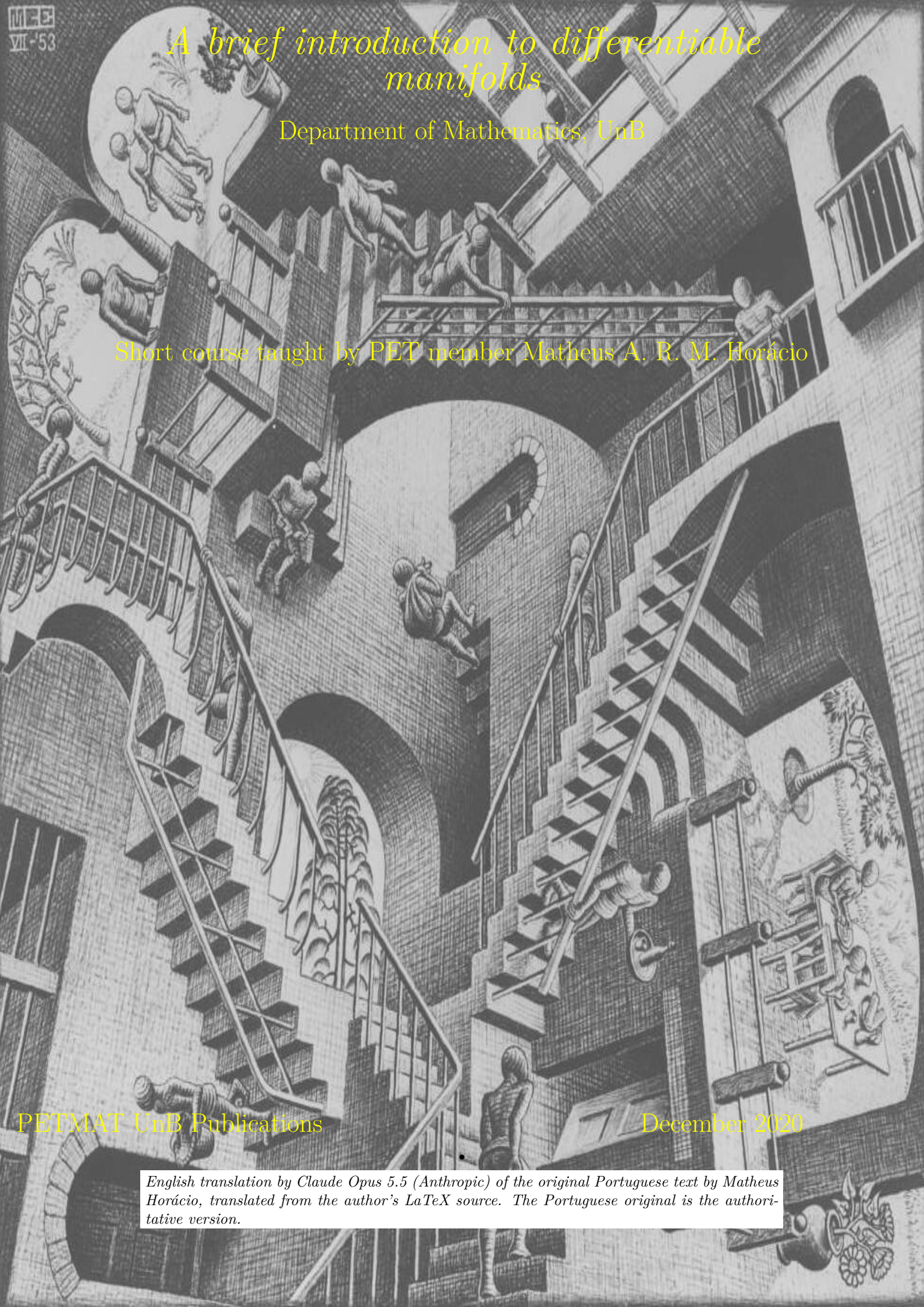


A brief introduction to differentiable manifolds

Department of Mathematics, UnB

Short course taught by PET member Matheus A. R. M. Horácio

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PET members:

Ayrton Anjos
Caio Tomás
Davi Batisaco
Giulia Albuquerque
Hebert Luan
Jorge Lucas
Matheus Freitas
Railandi Sousa
Thais Marçal
Thailany Machado
Carlos Campos
Manoel Fernando dos Reis
Bárbara Guerra Ribeiro
Matheus Andrade Ribeiro de Moura Horácio

PETMAT tutor: Luciana Maria Dias de Ávila Rodrigues

Preface and acknowledgments

I prepared the content of this short course always thinking about what I would have liked to see when I was learning the subject. Naturally, other people would have liked to see other perspectives, but I think the content here is, for the most part, satisfactory enough for basically everyone, especially as an introduction. I would have liked to have had more time to present more concepts, but at the end of the day it is a *mini*-course. The ideal prerequisites for studying this subject as seen here are only analysis in $\mathbb{R}^n$ and general topology (although in the near future, even if beyond the scope of this short course, a good knowledge of multilinear algebra is also necessary), but this is hardly an obstacle for the persistent reader, and I strongly encourage those who are interested to be persistent! I myself am a big fan of "taking leaps longer than your legs": even if you get hurt along the way, in the end the experience is well worth it.

I thank Rodrigo Duarte for the friendship that always encouraged me to go beyond where I thought possible, without which I certainly would not have learned everything I show here in time to present this short course. I also thank my undergraduate research advisor, Professor João Paulo dos Santos, for the guidance, for the patience, for the encouragement and for always having agreed to embark with me on adventures that seemed reckless at the undergraduate level, which made the preparation of this short course possible. I also thank Professor Luciana Ávila, tutor of PETMAT UnB, for the excellent work in having made the group what it was during the time I took part in it, without which this short course would probably not have been prepared either. I thank IMPA for having made available online the video lectures of the course Analysis on Manifolds, which were a main basis of this short course. Finally, I thank FNDE/MEC for the financial support while I was a scholarship holder of PETMAT UnB.

1 Preliminaries

1.1 A little bit of topology

Giving a topology to an arbitrary set $\mathcal{M}$ consists of specifying exactly which of its subsets are to be called "open". Such subsets should have familiar properties: $\mathcal{M}$ and $\emptyset$ are open, the intersection of any two open sets must remain open, and so must an arbitrary union (finite or infinite, countable or not). More precisely, we say that a topology on $\mathcal{M}$ is a collection $\tau \subset \mathcal{P}(\mathcal{M})$ (whose elements will be called open sets) such that

- $\mathcal{M}$ and $\emptyset$ are open
- $U_1 \cap U_2 \in \tau$ whenever $U_1, U_2 \in \tau$
- $\bigcup_{\alpha \in I} U_\alpha \in \tau$ for any collection $\{U_\alpha\}_{\alpha \in I}$ of open sets of $\mathcal{M}$ (where I is an arbitrary family of indices)

Exercise 1. *Show that any finite intersection of open sets is still an open set.*

But then you may ask yourself: why should the topology consist of open sets? And why do we want them to satisfy such properties? The following explanation, found in [1], does a good job of clearing up these doubts.

We can think of topology as "the art of imprecise measurements" or "geometry without distance", in a sense that can be made more (lol) precise as follows: in a perfect world, you could imagine rulers that can measure lengths with exact precision. That is, in this world there are rulers of exact length and each ruler comes with a length marking: so, to prove that a certain object has length ℓ, it would suffice to hold the ruler marked ℓ next to the object and it would be proved that the object indeed has length *exactly* ℓ.

But we live in an imperfect world, and in our imperfect world there are only rulers with tolerance. The best we can do is to associate to each ruler with tolerance a set U of real numbers with the property that whenever ℓ is in U, the ruler can tell you with certainty: indeed, $\ell \in U$. We will call such a ruler R_U.

Given two rulers R_U and R_V, you can easily prove that a given length lies within $U \cup V$. Just use the ruler R_U and then R_V: if one of the rulers gives a positive result, the length is certainly in $U \cup V$. We can then think of the ruler $R_{U \cup V}$ as a kind of virtual ruler. Analogously we can determine another ruler $R_{U \cap V}$.

Now, if we have an infinite family of rulers, say, $\{R_{U_i}\}_{i \in I}$, it is also easy to show that a given length lies in $\bigcup_{i \in I} U_i$: if it really is in this union, it is in some U_i, and you just need to exhibit the appropriate ruler R_{U_i} for the corresponding i.

But we cannot do the same for $\bigcap_{i \in I} U_i$: it may happen that we need an infinite proof to verify that all the R_{U_i} give us a positive result. Informally, we can then think of a topology as a (generalized) set of rulers that fits the description above.

Terminology. *A neighborhood of $p \in \mathcal{M}$ is an open set U_p of $\mathcal{M}$ such that $p \in U_p$. A topological space is a pair $(\mathcal{M}, \tau_{\mathcal{M}})$. When the topology of $\mathcal{M}$ is implicitly understood, we simply write $\mathcal{M}$ instead of the pair.*

Definition 1. *If X is a topological space and $U \subset X$, the topology on U induced by X is defined as $\tau_U \doteq \{U \cap A \mid A \in \tau_X\}$: that is, they are the open sets of X as seen by U.*

Definition 2. *A function $f : (X, \tau_X) \to (Y, \tau_Y)$ between two topological spaces is said to*

be continuous when $f^{-1}(O_Y) \in \tau_X$ *for every* $O_Y \in \tau_Y$.

Exercise 2. *Show that the previous definition is equivalent to the following: for each* $x \in X$ *and each neighborhood* V *of* $f(x)$*, there exists a neighborhood* U *of* x *such that* $f(U) \subset V$.

Exercise 3. *Let* X, Y *and* Z *be topological spaces. Show that the following maps are continuous:*

- *The identity map* $Id_X : X \to X$ *defined by* $\mathrm{Id}_X(x) = x$ *for each* $x \in X$*;*
- *Any constant function* $F : X \to Y$ *(that is, a function such that* $F(x) = F(y)$ *for all* $x, y \in X$*)*
- *Any composition* $G \circ F$ *of continuous maps* $F : X \to Y$ *and* $G : Y \to Z$

Exercise 4. *Show that continuity is a local phenomenon: that is,* $F : X \to Y$ *is continuous if and only if at each point* $p \in X$ *there exists a neighborhood* $X \supset U_p \ni p$ *such that* $F|_U$ *is continuous.*

In general we do not want to be stuck with having to define absolutely every open set of a space in order to endow it with some topology. This is where bases of topological spaces come in.

Definition 3. *A collection* $\mathcal{B}$ *of open subsets of a topological space* X *is said to be a basis for the topology of* X *if each open set of* X *can be written as a union of elements of* $\mathcal{B}$*. Even more generally, if* X *is merely a set (a priori without any topology) and* $\mathcal{B}$ *is a collection of subsets of* X *satisfying:*

- $X = \bigcup_{B \in \mathcal{B}} B$
- *If* $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 \cap B_2$*, there exists* $B_3 \in \mathcal{B}$ *such that* $x \in B_3 \subset B_1 \cap B_2$

Then the collection of all unions of elements of $\mathcal{B}$ *defines a topology on* X*, called the topology generated by* $\mathcal{B}$*, and* $\mathcal{B}$ *is naturally a basis for the topology so generated.*

Definition 4. *A topological space* X *is said to be Hausdorff when for all* $x, y \in X$ *with* $x \neq y$*, there exist neighborhoods* $U_x \ni x, U_y \ni y$ *satisfying* $U_x \cap U_y = \emptyset$.

