu)(X,Y)$;

(2) $\mathrm{Ric}(X,V) = 0$;

(3) $\mathrm{Ric}(V,W) = 0$ if $\alpha \neq \eta$;

(4)

$$\mathrm{Ric}(V,W) = \mathrm{Ric}^{\mathcal{N}_\alpha}(V,W) - \left(\begin{array}{l} \dfrac{\Delta_{\mathcal{B}} h_\alpha}{h_\alpha} + (d_\alpha - 1)\, \dfrac{\|\nabla^{\mathcal{B}} h_\alpha\|^2}{(h_\alpha)^2} \\ \\ + \displaystyle\sum_{\substack{1\leq\gamma\leq k \\ \gamma\neq\alpha}} d_\gamma\, \dfrac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}} h_\gamma, \nabla^{\mathcal{B}} h_\alpha\right)}{h_\gamma\, h_\alpha} \end{array} \right) g(V,W), \text{ if } \alpha = \eta.$$

Proof: First, we have

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \sum_{1\leq i\leq r} \mathrm{Rm}\left(\mathbf{e}_{(0,i)}, X, Y, \mathbf{e}_{(0,i)}\right) + \sum_{\substack{1\leq\mu\leq k \\ 1\leq\beta\leq d_\mu}} \mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\mu,\beta)}, X, Y, \widetilde{\mathbf{e}}_{(\mu,\beta)}\right) \\ &= \mathrm{Ric}^{\mathcal{B}}(X,Y) + \sum_{\substack{1\leq\mu\leq k \\ 1\leq\beta\leq d_\mu}} g\left(-\frac{\left(\nabla^2_{\mathcal{B}} h_\mu\right)(X,Y)}{h_\mu}\, \widetilde{\mathbf{e}}_{(\mu,\beta)}, \widetilde{\mathbf{e}}_{(\mu,\beta)} \right) \\ &= \mathrm{Ric}^{\mathcal{B}}(X,Y) - \sum_{1\leq\mu\leq k} \frac{d_\mu}{h_\mu} \left(\nabla^2_{\mathcal{B}} h_\mu\right)(X,Y). \end{aligned}$$

By the $\mathscr{C}^\infty(\mathcal{M})$-bilinearity of Ric, to prove the remaining items we may assume without loss of generality that $V = \widetilde{\mathbf{e}}_{(\alpha,\beta)}$ and $W = \widetilde{\mathbf{e}}_{(\eta,\xi)}$. We have

$$\begin{aligned} \mathrm{Ric}(X,V) &= \sum_{1\leq i\leq r} g\left(\mathrm{Rm}\left(\mathbf{e}_{(0,i)}, X\right) V, \mathbf{e}_{(0,i)}\right) + \sum_{\substack{1\leq\gamma\leq k \\ 1\leq\ell\leq d_\gamma}} g\left(\mathrm{Rm}\left(\mathbf{e}_{(\gamma,\ell)}, X\right) V, \mathbf{e}_{(\gamma,\ell)}\right) \\ &= -\sum_{1\leq i\leq r} \underbrace{g\left(\mathrm{Rm}\left(\mathbf{e}_{(0,i)}, X\right) \mathbf{e}_{(0,i)}, V\right)}_{=0, \text{ by } (1) \text{ of } (\mathbf{L.4.10})} - \sum_{\substack{1\leq\gamma\leq k \\ 1\leq\ell\leq d_\gamma}} \underbrace{g\left(\mathrm{Rm}\left(X, \mathbf{e}_{(\gamma,\ell)}\right) \mathbf{e}_{(\alpha,\beta)}, \mathbf{e}_{(\gamma,\ell)}\right)}_{=0, \text{ by } (3) \text{ and } (8) \text{ of } (\mathbf{L.4.10})} \\ &= 0. \end{aligned}$$