Exercise 5. *Show that the topology generated by* $\mathcal{B}$ *is indeed a topology on* X *and that* $\mathcal{B}$ *is a basis for the topology generated by* $\mathcal{B}$.

Example. Given $x = (x^1, x^2, \cdots, x^n) \in \mathbb{R}^n$, denote by $\|x\|$ the standard Euclidean norm of $\mathbb{R}^n$ and by $B_\varepsilon(x) \doteq \{y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon\}$ the ball with center x and radius ε. The collection $\mathcal{B} = \{B_\varepsilon(x) \mid x \in \mathbb{R}^n, \varepsilon > 0\}$ generates a topology on $\mathbb{R}^n$ called the standard topology on $\mathbb{R}^n$.

Exercise 6. *Prove that every open set of $\mathbb{R}^n$ can be written as a union of open balls with rational radii and centers all of whose coordinates are rational. Conclude that $\mathbb{R}^n$ has a countable basis that generates the same topology as in the previous example.*

Exercise 7. *Show that if $U \subset \mathbb{R}^n$ is open and $a \in U$, then there exists $\delta > 0$ such that every $v \in \mathbb{R}^n$ with $\|v\| < \delta$ satisfies $a + v \in U$ and the line segment (open or closed) $(a - \delta v, a + \delta v)$ remains in U.*

Definition 5. *Let X be a topological space. We say that X is connected when the only open sets of X whose complement is also open are X and $\emptyset$. Equivalently, X is said to be connected when $X = U \cup V$ (with $U, V \in \tau_X$) implies that U or V is empty.*

Exercise 8. *Prove the equivalence stated in the previous definition.*

Exercise 9. *Let X be a topological space and $x \in X$. The connected component of x, which we will call C_x, is defined as the union of all connected subsets of X that contain x. Show that*

1. *C_x is connected and is the largest (in the sense of inclusion) connected subset of X containing X*
2. *Any two connected components associated to two points either coincide or are disjoint*
3. *The relation $x \sim y \iff y \in C_x$ is an equivalence relation. Determine its classes.*

Definition 6. *Let X be a topological space and $A \subset X$. We say that A is compact when every open cover of A (i.e, a subset $\mathcal{C}$ of τ_A such that $\bigcup_{C \in \mathcal{C}} C = A$) admits a finite subcover (that is, a finite subset of $\mathcal{C}$ that is still an open cover).*

Exercise 10. *Prove the intermediate value theorem and the Weierstrass theorem: every continuous real function with compact domain attains (at least) a maximum and a minimum.*

We also need to talk about the (finite) product of topological spaces.

Definition 7. *If $X_1, \cdots, X_k$ are topological spaces, the collection of all subsets of the product $\prod_{i=1}^{k} X_i$ of the form $\prod_{i=1}^{k} U_i$ (where each U_i is open in X_i) forms a basis for a topology on $\prod_{i=1}^{k} X_i$, which is called the product topology. From now on, whenever we work with a product space this is the topology we will be considering.*

One easily verifies that the product topology satisfies reasonable properties: a function between a topological space and a product space (with finitely many factors) is continuous

if and only if its coordinate functions are continuous, the product of Hausdorff spaces is always Hausdorff, the product of bases is a basis, the finite product of spaces with countable bases has a countable basis, the product of connected (resp. compact) spaces is connected, and so on. If you have not verified these properties yet, you are advised to do so.

Remark. It is "easy" to differentiate functions between Euclidean spaces: besides having the structure of a topological space, they have a vector space structure that is entirely compatible with their topology. In general our life is not that easy: adding and subtracting (which is essential for us to define differentiability in Euclidean spaces) is something totally meaningless in arbitrary topological spaces; in that case we cannot talk about linearity.

But measuring the rate at which things change (that is, studying differentiability) is essential to any application of mathematics and is certainly not something we want to lose. This is where differential geometry comes in: one of our goals will be to define special topological spaces, where it will make sense to differentiate functions. Naturally, these special spaces (which we will call differentiable manifolds) should include open subsets of vector spaces as particular cases and at the same time not be too "crazy", so that the topological spaces we encounter naturally are differentiable manifolds (whatever that will mean). Before that, let us recall well how good life is in Euclidean spaces.

1.2 Differentiability of functions of several variables

We will denote the coordinates in $\mathbb{R}^n$ by $x^1, \cdots, x^n$ and write $p = (p^1, p^2, \cdots, p^n) \in U$ to denote a point of an open set U of $\mathbb{R}^n$. If $0 < k \in \mathbb{Z}$, we say that a function $f : U \to \mathbb{R}$ is C^k at p (a fact we denote by $f \in C^k_p(U)$) when the partial derivatives

$$\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}}$$

exist and are continuous at p for all $j \in \{1, \cdots, k\}, i_1, \cdots, i_j \in \{1, \cdots, n\}$. If it happens that f is C^k at p for every $0 \leq k \in \mathbb{Z}$ we say that f is C^∞ at p, or, equivalently, $f \in C^\infty_p(U)$. Extending this definition to functions $f : U \to V \subset \mathbb{R}^m$, we say that $f \in C^k_p(U)$ when $f^1, \cdots, f^m \in C^k_p(U)$ (where $f^1, \cdots, f^m$ are the coordinate functions of f, i.e, $f(p) = (f^1(p), \cdots, f^m(p))$). Finally, if f is C^k at every $p \in U$, we simply say that $f \in C^\infty(U)$. Let us now recall what the differentiability of f means and which properties its derivative $f' = \mathrm{d}f : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ enjoys.

Definition 8. *A continuous function $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ is said to be differentiable at $p \in U$ when there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ (that is, an m by n matrix) such that:*

$$\lim_{u \to 0} \frac{|f(p+u) - f(p) - Tu|}{|u|} = 0$$

We denote T by $\mathrm{d}f_p$.

Exercise 11. *For each $k \in \mathbb{N}$, construct a function that is C^k but not C^{k+1}.*

Exercise 12. *Prove that the matrix of* $\mathrm{d}f_p$ *is given by*

$$\mathrm{d}f_p = \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(p) & \cdots & \frac{\partial f^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(p) & \cdots & \frac{\partial f^m}{\partial x^n}(p) \end{pmatrix}$$

Conclude that $\mathrm{d}f_p$ *is well defined (that is, it is unique).*

Exercise 13. *Prove that* $\mathrm{d}f = f$ *whenever* f *is linear.*

Remark. The matrix above is usually called the *Jacobian matrix* of f at the point p, and its determinant the *Jacobian determinant* of f at the point p.

Exercise 14.

- *By the way it was constructed,* $\mathrm{d}f_p$ *is linear, and in the next item you are asked to prove that this definition coincides with the usual one when* $n = m = 1$*. A common mistake is to think: "well, consider the function* $t \mapsto t^3$*, for example: its derivative is* $t \mapsto 3t^2$*, which is evidently not linear, so the notions cannot coincide!" Explain why this reasoning is wrong.*
- *Make sense of the definition above when* $n = m = 1$*, explaining how it coincides with the definition of differentiability you know from Calculus 1.*
- *Show that differentiability is a local phenomenon: that is, if* U' *is another neighborhood of* p *and* V' *is another neighborhood of* $f(p)$*, then a function* $g : U' \to V'$ *that coincides with* f *on a neighborhood of* p *(that is, there exists* $\delta > 0$ *such that* $f|_{B_\delta(p)} = g|_{B_\delta(p)}$*) is differentiable at* p *if and only if* f *is, and, moreover, their derivatives at* p *coincide (that is,* $\mathrm{d}g_p = \mathrm{d}f_p$*).*

Exercise 15. *Suppose that* U*,* V *are open subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, and let* $f : U \to V$*,* $g : V \to \mathbb{R}^\ell$ *be differentiable. Prove and make sense of the equalities*

$$(g \circ f)'(p) = \mathrm{d}(g \circ f)_p = \mathrm{d}g_{f(p)} \circ \mathrm{d}f_p$$
$$\frac{\partial (g^i \circ f)}{\partial x^j}(p) = \sum_{r=1}^{m} \frac{\partial g^i}{\partial y^r}(f(p)) \frac{\partial f^r}{\partial x^j}(p), \text{ for every } (i,j) \in \{1, \cdots, \ell\} \times \{1, \cdots, n\}$$

As a particular case, prove that for any path $\lambda : (-\delta, \delta) \to U$ *of class* C^∞ *such that* $\lambda(0) = p$ *and* $\lambda'(0) = v$*, we have* $\mathrm{d}f_p(v) = (f \circ \lambda)'(0)$ *(where* $v \in \mathbb{R}^n$ *is arbitrary).*

1.3 Topological and differentiable notions of equivalence

In every area of mathematics we seek to distinguish our objects of study up to a certain relevant notion of equivalence (in fact, one can argue that this is essentially all that mathematicians do...). In topology and analysis it could not be otherwise: two topological

spaces X and Y will be called homeomorphic when there exists a continuous bijective function $F : X \to Y$ with continuous inverse whose inverse F^{-1} is also continuous, and two open sets $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ will be called diffeomorphic when there exists a differentiable function $F : U \to V$ that is a homeomorphism with inverse also differentiable. In these cases, we call F (respectively) a homeomorphism/diffeomorphism.

Exercise 16. *Let $f : X \to Y$ be a homeomorphism and $U \subset X$ an open set. Show that $f(U)$ is open in Y and that U is homeomorphic to $f(U)$.*

Exercise 17. *Let X and Y be homeomorphic topological spaces. Show that X is compact (resp. connected) if, and only if, Y is.*

Exercise 18. *Show that if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are two open sets diffeomorphic via some F, then $n = m$ and, moreover, $\mathrm{d}F_p$ is an isomorphism for every $p \in U$.*

Exercise 19. *Show that every finite composition of diffeomorphisms (resp. homeomorphisms) is still a diffeomorphism (resp. homeomorphism).*

Exercise 20. *Let X and Y be arbitrary topological spaces and $f : X \to Y$ a bijective function. Exhibit topologies on X and Y that make them homeomorphic.*

Exercise 21. *Show that translations and dilations are homeomorphisms, i.e: for any $A \subset \mathbb{R}^n$, $\lambda \neq 0$ and $p \in \mathbb{R}^n$, A is homeomorphic to $\lambda A = \{\lambda a \mid a \in A\}$ and to $A + p = \{a + p \mid a \in A\}$.*

Exercise 22. *Show that the topological spaces (A, τ_A) and (B, τ_B) are homeomorphic, where*

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 \mid \max\{|x|, |y|\} = 1\}$$

Think about it: are A and B diffeomorphic?

Exercise 23. *Show that any open ball of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$ itself.*

Exercise 24. *Show that if $f : X \to Y$ is continuous, then $G \doteq \{(x, f(x)) \mid x \in X\} \subset X \times Y$ is homeomorphic to X.*

Exercise 25. *Show that any differentiable cylinder in $\mathbb{R}^3$ (i.e, any subset of $\mathbb{R}^3$ diffeomorphic to $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$) is diffeomorphic to $\mathbb{R}^2 \setminus \{p\}$ (where $p \in \mathbb{R}^2$ is arbitrary), but $\mathbb{R}^2$ is not homeomorphic to any differentiable cylinder.*

Exercise 26. *Show that any differentiable cone in $\mathbb{R}^3$ (i.e, any subset of $\mathbb{R}^3$ diffeomorphic to $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$) is not diffeomorphic to $\mathbb{R}^2$.*

The definitions above guarantee that there is no topological (resp. differentiable) property that two homeomorphic (resp. diffeomorphic) spaces do not share simultaneously.

1.4 The most important local theorems

The two theorems below tell us (under reasonably relaxed hypotheses) the local behavior of differentiable functions in terms of their derivatives. More importantly, they essentially confirm our intuition of what differentiable functions look like on a small scale. Their proofs will not be presented here (just consult any good analysis book), but it is worth stressing that they fall into the famous class of "you need to have seen and understood this at least once in your life".

Theorem 1.1 (**Inverse function theorem**). *Let $U, V \subset \mathbb{R}^n$ be open and $F : U \to V$ differentiable. If $\mathrm{d}F_a$ is an isomorphism at some point $a \in U$, then there exist connected neighborhoods $a \in U_0 \subset U$ and $F(a) \in V_0 \subset V$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Theorem 1.2 (**Implicit function theorem**). *Let $U \subset \mathbb{R}^n \times \mathbb{R}^k$ be open and let us denote by $(x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^k)$ the standard coordinates on U. Suppose further that $\phi : U \to \mathbb{R}^k$ is differentiable, $(a, b) \in U$ and $c \doteq \phi(a, b)$. If the k by k matrix*

$$\left(\frac{\partial \phi^i}{\partial y^j}(a, b)\right)$$

is invertible, then there exist neighborhoods $a \in V_0 \subset \mathbb{R}^n$, $b \in W_0 \subset \mathbb{R}^k$ of b and a differentiable function $F : V_0 \to W_0$ such that $\phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F, i.e, $\phi(x, y) = c$ with $(x, y) \in V_0 \times W_0$ happens if, and only if, $y = F(x)$.

Remark. The two theorems above are, in fact, just one (that is, having one it is possible to prove the other and vice versa).

Exercise 27. *Suppose that $U \subset \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^n$ is differentiable with nonzero Jacobian determinant at every point of U. Prove that F maps open sets (of U) to open sets (of $\mathbb{R}^n$), and that if F is injective, it is also a diffeomorphism onto its image.*

Exercise 28. *Show that the restriction of the map F defined by*

$$(0, \infty) \times \mathbb{R} \ni (r, \theta) \mapsto F(r, \theta) = (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2$$

to any open subset of its domain on which it is injective is a diffeomorphism onto its image. Give an example of such a subset.

Exercise 29. *Exactly the same exercise as the previous one, but now for the map G defined by*

$$(0, \infty) \times (0, \pi) \times \mathbb{R} \ni (\rho, \varphi, \theta) \mapsto G(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \in \mathbb{R}^3$$

1.5 Einstein notation

From now on, we will adopt the Einstein convention when dealing with sums. It consists, essentially, of making the summation symbols disappear, agreeing on when certain indices

are being summed or not, and deducing in the obvious way how far the implicit sum goes. More explicitly, all summation symbols are omitted, with the common understanding that if the same index appears in a monomial expression once below and once above, we are in fact summing the expression over that index. For example:

- $v = \sum_{i=1}^{n} v^i e_i$ or $f = \sum_{i=1}^{n} f_i e^i$ simply become $v = v^i e_i$, $f = f_i e^i$
- We agree that if an index appears in the denominator of an expression, it is always below, which makes it completely consistent to write

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

  instead of

$$\frac{\partial}{\partial x^i} = \sum_{i=1}^{n} \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

- One must be careful with "dummy" indices when making substitutions. For example, if $p^i = a^i_j v^j$ and $v^j = b^j_i w^i$, it is not correct to write $p^i = a^i_j b^j_i w^i$ (in fact this expression is not even consistent with our notation). The correct thing would be to notice that in $v^j = b^j_i w^i$, the index i is dummy, so we could write $v^j = b^j_k w^k$ and substitute $p^i = a^i_j b^j_k w^k$.
- We define $\delta^j_i = \delta^j_i = \delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

At the moment this notation may seem hard to get used to and more laziness than something actually necessary, but believe me: it pays off (and a looot; maybe not now, but it is important to root good habits). Expressions become much less cluttered and it is a self-correcting notation, easy to remember, that practically writes itself. It saves us from having to write aberrations such as:

$$R = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} R^\ell_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \partial_\ell$$

which we trade for simply

$$R = R^\ell_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \partial_\ell$$

The "price" we have to pay for this convention to work is small: the choice of writing an index above or below is *not* arbitrary. Having made these necessary considerations, let us move on to a few more examples:

Example. Let us denote by $V \doteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ the space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will show that

$$\mathcal{B} = \{\pi_{ij} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \mid \pi_{ij}(e_i) = e_j \text{ and } \pi_{ij}(e_k) = 0 \text{ for all } k \neq i\}$$

is a basis of V using Einstein notation. Indeed, if $a^{lt} \in \mathbb{R}$ are such that $a^{lt}\pi_{lt} = 0$, in particular we will have

$$a^{lt}\pi_{lt}(e_s) = a^{lt}\delta_{sl}e_t = 0$$

for every $s \in \{1, \cdots, n\}$ (where we define $\delta_{sl} = 1$ if $s = l$ and $\delta_{sl} = 0$ otherwise, and $\{e_i\}$ in this case denotes the canonical basis). In particular, since $\{e_t\}_{1\leq t\leq m}$ is a basis of $\mathbb{R}^m$, we have

$$a^{lt}\delta_{sl} = a^{st} = 0$$

for all $l \in \{1, \cdots, n\}$ and $s \in \{1, \cdots, n\}$, $t \in \{1, \cdots m\}$, showing that $\mathcal{B}$ is linearly independent. Now, denoting by $\mathrm{d}w^t$ the map $\mathbb{R}^m \ni v = v^i e_i \mapsto \mathrm{d}w^t(v) = v^t$, we have

$$\begin{aligned} T(y) = T(y^i e_i) &= y^i T(e_i) \\ &= y^i \mathrm{d}w^t(Te_i) e_t \\ &= \mathrm{d}w^t(Te_i)\pi_t^i(y) \end{aligned}$$

for all $T \in V$ and $y \in \mathbb{R}^n$, showing that any $T \in V$ can be written as

$$T = c^{it}\pi_{it}$$

where the constants depend on T. This proves that $\mathcal{B}$ spans V.

Example. Let V be a real vector space of finite dimension n endowed with an inner product g and let $\{e_i\}_{i=1}^n$ be an **arbitrary** basis of V. We will show that every $v \in V$ is entirely determined by the values $\{g(v, e_i)\}_{1 \leq i \leq n}$. Note that since every $v \in V$ is entirely determined by its components in the basis $\{e_i\}_{i=1}^n$, it suffices to find an expression for these in terms of $\{g(v, e_i)\}_{1 \leq i \leq n}$. First, denoting $g_{ij} \doteq g(e_i, e_j)$ and by g^{ij} the coefficients of the inverse matrix of $\tilde{g} = (g_{ij})$ (which is well defined, since $\tilde{g}v = 0 \iff v = 0$), observe that the equality

$$g(v, e_k) = g(v^i e_i, e_k) = v^i g_{ki}$$

holds. And therefore

$$g^{kt} g(v, e_k) = v^i g_{ki} g^{kt} = v^i \delta_{it} = v^t$$

as desired. This process is entirely reversible. Indeed, knowing that $g^{ls} g(v, e_\ell) = v^s$, we recover

$$\begin{aligned} g^{\ell s} g_{sm} g(v, e_\ell) = \delta_m^\ell g(v, e_\ell) &= g(v, e_m) \\ &= v^s g_{sm} \end{aligned}$$

Exercise 30. *Redo the two previous examples, but now writing out **all** the sums, indices and the sets over which they range. Moreover, justify all the equalities used.*

2 Differentiable manifolds

Although the intent of this very brief introduction is to clarify a subject that should be almost common knowledge, it is impossible to motivate how manifolds definitively entered our lives without talking about geometry. The discussion we present below tries to explain why manifolds are not only useful but, moreover, *necessary* in a certain sense, and it is important (for obvious reasons) to stress that a large part of it also appears almost *verbatim* in [2].

2.1 Why do we need manifolds?

For a looong time the (false) idea that Euclidean geometry models our universe with complete fidelity persisted in the minds of several giants. It is fitting to insert now a paragraph eloquently written by the fantastic physicist and mathematician Roger Penrose in [3]:

> The fact that Euclidean geometry seems to reflect so accurately the structure of the "space" of our world has fooled us (or, at least, our ancestors!) into thinking that this geometry is a logical necessity, or into thinking that we have an innate *a priori* intuitive understanding that Euclidean geometry *must* apply to the world in which we live (even the great philosopher Immanuel Kant claimed this!). Our real break with Euclidean geometry only came with Einstein's general relativity, which was put forward many years later. Far from Euclidean geometry being a logical necessity, it is an *empirical observational fact* that this geometry applies so accurately (though not exactly) to the structure of our physical space! Euclidean geometry was, indeed, always an amazing *physical* theory. This in addition to being an elegant and logical part of pure mathematics.

Kant's perception was frustrated: shortly afterwards, the first problems of Euclidean geometry were discovered in the mind of Gauss. Soon after, Riemann laid the essential foundations of the area that we know today as *Riemannian geometry*, which strongly generalizes all the work of Gauss and his predecessors. Some 100 years later, Einstein (crucially using the work of Riemann) showed that our universe is indeed non-Euclidean (for example, where there are large concentrations of mass, as in black holes, this is much more noticeable); among other things, his theory describes the curvature and geometry of small regions of space in terms of how much matter and energy they contain. But what does curvature really mean from an intrinsic perspective? Why is Euclidean geometry not general enough? These questions, more philosophical than mathematical, were the essential sparks for manifolds to enter our story for good. Let us now recall a very controversial postulate of Euclid, which started all this "turmoil":

The fifth postulate. *If a straight line intersects two other straight lines making two interior angles on the same side that add up to less than two right angles, then the two lines, when extended indefinitely, meet on that side on which the angles add up to less than two right angles.*

By the end of the 18th century, the structure of Euclidean geometry was still obscure. Although many people (perhaps even Euclid) believed that Euclid's fifth postulate was a consequence of the four others, possibly together with some additional postulate that would be simpler and more elegant, nobody had managed to find any candidate for such an additional postulate. The discomfort with the fifth postulate arises from its complexity in contrast to the others, and also because one cannot verify it (or even test its approximate

accuracy) by direct physical measurements, since the point where the two lines meet could escape the limits of your sheet of paper.