If $\alpha \neq \eta$, we have:

$$\mathrm{Ric}\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)}, \widetilde{\mathbf{e}}_{(\eta,\xi)}\right) = \sum_{1\leq i\leq r} \underbrace{g\left(\mathrm{Rm}\left(\mathbf{e}_{(0,i)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\xi)}, \mathbf{e}_{(0,i)}\right)}_{=0, \text{ by } (3) \text{ of } (\mathbf{L.4.10})} + \sum_{\substack{1\leq\gamma\leq k \\ 1\leq p\leq d_\gamma}} g\left(\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\gamma,p)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\xi)}, \widetilde{\mathbf{e}}_{(\gamma,p)}\right).$$

It is obvious that the terms of the last sum on the right-hand side above all vanish. Indeed, if $\gamma = \alpha$ this is absolutely trivial, and when γ, α and η are all distinct, item **(3)** of lemma **(L.4.8)** guarantees that

$$\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\gamma,p)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\eta,\xi)} = 0.$$

Now, when $\alpha = \eta$, we have

$$\begin{aligned}
\mathrm{Ric}\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)}, \widetilde{\mathbf{e}}_{(\alpha,\xi)}\right) &= \sum_{1\le i\le r} g\left(\mathrm{Rm}\left(\mathbf{e}_{(0,i)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)}, \mathbf{e}_{(0,i)}\right) + \sum_{\substack{1\le\gamma\le k\\ 1\le p\le d_\gamma}} g\left(\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\gamma,p)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)}, \widetilde{\mathbf{e}}_{(\gamma,p)}\right)\\
&= \sum_{1\le i\le r} g\left(\frac{-\delta_{\beta\xi}}{h_\alpha}\nabla^{\mathcal{B}}_{\mathbf{e}_{(0,i)}}\nabla^{\mathcal{B}} h_\alpha, \mathbf{e}_{(0,i)}\right) + \sum_{1\le p\le d_\alpha} g\left(\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\alpha,p)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)}, \widetilde{\mathbf{e}}_{(\alpha,p)}\right)\\
&+ \sum_{\substack{1\le\gamma\le k\\ \gamma\neq\alpha\\ 1\le p\le d_\gamma}} g\left(\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\gamma,p)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)}, \widetilde{\mathbf{e}}_{(\gamma,p)}\right)\\
&= -\frac{\delta_{\beta\xi}}{h_\alpha}\Delta_{\mathcal{B}} h_\alpha\\
&+ \sum_{1\le p\le d_\alpha} g\left(\mathrm{Rm}^{\mathcal{N}_\alpha}\left(\widetilde{\mathbf{e}}_{(\alpha,p)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)} + \frac{\|\nabla^{\mathcal{B}} h_\alpha\|^2}{(h_\alpha)^2}\left\{\delta_{p\xi}\,\widetilde{\mathbf{e}}_{(\alpha,\beta)} - \delta_{\beta\xi}\,\widetilde{\mathbf{e}}_{(\alpha,p)}\right\}, \widetilde{\mathbf{e}}_{(\alpha,p)}\right)\\
&- \sum_{\substack{1\le\gamma\le k\\ \gamma\neq\alpha\\ 1\le p\le d_\gamma}} g\left(\delta_{\beta\xi}\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}} h_\gamma, \nabla^{\mathcal{B}} h_\alpha\right)}{h_\gamma\, h_\alpha}\widetilde{\mathbf{e}}_{(\gamma,p)}, \widetilde{\mathbf{e}}_{(\gamma,p)}\right)\\
&= -\frac{\delta_{\beta\xi}}{h_\alpha}\Delta_{\mathcal{B}} h_\alpha + \mathrm{Ric}^{\mathcal{N}_\alpha}\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)}, \widetilde{\mathbf{e}}_{(\alpha,\xi)}\right) + \frac{\|\nabla^{\mathcal{B}} h_\alpha\|^2}{(h_\alpha)^2}\delta_{\beta\xi} - \frac{\|\nabla^{\mathcal{B}} h_\alpha\|^2}{(h_\alpha)^2}\delta_{\beta\xi}\, d_\alpha\\
&- \delta_{\beta\xi}\sum_{\substack{1\le\gamma\le k\\ \gamma\neq\alpha}} d_\gamma\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}} h_\gamma, \nabla^{\mathcal{B}} h_\alpha\right)}{h_\gamma\, h_\alpha}\\
&= \mathrm{Ric}^{\mathcal{N}_\alpha}\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)}, \widetilde{\mathbf{e}}_{(\alpha,\xi)}\right) - \left(\begin{array}{l}\dfrac{\Delta_{\mathcal{B}} h_\alpha}{h_\alpha} + (d_\alpha - 1)\,\dfrac{\|\nabla^{\mathcal{B}} h_\alpha\|^2}{(h_\alpha)^2}\\ \qquad + \displaystyle\sum_{\substack{1\le\gamma\le k\\ \gamma\neq\alpha}} d_\gamma\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}} h_\gamma, \nabla^{\mathcal{B}} h_\alpha\right)}{h_\gamma\, h_\alpha}\end{array}\right) g\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)}, \widetilde{\mathbf{e}}_{(\alpha,\xi)}\right).
\end{aligned}$$