In 1817, Gauss had already convinced himself that there should exist other geometries satisfying Euclid's first four postulates (and, of course, also satisfying the common notions), but not the fifth. He did not publish his ideas. One reason for his reticence is quite plausible: Gauss could not clearly define his new geometry, and he often described himself in letters to friends as a "perfectionist". Also for this reason he delayed for years the publication of his contributions to the theory of surfaces.

Several partial constructions of non-Euclidean geometries appear as early as 1829: in the works of Lobachevski, followed by a work of Bolyai. They satisfied the first four postulates, but not the fifth; however, they had no solid mathematical foundation. The correct construction of non-Euclidean geometry requires abstraction, since (as we see in a good course on classical differential geometry and as Hilbert proved) there are no candidates for any such geometry inside Euclidean space: this is our first motivation for Riemannian geometry (and, of course, for differentiable manifolds before it).

Moreover, even if you understand surfaces in $\mathbb{R}^3$ very well, there are many reasons to work in dimensions 3, 4, etc. For example, to treat time as an additional coordinate, or to investigate the space of lines, the space of circles, the space of spheres, etc. There is no limit to the number of parameters, i.e dimensions, needed to describe sets of geometric objects. Even if there is some elementary method that manages to place such a geometric object in some Euclidean space $\mathbb{R}^n$, such an embedding may not be geometrically equivariant, i.e it may happen that symmetries of the geometric set do not extend to symmetries of Euclidean space.

Such situations appear, for example, in classical mechanics. Start with the notion of a rigid body. It has six parameters: three for the location of the center of gravity, and three to say how the body has been rotated about that center. You might even manage to avoid having to work in a Euclidean space of dimension 6, since the center of gravity lives in a three-dimensional Euclidean space, but what is the set of rotations, as a three-dimensional object? How can we explore geometry there? The reader who has struggled with mechanics books may have suffered with the *Euler angles*, and for such a reader this question is not superfluous. We would like to have a more general theory to deal with these questions, in which the motions of the rigid body will be curves with geometric meaning.

More generally, we need to define the set of positions of a mechanical object, such as a multiple pendulum, and see what the trajectories of this object are, as curves in that set. For a double pendulum, it is possible to show that the space of such positions is a surface (in fact, a torus), but that the motions of the pendulum do not correspond to geodesics of the torus for any torus in $\mathbb{R}^3$: they will only be geodesics for a differentiable manifold endowed with an abstract Riemannian geometry.

In statistical mechanics, physicists consider systems of N particles, with N large. The space of their positions has dimension $3N$, and to carry complete information about their motions, one needs to know their positions and velocities, whence arises a *phase space* of dimension $6N$, which is not simply $\mathbb{R}^{6N}$, due to collisions. Therefore our third reason for abstraction is that we are already familiar with manipulating such abstract spaces, and were trying to pretend that we were not (i.e trying to parametrize them).

Hilbert showed that no complete surface immersed in $\mathbb{R}^3$ can have constant negative Gaussian curvature, and this tells us that the abstract Riemannian geometry of surfaces, in general, can only be represented locally, and even then only subject to some nondegeneracy

hypothesis, in $\mathbb{R}^3$. The program of abstract Riemannian geometry received two decisive thrusts, the first by Gauss in 1827, and the second by Riemann in 1854. Essentially, Gauss conceived the concept of the intrinsic metric of a surface in $\mathbb{R}^3$, and then realized that the *first fundamental form* of a surface (the Euclidean inner product restricted to its tangent spaces) and the intrinsic geometry of the surface (i.e, the metric obtained by measuring distances *only* within the surface, without ever leaving it) determine each other.

Riemann made two simultaneous innovations: first, he defined (not very rigorously) a differentiable manifold as any set of any dimension n where one can do differential calculus, change coordinates, etc. In particular, we get differentiable curves, tangent vectors (velocities) of such curves, and a tangent space at each point (i.e all possible velocities of any curves passing through that point). He then asked that a geometry on a manifold be simply an arbitrary positive definite quadratic form on each of these tangent spaces, thought of as an analogue of Gauss's first fundamental form. One could then use the same expression to define lengths of curves, look for the curves of least distance between two points, etc.

In this short course, we will not concern ourselves with the immense task of how to define the properties of such an abstract geometry: this involves searching for geometric invariants that generalize the invariants of Gauss, the Gaussian curvature, and extensions of the results of Gauss (Riemann managed to find the right invariant, known today as the *Riemann curvature tensor*). Technical tools such as parallel transport and absolute differential calculus were developed at the end of the 19th century, culminating in the work of the Italian school of Ricci and Levi-Civita, but the underlying concept of manifold was only definitively understood by Whitney in 1936.

Instead, we will treat (very briefly) only the absolute essentials so that one can actually generalize the classical differential geometry that we know from $\mathbb{R}^3$ to general topological spaces (sufficiently well behaved in a way that we will make precise later) in the near future. This involves, among a *manifold* (lol) of other things, explaining how to generalize the concepts of differential surface, tangent vectors, orientation of surfaces, vector fields and their variations, etc., to such spaces.

2.2 What exactly are manifolds?

From our experience in $\mathbb{R}^3$ we are already familiar with several surfaces: spheres, cylinders, tori, planes, etc. We have always thought of these spaces extrinsically: as subsets that live in a larger *ambient space*, in this case a three-dimensional Euclidean space. On the other hand, we can think of surfaces from another, *intrinsic*, point of view: seeing locally how the surface behaves at different points, but without introducing any external space (for example, it makes perfect sense to talk about the Klein bottle as an intrinsic abstract surface, but it cannot be seen extrinsically in the three-dimensional space $\mathbb{R}^3$, only in $\mathbb{R}^4$). In the end the two points of view are essentially equivalent, but for several reasons it is generally easier (and more interesting!) to adopt the intrinsic perspective. In certain situations, such as in Einstein's theory of relativity, the extrinsic perspective makes no sense at all, since there is no physical sense whatsoever in "seeing the universe from an extrinsic perspective".

A good example of a surface is the surface of the Earth. Extrinsically, it can be (naively) seen as a subspace of the three-dimensional space $\mathbb{R}^3$. But it can also be seen intrinsically as a two-dimensional surface by means of an *atlas*: a collection of *maps* or *charts* that describe various regions of the surface by identifying them with a subset of a two-dimensional plane. As long as we have enough charts to cover the original surface,

the resulting atlas is enough to describe the surface. Of course, even this perspective is not completely intrinsic (in fact, coordinate-free approaches *a la Cartan* are much more elegant, but that is another story), since there is (much) more than one atlas that can be associated with the surface of the Earth, and they may all differ[1] significantly or not. But there are several facts that can be deduced with an atlas that do not depend on the specific atlas chosen: for example, using any accurate atlas of the Earth we can see that it is not possible to go from Brasília to Lisbon without crossing at least one ocean. It is *this* kind of *intrinsic* or *coordinate-free* property that we will be interested in studying. When we define differentiability, for example, we will use coordinates, but we will have to prove that our definition is indeed coordinate-free. This kind of situation should already be familiar to you if you have studied classical differential geometry of surfaces.

Gauss, in his fantastic work *Disquisitiones generales circa superficies curvas* ("General Investigations of Curved Surfaces") was the first to adopt an intrinsic perspective in the study of surfaces, freely using local coordinates (and therefore already having the notion of charts), treating surfaces as abstract spaces that exist in their own right, independently of any external Euclidean space.

His student, Riemann, in probably the most influential lecture of the last centuries (which managed to amaze even Gauss himself) *Uber die Hypothesen, welche der Geometrie zu Grunde liegen* ("On the hypotheses on which geometry is founded"), laid the foundation of differential geometry generalized to arbitrary dimensions, known today as *Riemannian geometry*. In fact, the Portuguese word "variedade" itself is a translation of the English "manifold", which in turn is a translation of the German word "Mannigfaltigkeit", which Riemann used to describe his objects of study.

Having made these considerations, let us now see what it precisely means for a topological space to be locally Euclidean before we define manifolds.

2.2.1 Topological manifolds

Definition 9. *A topological space $\mathcal{M}$ is locally Euclidean of dimension n if every $p \in \mathcal{M}$ has a neighborhood $U_p \in \mathcal{M}$ that is homeomorphic to an open set V_p of $\mathbb{R}^n$.*
Terminology. *If $p \in \mathcal{M}$, $U \ni p$ is a neighborhood of p and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism, we say that φ is a chart or a coordinate system around p.*

> **Exercise 31.** *Using exercise 23, show that this definition is unchanged if we replace "an open set V_p" by "an open ball B_p" or by $\mathbb{R}^n$ itself (that is, $V_p = \mathbb{R}^n$).*

With this, we can finally define what a *topological manifold* is.
Definition 10. *A topological manifold of dimension n is a topological space $(\mathcal{M}, \tau)$ with a countable basis, Hausdorff and locally Euclidean of dimension n.*
Remark. Although it may initially seem that being locally Euclidean, being Hausdorff and having a countable basis are related, this is *not* true! It turns out that these three conditions are entirely independent.
Remark. We will now see that the dimension of a connected topological manifold is indeed well defined (i.e, a connected topological manifold cannot have dimensions m and n at the same time if $m \neq n$): this is due to the fact called *invariance of dimension*: two Euclidean open sets of different dimensions are never homeomorphic. This result is not

[1] But because of the Theorema Egregium they all have one thing in common: none of them is completely faithful; they will always distort angles or areas or some other geometric property.

trivial and its proof is beyond the scope of this material, so it will be used freely here.

Indeed, let $(\mathcal{M}, g)$ be a connected topological manifold of dimension n and consider on it the equivalence relation $\sim$, where $p \sim q$ if there exist charts around both with images in the same $\mathbb{R}^n$ (note that the charts need not be the same here, we are only asking that the images of the charts have the same dimension). It is clear that $\sim$ really is an equivalence relation. Moreover, for any $p \in \mathcal{M}$, by definition there exists a chart around p that lands in $\mathbb{R}^n$, and the equivalence class $[p] = \{q \in \mathcal{M} \mid \text{there exists a chart } \varphi_q : U_q \to \mathbb{R}^n\}$ is open (it is easy to see that $[p]$ is the union of the U_q above, which are open by definition). By connectedness, the partition associated with this equivalence relation consists of a single class. And since two open sets of different dimensions cannot be homeomorphic, it is clear that all the charts land only in $\mathbb{R}^n$, which is what we wanted to prove.

What we have just done shows that the topological dimension of a manifold is one of its *intrinsic* properties. It is also worth observing that our definition is *born* excluding "oddities" such as the disjoint union of a plane and a line, for example: one cannot have different dimensions at different points. It is also obvious that in these circumstances the empty set can be considered as an n-dimensional manifold for any n. Another interesting question that can be asked is the following: are we "gaining" more spaces with this definition? Are there manifolds that are not simply reasonably well-behaved subsets of $\mathbb{R}^n$ for some sufficiently large n? The answer, for better or worse, is no: every topological manifold, of any dimension, is homeomorphic to some subset of some $\mathbb{R}^n$. This also holds for differentiable and Riemannian manifolds, replacing homeomorphisms by diffeomorphisms and isometries, respectively (see the Whitney and Nash embedding theorems).

Exercise 32. *Show that every open subset of a topological manifold is also a topological manifold in its own right.*

Exercise 33. *Show that $\mathbb{R}^n$ is a topological manifold for every $n \in \mathbb{N}$.*

Exercise 34. *Let X be a topological space, $x \in X$ arbitrary and let U be a neighborhood of x. If there always exists an open set V such that $x \in V \subset \overline{V} \subset U$ (for any x and U), we say that X is regular. Show that every topological manifold is regular.*

Exercise 35. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Show that*

$$\text{graph}(f) = \{(x, f(x)) \mid x \in U\} \subset \mathbb{R}^{n+1}$$

is a topological manifold. What is its dimension?

Exercise 36. *If $p \in \mathcal{M}$ and φ is a chart around p, we say that φ is centered at p if $\varphi(p) = 0$. Show that every point of a manifold has a chart centered at it.*

Exercise 37. *Show that*

$$A = \{(x, 0] \mid x \in (-1, 0]\} \cup \{(x, x) \mid x \in (0, 1)\} \cup \{(x, -x) \mid x \in (0, 1)\} \subset \mathbb{R}^2$$

is not a topological manifold with the topology induced from $\mathbb{R}^2$.

Example. The cross

$$C = \{(x, 0) \mid x \in (-1, 1)\} \cup \{(0, x) \mid x \in (-1, 1)\} \subset \mathbb{R}^2$$

is not a topological manifold with the topology induced from $\mathbb{R}^2$. Indeed, suppose by contradiction that it were: then there would exist a neighborhood of $p \doteq (0, 0)$, say U_p, that is homeomorphic to $B \doteq B(0, \varepsilon) \subset \mathbb{R}^n$ (for some $\varepsilon > 0$ and some n) centered at p. This chart induces an obvious homeomorphism between $U_p \setminus \{p\}$ and $B \setminus \{0\}$. But $U_p \setminus \{p\}$ has four connected components while $B \setminus \{0\}$ either is connected (if $n \geq 2$) or has two connected components (if $n = 1$). This contradiction shows that C is not locally Euclidean at $(0, 0)$.

Example. Let $\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} \mid \|v\| = 1\} \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere, and let us denote by N its *north pole* $(0, \cdots, 0, 1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, and by S the *south pole* $(0, \cdots, 0, -1)$. It is clear that $\mathbb{S}^n$ with the topology induced from $\mathbb{R}^n$ has a countable basis and is Hausdorff, and since the stereographic projections:

$$\begin{aligned} \pi_N : \mathbb{S}^n \setminus \{N\} &\to \mathbb{R}^n \\ (x^1, \cdots, x^n) &\mapsto \frac{(x^1, \cdots, x^n)}{1 - x^{n+1}} \end{aligned}$$

and

$$\begin{aligned} \pi_S : \mathbb{S}^n \setminus \{S\} &\to \mathbb{R}^n \\ (x^1, \cdots, x^n) &\mapsto \frac{(x^1, \cdots, x^n)}{1 + x^{n+1}} \end{aligned}$$

are homeomorphisms, it follows that the sphere is a topological manifold.

Exercise 38. *Verify that the inverse functions of π_N and π_S are given (respectively) by*

$$\mathbb{R}^n \ni (u^1, \cdots, u^n) \mapsto \left(\frac{2u^1}{\|u\|^2 + 1}, \cdots, \frac{2u^n}{\|u\|^2 + 1}, \frac{\|u\|^2 - 1}{\|u\|^2 + 1} \right) \in \mathbb{S}^n$$

$$\mathbb{R}^n \ni (u^1, \cdots, u^n) \mapsto \left(\frac{2u^1}{\|u\|^2 + 1}, \cdots, \frac{2u^n}{\|u\|^2 + 1}, \frac{\|u\|^2 - 1}{1 - \|u\|^2} \right) \in \mathbb{S}^n$$

2.2.2 Differentiable manifolds

Note that it still makes no sense at all to talk about any kind of differentiability of functions between manifolds. But, since we can locally identify neighborhoods of topological manifolds with Euclidean neighborhoods, it is natural to try to define differentiability in this context with the charts in mind, locally identifying open sets of the manifold with Euclidean open sets, where we know how to get by.

Suppose that $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ are two coordinate systems of a topological manifold. Since $U \cap V$ is open in U and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, the image $\phi(U \cap V)$ is also an open subset of $\mathbb{R}^n$. Obviously $\psi(U \cap V)$ is also an open subset of $\mathbb{R}^n$.

Definition 11. *Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are said to be $\mathcal{C}^\infty$-compatible when the maps*

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞. These two maps are called transition functions between the two coordinate systems.

There is a rich and complex theory for $\mathcal{C}^k$ manifolds, but it is more complicated and not as commonly studied. We will be interested here only in $\mathcal{C}^\infty$ things, so whenever we talk about differentiability, assume $\mathcal{C}^\infty$ (unless, of course, explicitly stated otherwise).

Definition 12. *A $\mathcal{C}^\infty$ (compatible) atlas, or simply an atlas, on a locally Euclidean space $\mathcal{M}$ is a collection $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ of pairwise $\mathcal{C}^\infty$ compatible charts that cover $\mathcal{M}$, that is, that satisfy $\mathcal{M} = \bigcup_{\alpha \in I} U_\alpha$.*

Example. The unit circle $\mathbb{S}^1$ in the complex plane $\mathbb{C}$ can be described as the set of points $\{e^{it} \in \mathbb{C} \mid 0 \leq t \leq 2\pi\}$. Let U_1 and U_2 be the two open subsets of $\mathbb{S}^1$ defined by

$$U_1 = \{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\}$$
$$U_2 = \{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\}$$

And let us define $\phi_\alpha : U_\alpha \to \mathbb{R}$ for $\alpha \in \{1, 2\}$ by

$$\phi_1\left(e^{it}\right) = t, -\pi < t < \pi$$
$$\phi_2\left(e^{it}\right) = t, 0 < t < 2\pi$$

Since ϕ_1 and ϕ_2 are branches of the complex logarithm $z \mapsto \frac{1}{i}\log(z)$ and are homeomorphisms onto their respective images, it follows that (U_1, ϕ_1) and (U_2, ϕ_2) are charts on $\mathbb{S}^1$. The intersection $U_1 \cap U_2$ consists of two connected components given by

$$A = \{e^{it} \mid -\pi < t < 0\}$$
$$B = \{e^{it} \mid 0 < t < \pi\}$$

satisfying

$$\phi_1(U_1 \cap U_2) = \phi_1(A \cup B) = \phi_1(A) \cup \phi_1(B) = (-\pi, 0) \cup (0, \pi)$$
$$\phi_2(U_1 \cap U_2) = \phi_2(A \cup B) = \phi_2(A) \cup \phi_2(B) = (\pi, 2\pi) \cup (0, \pi)$$

where the unions above are disjoint. The transition functions are given by

$$\left(\phi_2 \circ \phi_1^{-1}\right)(t) = \begin{cases} t + 2\pi & \text{if } t \in (-\pi, 0) \\ t & \text{if } t \in (0, \pi) \end{cases}$$

and

$$\left(\phi_1 \circ \phi_2^{-1}\right)(t) = \begin{cases} t - 2\pi & \text{if } t \in (\pi, 2\pi) \\ t & \text{if } t \in (0, \pi) \end{cases}$$

Therefore (U_1, ϕ_1) and (U_2, ϕ_2) are $\mathcal{C}^\infty$-compatible charts and constitute a $\mathcal{C}^\infty$ atlas on $\mathbb{S}^1$.

Although $\mathcal{C}^\infty$ compatibility is clearly reflexive and symmetric, it is not transitive. Indeed, suppose that (U_1, ϕ_1) is $\mathcal{C}^\infty$-compatible with (U_2, ϕ_2) and that (U_2, ϕ_2) is $\mathcal{C}^\infty$-compatible with (U_3, ϕ_3). Note that these 3 coordinate functions are simultaneously defined only on the triple intersection $U_{123} = U_1 \cap U_2 \cap U_3$. Hence, the composition

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$$

is $\mathcal{C}^\infty$, but only on $\phi_1(U_{123})$ and not necessarily on $\phi_1(U_{13})$. A priori we know nothing about $\phi_3 \circ \phi_1^{-1}$ on $\phi_1(U_{13} \backslash U_{123})$ and therefore we cannot conclude that (U_1, ϕ_1) and (U_3, ϕ_3) are $\mathcal{C}^\infty$-compatible.

Definition 13. *We say that (V, ψ) is compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$ when it is compatible with all the charts (U_α, ϕ_α) of the atlas.*

It is tempting to say now that a differentiable manifold is simply a topological manifold with a compatible atlas... but if the atlas does not have all the charts it could have, this causes problems. That is why we will need the following:

Definition 14. *A compatible atlas $\mathcal{A}$ on $\mathcal{M}$ is said to be maximal when it is not contained in any other. Equivalently, given any homeomorphism $\varphi : U \subset \mathcal{M} \to \varphi(U) \subset \mathbb{R}^n$ and any $\varphi_V \in \mathcal{A}$ with $V \cap U \doteq W \neq \emptyset$ such that the compositions $\varphi_V \circ \varphi^{-1}$ and $\varphi \circ \varphi_V^{-1}$ are diffeomorphisms, then $\varphi \in \mathcal{A}$.*

Evidently, if $\mathcal{A}$ is an atlas, there exists a (unique) maximal atlas $\overline{A}$ such that $\mathcal{A} \subset \overline{\mathcal{A}}$: just add to $\mathcal{A}$ all the charts that satisfy the property of the definition above.

Terminology. *A maximal atlas is also called a differentiable structure.*

Now we can indeed introduce the

Definition 15. *A differentiable manifold $\mathcal{M}^n$ is a topological manifold endowed with a differentiable structure.*

As is evident from the definitions, one expects all the local theorems seen in the previous section to continue to hold for manifolds: after all, they are simply "little pieces of $\mathbb{R}^n$ glued together in a differentiable way". We will confirm this intuition shortly. Before that, let us see some examples (it is worth observing that in general we do not give a maximal atlas when proving that a space is a differentiable manifold: we only give an atlas, and the maximal atlas associated with it is implicitly assumed):

Example. $\mathbb{R}^n$ is a differentiable manifold. Indeed, it is trivially locally Euclidean, and $\mathcal{A} = \{\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n\}$ is an atlas for $\mathbb{R}^n$, whose maximal atlas is evidently $\bar{\mathcal{A}} = \{\varphi : U \to \varphi(U) \mid \varphi \text{ is a diffeomorphism}\}$.

Exercise 39. *Show that every open subset of a differentiable manifold is itself also a differentiable manifold, and use this fact to prove that the set of invertible n by n matrices (considered as a subset of $\mathbb{R}^{n^2}$) is a differentiable manifold.*

Example. $\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} \mid \|v\| = 1\}$ with the topology induced from $\mathbb{R}^{n+1}$ is a differentiable manifold: as an atlas, just take the stereographic projections from the north pole and from the south pole.