■

Lemma (L.4.12). Let $\mathcal{M} = B \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiple warped product with metric $g = g_B \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$. Then the scalar curvature of $\mathcal{M}$ is given by

$$\begin{aligned}
R = R_B - 2\sum_{1\le i\le k} d_i\frac{\Delta_B h_i}{h_i} + \sum_{1\le i\le k}\frac{R_{\mathcal{N}_i}}{h_i^2} - \sum_{1\le i\le k} d_i(d_i-1)\frac{\|\mathrm{grad}_B h_i\|_B^2}{h_i^2}\\
+ \sum_{1\le i\le k}\sum_{\substack{1\le\ell\le k\\ \ell\neq i}} d_i d_\ell\frac{g_B(\mathrm{grad}_B h_i, \mathrm{grad}_B h_\ell)}{h_i h_\ell}.
\end{aligned} \tag{4.37}$$

Proof: We have

$$
\begin{aligned}
R &= \sum_{1\leq i\leq r} \mathrm{Ric}\left(\mathbf{e}_{(0,i)},\mathbf{e}_{(0,i)}\right) + \sum_{\substack{1\leq\alpha\leq k\\ 1\leq\varphi\leq d_\alpha}} \mathrm{Ric}\left(\widetilde{\mathbf{e}}_{(\alpha,\varphi)},\widetilde{\mathbf{e}}_{(\alpha,\varphi)}\right)\\
&= R_{\mathcal{B}} - \sum_{\substack{1\leq\mu\leq k\\ 1\leq i\leq r}} \frac{d_\mu}{h_\mu}\left(\nabla^2_{\mathcal{B}} h_\mu\right)\left(\mathbf{e}_{(0,i)},\mathbf{e}_{(0,i)}\right)\\
&\quad + \sum_{\substack{1\leq\alpha\leq k\\ 1\leq\varphi\leq d_\alpha}} \mathrm{Ric}^{\mathcal{N}_\alpha}\left(\widetilde{\mathbf{e}}_{(\alpha,\varphi)},\widetilde{\mathbf{e}}_{(\alpha,\varphi)}\right) - \sum_{\substack{1\leq\alpha\leq k\\ 1\leq\varphi\leq d_\alpha}} \left(\begin{array}{l}\dfrac{\Delta_{\mathcal{B}} h_\alpha}{h_\alpha} + (d_\alpha - 1)\,\dfrac{\|\nabla^{\mathcal{B}} h_\alpha\|^2}{(h_\alpha)^2}\\ \qquad + \displaystyle\sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha}} d_\gamma\,\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}} h_\gamma,\nabla^{\mathcal{B}} h_\alpha\right)}{h_\gamma\, h_\alpha}\end{array}\right) g\left(\widetilde{\mathbf{e}}_{(\alpha,\varphi)},\widetilde{\mathbf{e}}_{(\alpha,\varphi)}\right)\\
&= R_{\mathcal{B}} - 2\sum_{1\leq\mu\leq k} d_\mu \frac{\Delta_{\mathcal{B}} h_\mu}{h_\mu} + \sum_{1\leq\mu\leq k} \frac{R_{\mathcal{N}_\mu}}{(h_\mu)^2} - \sum_{1\leq\mu\leq k} d_\mu\,(d_\mu - 1)\,\frac{\|\mathrm{grad}_{\mathcal{B}} h_\mu\|^2}{(h_\mu)^2}\\
&\quad + \sum_{1\leq\alpha\leq k}\sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha}} d_\alpha\, d_\gamma\,\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}} h_\gamma,\nabla^{\mathcal{B}} h_\alpha\right)}{h_\gamma\, h_\alpha}.
\end{aligned}
$$