Exercise 40. *Verify the details of the previous example. And answer: why does any atlas of the previous manifold have at least two charts?*

Example. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^\infty$. Then

$$\text{graph}(f) = \{(x, f(x)) \mid x \in U\}$$

is a differentiable manifold, and an atlas for it is $\text{graph}(f) \ni (x, f(x)) \mapsto \pi(x) = x \in U$
Example. Let $\mathcal{M}^m$ and $\mathcal{N}^n$ be differentiable manifolds and consider the space $\mathcal{M}^m \times \mathcal{N}^n$ with the product topology. Evidently it is locally Euclidean. Moreover, given $(p_1, p_2) \in \mathcal{M} \times \mathcal{N}$, we know that there exist charts (φ_1, U_1) and (φ_2, U_2) around p_1 and p_2, respectively, and we can then define

$$\begin{aligned}\varphi_{U\times V} : U_1 \times U_2 &\to \mathbb{R}^{n+m} \\ (p_1, p_2) &\mapsto (\varphi_1(p_1), \varphi_2(q_2)) = (\varphi_1 \times \varphi_2)(p_1, p_2)\end{aligned}$$

Letting then $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ be $\mathcal{C}^\infty$ atlases for $\mathcal{M}$ and $\mathcal{N}$ respectively, this motivates us to define the possible atlas for the product given by

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i)\}$$

which is indeed a $\mathcal{C}^\infty$ atlas, since given any charts (U, φ_U), $(U', \varphi_{U'})$, (V, φ_V), $(V', \varphi_{V'})$ around $p_1 \in \mathcal{M}$ and $p_2 \in \mathcal{N}$, the change of coordinates satisfies:

$$\varphi_{U\times V} \circ \varphi^{-1}_{U'\times V'} = (\varphi_U \times \varphi_V) \circ (\varphi_{U'} \times \varphi_{V'})^{-1} = (\varphi_U \circ \varphi^{-1}_{U'}) \times (\varphi_V \circ \varphi^{-1}_{V'})$$

and is therefore smooth, since it is a function between Euclidean open sets whose component functions are all smooth.

2.3 How to do calculus on manifolds

2.3.1 Differentiable maps between manifolds

Definition 16. *A continuous function $F : M_1 \to M_2$ between two manifolds is said to be differentiable (or $\mathcal{C}^\infty$) at $p \in M_1$ if there exist a chart (φ, U) around p and a chart (ξ, V) around $F(p)$ such that the composition $\xi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\varphi(p)$. When this happens at every $p \in M_1$, we say that F is differentiable.*
Remark. This is indeed well defined, that is, differentiability does not depend on the charts chosen. Indeed, suppose that $F : \mathcal{M} \to \mathcal{N}$ is smooth at $p \in \mathcal{M}$ and let (U, ϕ), (V, ψ) be arbitrary charts around p and $F(p)$, respectively. Since F is smooth at p, there exist charts (U_α, ϕ_α) around $p \in \mathcal{M}$ and (V_β, ψ_β) around $F(p) \in \mathcal{N}$ such that $\psi_\beta \circ F \circ \phi^{-1}_\alpha$ is smooth at $\phi_\alpha(p)$. By the $\mathcal{C}^\infty$ compatibility of charts in a differentiable structure, the compositions $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi^{-1}_\beta$ are $\mathcal{C}^\infty$ on open subsets of Euclidean spaces. It then follows that the composition:

$$\psi \circ F \circ \phi^{-1} = (\psi \circ \psi^{-1}_\beta) \circ (\psi_\beta \circ F \circ \phi^{-1}_\alpha) \circ (\phi_\alpha \circ \phi^{-1})$$

is $\mathcal{C}^\infty$ at $\phi(p)$.

Note that we have to be careful with the domains of the composition above; evidently it is only defined when we restrict the domains of the charts. But we need not worry about this, since there is no loss of generality in assuming, for example, that $U \subset U_\alpha$, $F(U_\alpha) \subset V_\beta$ and $U_\beta \subset U$ (otherwise, let us work with the chart ϕ' obtained by restricting ϕ to $U' = U \cap U_\alpha \subset U_\alpha$, which we can do because we are working with a maximal atlas; analogously we can restrict to the open sets $F^{-1}(V_\beta) \cap U_\alpha$ and $V \cap V_\beta$).

Terminology. *If $F : \mathcal{M} \to \mathcal{N}$ is a smooth map and (U, φ) and (V, ψ) are smooth charts for $\mathcal{M}$ and $\mathcal{N}$, respectively, we call $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ the coordinate representation of F with respect to these coordinates. It maps (diffeomorphically) the Euclidean open set $\varphi(U \cap F^{-1}(V))$ into the Euclidean open set $\psi(V)$.*

Exercise 41. *Specialize the definition of differentiability of maps between manifolds to a definition of differentiability of maps between a manifold and $\mathbb{R}^n$.*

Definition 17. *We say that a differentiable function f between two manifolds is a diffeomorphism when f is bijective and f^{-1} is also differentiable.*
Definition 18. *We say that $f : M_1 \to M_2$ is a local diffeomorphism when $\forall p \in M_1$, there exist a chart (φ, U) around p and an open set $M_2 \supset V \ni f(p)$ such that $f|_U : U \to V$ is a diffeomorphism.*

Exercise 42. *Verify that* $\exp : \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C}$ *defined by* $\exp(\theta) = e^{i\theta}$ *is a local diffeomorphism but not a global one.*

Exercise 43. *Verify that the projections*

$$\pi_i : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i$$
$$(p_1, p_2) \mapsto p_i, i \in \{1, 2\}$$

are differentiable.

Exercise 44. *Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds. Show that a constant map between $\mathcal{M}$ and $\mathcal{N}$ is smooth, and that the identity map of $\mathcal{M}$ is smooth.*

Exercise 45. *Verify that the composition of differentiable maps between manifolds is differentiable.*

Proposition 2.1. *Let (U, ϕ) be a coordinate system on a manifold $\mathcal{M}^n$. Then the chart $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.*

Proof: ϕ was born a homeomorphism; it remains to show that it and its inverse are smooth. To test the smoothness of $\phi : U \to \phi(U)$, we can use the one-chart atlas $\{(U, \phi)\}$ on U and the atlas $\{(\phi(U), \mathrm{Id}_{\phi(U)})\}$ on the Euclidean open set $\phi(U)$. Since $\mathrm{Id}_{\phi(U)} \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U)$ is the identity map, ϕ is smooth. Analogously one proves that ϕ^{-1} is smooth.

■

Proposition 2.2. *Let U be an open subset of a manifold $\mathcal{M}^n$. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a coordinate system in the differentiable structure of $\mathcal{M}$.*

Proof: Whatever the coordinate system (U_α, ϕ_α) in the maximal atlas of $\mathcal{M}$, both ϕ_α

and its inverse are smooth by the previous proposition. As compositions of smooth maps, both $F \circ \phi_\alpha^{-1}$ and its inverse are smooth, whence it follows that (U, F) is compatible with the maximal atlas of $\mathcal{M}$ and therefore (since the atlas is maximal) that F is a chart of a coordinate system around U.

■

Theorem 2.3 (Invariance of Dimension under Diffeomorphisms)**.** *If $m \neq n$, a smooth differentiable manifold of dimension m is not diffeomorphic to any differentiable manifold of dimension n.*

Proof: Let $\emptyset \neq \mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds of dimension m and n. Suppose that $F : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism. Let us choose any point $p \in \mathcal{M}$ and let (U, φ) and (V, ψ) be smooth coordinate systems around p and $F(p)$, respectively. Then the coordinate representation of F is a diffeomorphism between Euclidean open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, therefore $m = n$.

■

Exercise 46. *Prove that the map $\mathbb{R} \ni t \mapsto (\cos(t), \sin(t)) \in \mathbb{S}^1$ is smooth.*

Exercise 47. *Show that every finite composition or product of diffeomorphisms is a diffeomorphism and that diffeomorphisms induce an equivalence relation on the set of all differentiable manifolds.*

Exercise 48. *Let $\mathcal{M}_1, \cdots, \mathcal{M}_k, \mathcal{N}$ be differentiable manifolds, and for each $1 \leq i \leq k$, let us denote by $\pi_i : \mathcal{M}_1 \times \cdots \times \mathcal{M}_k$ the projection onto the factor $\mathcal{M}_i$. Prove that a map $F : \mathcal{N} \to \mathcal{M}_1 \times \cdots \times \mathcal{M}_k$ is smooth if, and only if, each component function $F_i = \pi_i \circ F$ is smooth.*

Exercise 49. *Let $\widetilde{\mathbb{R}}$ be the topological manifold $\mathbb{R}$ with the differentiable structure defined by the global chart $x \mapsto x^3$. Verify that this is indeed a differentiable structure on $\mathbb{R}$ and that $\widetilde{\mathbb{R}}$ is diffeomorphic to $\mathbb{R}$ with its standard differentiable structure (given by the global chart $x \mapsto x$).*

2.4 The tangent space as an abstract entity

Let $\mathcal{M}$ be a differentiable manifold and $W \subset \mathcal{M}$ an open set. From now on, we will denote by $\mathcal{C}^\infty(W)$ the space of smooth functions defined on the manifold W (of course, with the natural differentiable structure that W inherits from $\mathcal{M}$). The space $\mathcal{C}^\infty(\mathcal{M})$ is an *algebra* (that is, a vector space where we can multiply elements) with the obvious operations of multiplication and addition defined pointwise.

Exercise 50. *Prove that $\mathcal{C}^\infty(W)$ always has infinite dimension.*

Before proceeding, let us return to $\mathbb{R}^n$ to introduce a bit more notation. Let us denote by $r^i : \mathbb{R}^n \to \mathbb{R}$ the function $\mathbb{R}^n \ni (x^1, \cdots, x^n) \mapsto r^i(x^1, \cdots, x^n) = x^i$ and by e_i the i-th

vector of the canonical basis of $\mathbb{R}^n$, so that $r^i(e_j) = \delta^i_j$.

Given a manifold $\mathcal{M}$ and a chart σ defined on a neighborhood of a point $p \in \mathcal{M}$, we will denote the function $r^i \circ \sigma$ simply by x^i, and we will call the functions x^i the *coordinates* of the chart σ or *local coordinates around* p. We often abuse this language and write things like *let* $p = (p^1, \cdots, p^n)$ *be a point in* $\mathcal{M}$.

Definition 19. *Let $\mathcal{M}$ be a differentiable manifold, $x \in \mathcal{M}$, U, V two neighborhoods of x and $f \in \mathcal{C}^\infty(U)$, $g \in \mathcal{C}^\infty(V)$. We say that f and g have the same germ at x if there exists a smaller neighborhood $W \subset U \cap V$ such that f and g coincide when restricted to W.*

Defining an obvious equivalence relation on the set of smooth functions around x, by setting $(U, f) \sim (V, g)$ if there exists a neighborhood $W \subset U \cap V$ such that f and g coincide when restricted to W, we can think of germs as equivalence classes. The germ of f is denoted by $[f]$, and the set of germs at x is denoted by $\mathcal{F}_x(\mathcal{M})$.

A germ at x has a well-defined value (only) at x, and this then induces a map $\mathcal{F}_x\mathcal{M} \ni [f] \mapsto \mathrm{eval}_x([f]) = f(x)$.

Exercise 51. *Show that $\mathcal{F}_x\mathcal{M} \cap \mathcal{F}_y\mathcal{M} = \emptyset$ whenever $x \neq y$.*

We can then define what a tangent vector at a point $x \in \mathcal{M}$ is.

Definition 20. *Let $\mathcal{M}$ be a differentiable manifold and $x \in \mathcal{M}$. A tangent vector at x is a linear map*

$$v : \mathcal{F}_x(\mathcal{M}) \to \mathbb{R}$$

that satisfies the derivation property:

$$v([fg]) = \mathrm{eval}_x([f])v(g) + \mathrm{eval}_x([g])v(f)$$

Since a tangent vector is a linear map between vector spaces, the set of all tangent vectors is itself a vector space, which we denote by $T_x(\mathcal{M})$. We will prove shortly that it has the same dimension as $\mathcal{M}$.

Exercise 52. *Let v be a tangent vector at $x \in \mathcal{M}$ and c a constant function defined on $\mathcal{M}$. Prove that $v([c]) = 0$.*

Since germs are defined via equivalence classes, it ends up being a bit more annoying to work with them. That is why we will reformulate the definition of a tangent vector in a somewhat more convenient way.

Definition 21. *Let $\mathcal{M}$ be a differentiable manifold, $x \in \mathcal{M}$, and let W be any neighborhood of x. A derivation of $\mathcal{C}^\infty(W)$ at x is a linear map $w : \mathcal{C}^\infty(W) \to \mathbb{R}$ that satisfies the derivation property*

$$w(fg) = f(x)w(g) + g(x)w(f)$$

Every vector $v \in T_x(\mathcal{M})$ naturally induces a derivation w of $\mathcal{C}^\infty(W)$, for any neighborhood W of x, by setting

$$w(f) \doteq v([f])$$

In fact, the converse also holds, as we will prove next. We first need a preliminary lemma, which is the most important reason why we ask manifolds to be Hausdorff and to have a countable basis: this guarantees that they are *paracompact*, which in turn guarantees

that the lemma below is true. Recall that for a smooth function $f : \mathcal{M} \to \mathbb{R}$, we denote by $\text{supp}(f)$ the *support* of f, defined by

$$\text{supp}(f) \doteq \overline{\{x \in \mathcal{M} \mid f(x) \neq 0\}}$$

Lemma 2.4 (**Bump functions**). *Let $\mathcal{M}$ be a differentiable manifold, and $K \subset U \subset \mathcal{M}$ subsets, where K is closed and U is open. Then there exists a smooth function $f : \mathcal{M} \to \mathbb{R}$ satisfying:*

1. $0 \leq f(x) \leq 1$ *for every* $x \in \mathcal{M}$
2. $\text{supp}(f) \subset U$
3. $f(x) = 1$ *for every* $x \in K$

Proof: See [4].

■

Proposition 2.5. *Let $\mathcal{M}$ be a differentiable manifold, $x \in \mathcal{M}$, and let W be any neighborhood of x. Then there exists a linear isomorphism between $T_x(\mathcal{M})$ and the space of derivations of $\mathcal{C}^\infty(W)$ at x.*

Proof: Let W be any neighborhood of x. We will prove the result in three steps:

1. Let $w : \mathcal{C}^\infty(W) \to \mathbb{R}$ be a derivation at x and suppose that $f \in \mathcal{C}^\infty(W)$ is identically zero on a neighborhood $V \subset W$ of x. We claim that $w(f) = 0$. To see this, let us choose a smooth function $\eta : \mathcal{M} \to \mathbb{R}$ such that $\eta(x) = 1$ and $\text{supp}(\eta) \subset V$ (whose existence is guaranteed by the previous lemma with $K = \{x\}$ and "U" equal to V). Let us define $g = \eta f$, thought of as a function from W to $\mathbb{R}$. Evidently g is identically zero (by the construction of f it vanishes on V, and since the support of η is in V, it also vanishes outside V and therefore vanishes on all of W), therefore $w(g) = 0$ by linearity (letting f_1 be the constant function equal to 1 on W, note that $w(g) = w(0 \cdot f_1) = 0w(f_1) = 0$). But by the derivation property:

$$w(g) = w(\eta f) = \eta(x)w(f) + f(x)w(\eta) = w(f)$$

   since $\eta(x) = 1$ and $f(x) = 0$. Hence $w(f) = 0$.

2. Now let $[f] \in \mathcal{F}_x(\mathcal{M})$. We claim that we can always find a representative of $[f]$ with domain W, that is, a smooth function $g : W \to \mathbb{R}$ satisfying $[g] = [f]$. For this, let (V, f) be any representative of $[f]$. Shrinking V if necessary, we can assume that $V \subset W$ (if it were not, it would suffice to intersect with W and it would be, where evidently we lose nothing by doing so, by the very definition of germs). Let us now choose a smaller neighborhood U of x such that $\overline{U} \subset V \subset W$ (which exists since every manifold is regular). Our goal now is to *extend* f to a smooth function g defined on W such that $g|_U = f$. For this, let us apply lemma 2.4 again, this time with $K = \bar{U}$ and "U" equal to V. Let us now consider the smooth function

$$g : W \to \mathbb{R}, \quad g(x) := \begin{cases} \eta(x)f(x), & x \in V \\ 0, & x \in W \backslash V \end{cases}$$

   Since $g|_U = f$, it is clear that $[g] = [f]$.

3. Now we finish the proof. Let $w : \mathcal{C}^\infty(W) \to \mathbb{R}$ be a derivation at x. We define a linear map $v : \mathcal{F}_x(\mathcal{M}) \to \mathbb{R}$ by setting

$$v([f]) \doteq w(f), \text{ where } (W, f) \text{ is any representative of } [f]$$

Which exists by step 2 and is well defined by step 1. Indeed, if (W, h) were another representative of f, then by hypothesis there would exist a smaller neighborhood V of x such that f and g coincide on V. Then by linearity $w(f)-w(h) = w(f-h) = 0$, where $w(f-h) = 0$ by step 1. Finally, it is clear that v is a derivation, and evidently the association $w \mapsto v$ inverts the association introduced right after definition 20. The desired isomorphism is then exactly $w \mapsto v$.

■

Thanks to this proposition, from now on we will always consider a tangent vector v as a derivation of $\mathcal{C}^\infty(W)$ at x for any open set W containing x. We usually take W as the domain of a chart or as the whole manifold $\mathcal{M}$. We emphasize that because of this proposition, it does not matter which W we choose:

Corollary 2.6. *Let $\mathcal{M}$ be a differentiable manifold and $\emptyset \neq W \subset \mathcal{M}$ an open set. Considering W as a differentiable manifold with the structure inherited from $\mathcal{M}$, then for any $x \in W$ there exists a canonical identification $T_x\mathcal{M} \cong T_xW$.*

The following corollary is also immediate from exercise 50:

Corollary 2.7. *Let $\mathcal{M}$ be a differentiable manifold, $p \in \mathcal{M}$, and let $f \in \mathcal{C}^\infty(W)$ for some open set $W \ni x$. If f is constant on some neighborhood of p then $v(f) = 0$ for every $v \in T_p(\mathcal{M})$.*

A concrete example of a tangent vector is the following:

Example. Let $\mathcal{M}$ be a differentiable manifold of dimension n, $\sigma : U \to O$ a chart on $\mathcal{M}$, and let us write $x^i = r^i \circ \sigma$ to denote the local coordinates of σ. Let $p \in U$ be arbitrary. We define a derivation of $\mathcal{C}^\infty(U)$ at p by:

$$\left.\frac{\partial}{\partial x^i}\right|_p : \mathcal{C}^\infty(U) \to \mathbb{R}$$

$$f \mapsto \left.\frac{\partial}{\partial x^i}\right|_p (f) \doteq \frac{\partial(f \circ \sigma^{-1})}{\partial r^i}(p) = \mathrm{d}(f \circ \sigma^{-1})_{\sigma(p)}(e_i)$$

We will prove shortly that the collection of the $\left.\frac{\partial}{\partial x^i}\right|_p$, with $i \in \{1, \cdots, n\}$, forms a basis of $T_p(\mathcal{M})$.

We will need the lemma below to establish the dimension of the tangent space. Before that, it will be useful to introduce the following notation: if $f : O \subset \mathbb{R}^n \to \mathbb{R}^k$ is a smooth map defined on an open subset of $\mathbb{R}^n$ and $x \in O$, we denote the partial derivative of f at x with respect to the j-th canonical vector by:

$$D_j f(x) = \mathrm{d}f_x(e_j) = \lim_{t \to 0} \frac{f(x + te_j) - f(x)}{t}$$

Lemma 2.8. *Let $O \subset \mathbb{R}^n$ be a star-shaped open set with $0 \in O$ (that is, given any $x \in O$, the line segment from 0 to x is contained in O). Suppose that $h : O \to \mathbb{R}$ is a smooth function. Then there exist n smooth functions $g_i : O \to \mathbb{R}$ with $i \in \{1, \cdots, n\}$ such that $g_i(0) = D_i h(0)$ satisfying*

$$h = h(0) + r^i g_i$$

Proof: Fix $x = (x^1, \cdots, x^n) \in O$ and consider the line segment from 0 to x given by $\gamma(t) = tx$. Setting $\delta = h \circ \gamma : [0, 1] \to \mathbb{R}$, the chain rule tells us that

$$\delta'(t) = \sum_{i=1}^{n} x^i D_i h(tx)$$

Hence

$$h(x) - h(0) = \delta(1) - \delta(0) = \int_0^1 \delta'(t)\ \mathrm{d}t = \sum_{i=1}^{n} x^i \int_0^1 D_i h(tx)\ \mathrm{d}t$$

Since $x^i = r^i(x)$ by definition, the desired result follows by defining $g_i(x) = \displaystyle\int_0^1 D_i h(tx)\ \mathrm{d}t$.

■

We are then finally ready to prove the

Theorem 2.9. *Let $\mathcal{M}$ be a differentiable manifold of dimension n, $\sigma : U \to O$ a chart on $\mathcal{M}$ and $x \in U$. Then any $v \in T_x(\mathcal{M})$ can be written uniquely as a linear combination of the form*

$$v = a^i \left.\frac{\partial}{\partial x^i}\right|_x$$

where $a^i = v(x^i)$. Therefore

$$\left\{ \left.\frac{\partial}{\partial x^i}\right|_x \ \textit{with}\ 1 \leq i \leq n \right\}$$

is a basis of $T_x(\mathcal{M})$.