■

Example (E.4.6). For this example, we first recall that a Riemannian manifold $(\mathcal{M}, g)$ has constant sectional curvature equal to $S \in \mathbb{R}$ if, and only if, its curvature tensor is given by

$$\mathrm{Rm} = S\,(g \owedge g)\,,$$

or, equivalently, if and only if,

$$\mathrm{Rm}(X,Y,Z,W) = S\,(g(X,W)\,g(Y,Z) - g(X,Z)\,g(Y,W))\,,$$

for all $X, Y, Z, W \in \Gamma(T\mathcal{M})$. Now, denoting

$$\mathbb{Q}^n(\kappa) \doteq \begin{cases}\mathbb{R}^n, & \text{if } \kappa = 0;\\ \mathbb{S}^n, & \text{if } \kappa = 1;\\ \mathbb{H}^n, & \text{if } \kappa = -1;\end{cases}$$

we have (as proved in [30]) that the Riemannian tensor of $\mathcal{I} \times_h \mathbb{Q}^n(\kappa)$ (where $\mathcal{I} \subset \mathbb{R}$ is an open interval and h is a smooth positive real function defined on $\mathcal{I}$) is given by

$$
\begin{aligned}
\mathrm{Rm}(X,Y,Z,W) &= \left(\frac{(h')^2 - \kappa}{h^2}\circ\pi_{\mathcal{I}}\right)(g(X,Z)\,g(Y,W) - g(X,W)\,g(Y,Z))\\
&\quad + \left(\frac{h\,h'' - (h')^2 + \kappa}{h^2}\circ\pi_{\mathcal{I}}\right)\left(\begin{array}{l}g(X,Z)\,g\left(Y,\partial_t\right)\,g\left(W,\partial_t\right) - g(Y,Z)\,g\left(X,\partial_t\right)\,g\left(W,\partial_t\right)\\ \\ - g(X,W)\,g\left(Y,\partial_t\right)\,g\left(Z,\partial_t\right) + g(Y,W)\,g\left(X,\partial_t\right)\,g\left(Z,\partial_t\right)\end{array}\right).
\end{aligned}
$$

In particular, $\mathcal{I} \times_h \mathbb{Q}^n(\kappa)$ has constant sectional curvature equal to $C \in \mathbb{R}$ if, and only if,

$$-C = \frac{(h')^2 - \kappa}{h^2} \text{ is constant and } \frac{h''}{h} = \frac{(h')^2 - \kappa}{h^2}.$$

Consequently,

$\mathbb{S}^n$ is locally isometric to $\mathcal{I} \times_{\sin(t)} \mathbb{S}^{n-1}$;
$\mathbb{H}^n$ is locally isometric to $\mathcal{I} \times_{e^t} \mathbb{R}^{n-1}$ and to $\mathcal{I} \times_{\sinh(t)} \mathbb{S}^{n-1}$;
$\mathbb{R}^n$ is locally isometric to $\mathcal{I} \times_t \mathbb{S}^{n-1}$.

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