Proof: We can assume without loss of generality that σ is centered at x and that O is a star-shaped open set (note that we are using corollary 2.6 here). Let $f \in \mathcal{C}^\infty(U)$ and let us apply the previous lemma with $h = f \circ \sigma^{-1}$. We then obtain $f = f(x) + x^i(g_i \circ \sigma)$, where

$$g_i(0) = D_i \left(f \circ \sigma^{-1}\right)(0) = \left.\frac{\partial}{\partial x^i}\right|_x (f)$$

Then for any derivation v, we have

$$v(f) = \underbrace{v(f(x))}_{=0} + \sum_{i=1}^{n} \left(v\left(x^i\right) g_i(0) + \underbrace{x^i(x)}_{=0} v\left(g_i \circ \sigma\right)\right) = \sum_{i=1}^{n} v\left(x^i\right) \left.\frac{\partial}{\partial x^i}\right|_x (f)$$

where we used corollary 2.7 and the hypothesis $\sigma(x) = 0$.

It remains to show that the vectors are linearly independent. Note that:

$$\left.\frac{\partial}{\partial x^i}\right|_x \left(x^j\right) = \left.\frac{\partial}{\partial x^i}\right|_x \left(r^j \circ \sigma\right) = D_i \left(r^j \circ \sigma \circ \sigma^{-1}\right)(\sigma(x)) = D_i r^j(\sigma(x)) = \delta_i^j$$

where we used the fact that $\mathrm{d}r^j = r^j$, since r^j is linear. Thus, if $v = a^i \left.\dfrac{\partial}{\partial x^i}\right|_x = 0$, we have $v(x^j) = a^j = 0$.

■

Remark. Suppose that σ and τ are two charts around a point x, with corresponding coordinate systems $x^i = r^i \circ \sigma$ and $y^i = r^i \circ \tau$. Taking $v = a^i \left.\dfrac{\partial}{\partial y^j}\right|_x$ in the previous proposition we then obtain

$$\left.\frac{\partial}{\partial y^j}\right|_x = \left.\frac{\partial}{\partial y^j}\right|_x \left(x^i\right) \left.\frac{\partial}{\partial x^i}\right|_x$$

But unwinding the definitions,

$$\left.\frac{\partial}{\partial y^j}\right|_x \left(x^i\right) = D_j \left(x^i \circ \tau^{-1}\right)(\tau(x)) = D_j \left(u^i \circ \sigma \circ \tau^{-1}\right)(\tau(x))$$

which is simply the (i,j)-th entry of the matrix of $\mathrm{d}(\sigma\circ\tau^{-1})_{\tau(x)}$, which shows that the change-of-basis matrix between the bases $\left\{\left.\frac{\partial}{\partial y^i}\right|_x\right\}_{1\leq 1\leq n}$ and $\left\{\left.\frac{\partial}{\partial x^i}\right|_x\right\}_{1\leq 1\leq n}$ is given by the matrix of $\mathrm{d}(\sigma\circ\tau^{-1})_{\tau(x)}$.

2.4.1 The differential and its expression in local coordinates

Definition 22. *Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds, and let $\varphi:\mathcal{M}\to\mathcal{N}$ be a smooth map. Fix $x\in\mathcal{M}$ and $v\in T_x\mathcal{M}$. We define a tangent vector $w\in T_{\varphi(x)}\mathcal{N}$ by*

$$w(f)\doteq v(f\circ\varphi),\ \textit{for each } f\in\mathcal{C}^\infty(\mathcal{N})$$

It is clear that w is a linear derivation of $\mathcal{C}^\infty(\mathcal{N})$ at $\varphi(x)$, and therefore $w\in T_{\varphi(x)}\mathcal{N}$. Moreover, if we denote w by $\mathrm{d}\varphi_x(v)$, evidently the map $T_x\mathcal{M}\ni v\mapsto\mathrm{d}\varphi_x(v)\in T_{\varphi(x)}\mathcal{N}$ is linear. We say that $\mathrm{d}\varphi_x$ is the derivative (or differential) of φ at x.

Theorem 2.10 (*The chain rule on manifolds*)**.** *Let $\mathcal{M},\mathcal{N}$ and $\mathcal{V}$ be differentiable manifolds, and suppose that $\varphi:\mathcal{M}\to\mathcal{N}$ and $\psi:\mathcal{N}\to\mathcal{V}$ are smooth maps. Then:*

$$\mathrm{d}(\psi\circ\varphi)_x=\mathrm{d}\psi_{\varphi(x)}\circ\mathrm{d}\varphi_x,\forall x\in\mathcal{M}$$

Proof: Let $v\in T_x\mathcal{M}$ and $f\in\mathcal{C}^\infty(\mathcal{V})$ be arbitrary. Then:

$$\mathrm{d}(\psi\circ\varphi)_x[v](f)=v(f\circ\psi\circ\varphi)=\mathrm{d}\varphi_x[v](f\circ\psi)=\mathrm{d}\psi_{\varphi(x)}[\mathrm{d}\varphi_x(v)](f)$$

■

Exercise 53. *Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds, $F:\mathcal{M}\to\mathcal{N}$ a smooth map and $p\in\mathcal{M}$. Prove that*

1. *$\mathrm{d}F_p$ is linear*
2. *$\mathrm{d}(\mathrm{Id}_{\mathcal{M}})_p=\mathrm{Id}_{T_p\mathcal{M}}:T_p\mathcal{M}\to T_p\mathcal{M}$*
3. *If F is a diffeomorphism, then $\mathrm{d}F_p$ is an isomorphism and satisfies $(\mathrm{d}F_p)^{-1}=\mathrm{d}(F^{-1})_{F(p)}$*

Let us now compute the map $\mathrm{d}\varphi_x$ in coordinates:

Lemma 2.11. *Let $\varphi:\mathcal{M}\to\mathcal{N}$ be a smooth map between two differentiable manifolds, of dimension n and k, respectively. Let $x\in\mathcal{M}$, $\sigma:U\to O$ a chart on $\mathcal{M}$ around x, and let $\tau:V\to\Omega$ be a chart on $\mathcal{N}$ around $\varphi(x)$. Let us denote the local coordinates of σ by (x^i) and those of τ by (y^i). Then the matrix of $\mathrm{d}\varphi_x$ with respect to the bases $\left\{\left.\frac{\partial}{\partial x^j}\right|_x\right\}_{1\leq j\leq n}$ of $T_x\mathcal{M}$ and $\left\{\left.\frac{\partial}{\partial y^i}\right|_{\varphi(x)}\right\}_{1\leq 1\leq k}$ of $T_{\varphi(x)}\mathcal{N}$ is given by the matrix of $\mathrm{d}(\tau\circ\varphi\circ\sigma^{-1})_{\sigma(x)}$.*

Proof: Note that

$$\begin{aligned}\mathrm{d}\varphi_x\left[\left.\frac{\partial}{\partial x^j}\right|_x\right]&=\mathrm{d}\varphi_x\left[\left.\frac{\partial}{\partial x^j}\right|_x\right](y^i)\left.\frac{\partial}{\partial y^i}\right|_{\varphi(x)}\\&=\left.\frac{\partial}{\partial x^j}\right|_x(y^i\circ\varphi)\left.\frac{\partial}{\partial y^i}\right|_{\varphi(x)}\\&=D_j\left(r^i\circ\tau\circ\varphi\circ\sigma^{-1}\right)(\sigma(x))\left.\frac{\partial}{\partial y^i}\right|_{\varphi(x)}\end{aligned}$$

Since the number $D_j\left(r^i \circ \tau \circ \varphi \circ \sigma^{-1}\right)(\sigma(x))$ is simply the (i,j)-th entry of the matrix $\mathrm{d}\left(\tau \circ \varphi \circ \sigma^{-1}\right)_{\sigma(x)}$, we have finished the proof.

■

Theorem 2.12 (**The Inverse Function Theorem for Manifolds**). *Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds, and suppose that $F : \mathcal{M} \to \mathcal{N}$ is smooth. If $p \in \mathcal{M}$ is a point such that $\mathrm{d}F_p$ is invertible, then there exist connected neighborhoods $U_0 \ni p$ and $V_0 \ni F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Proof: The fact that $\mathrm{d}F_p$ is bijective implies that $\mathcal{M}$ and $\mathcal{N}$ have the same dimension, say n. Let us choose smooth charts (U,φ) and (V,ψ) centered at p and at $F(p)$, respectively, with $F(U) \subset V$. Then the coordinate representation of F, given by $\hat{F} = \psi \circ F \circ \varphi^{-1}$, is a smooth map from a Euclidean open set (namely, $\hat{U} = \varphi(U) \subset \mathbb{R}^n$) into another Euclidean open set (namely, $\hat{V} = \psi(V) \subset \mathbb{R}^n$) satisfying $\hat{F}(p) = 0$. Since φ and ψ are diffeomorphisms, the differential $\mathrm{d}\hat{F}_0 = \mathrm{d}\psi_{F(p)} \circ \mathrm{d}F_p \circ \mathrm{d}(\varphi^{-1})_0$ has nonzero determinant. The inverse function theorem for $\mathbb{R}^n$ then guarantees the existence of connected open subsets $\hat{U}_0 \subset \hat{U}$ and $\hat{V}_0 \subset \hat{V}$ containing 0 such that $\hat{F}$ induces a diffeomorphism between $\hat{U}_0$ and $\hat{V}_0$. Then $U_0 = \varphi^{-1}\left(\hat{U}_0\right)$ and $V_0 = \psi^{-1}\left(\hat{V}_0\right)$ are connected neighborhoods of p and $F(p)$, respectively, and it follows that $F = \psi^{-1} \circ \hat{F} \circ \varphi$ is a diffeomorphism between U_0 and V_0.

■

Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds of dimension n and k respectively, $\varphi : \mathcal{M} \to \mathcal{N}$ a smooth map, and let us take charts σ around a point x of $\mathcal{M}$ and τ around $\varphi(x)$, and define linear isomorphisms

$$\begin{aligned} T_x : \mathbb{R}^n &\to T_x\mathcal{M} \\ e_i &\mapsto T_x e_i = \left.\frac{\partial}{\partial x^i}\right|_x \end{aligned}$$

Analogously, denoting by $e_j^{'}$ the canonical basis of $\mathbb{R}^k$, we have a linear isomorphism

$$\begin{aligned} T_{\varphi(x)} : \mathbb{R}^k &\to T_{\varphi(x)}\mathcal{N} \\ e_j^{'} &\mapsto T_{\varphi(x)} e_j^{'} = \left.\frac{\partial}{\partial y^j}\right|_{\varphi(x)} \end{aligned}$$

Exercise 54. *Prove that the following diagram commutes*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathrm{d}\left(\tau \circ \varphi \circ \sigma^{-1}\right)_{\sigma(x)}} & \mathbb{R}^k \\ \Big\downarrow{\scriptstyle T_x} & & \Big\downarrow{\scriptstyle T_{\varphi(x)}} \\ T_x\mathcal{M} & \xrightarrow[\mathrm{d}\varphi_x]{} & T_{\varphi(x)}\mathcal{N} \end{array}$$

In particular, explain why for a smooth function $f : O \subset \mathbb{R}^n \to \mathbb{R}^k$ our first definition of $\mathrm{d}f_p$ coincides with the definition now given for the differential of a smooth map between manifolds.

We will now give a different way of defining tangent vectors which, although not as aesthetically pleasing as derivations, has the advantage of making calculations easier.

Suppose that $\gamma : (a, b) \to \mathbb{R}^n$ is a smooth function. We usually write the coordinate function on $\mathbb{R} = \mathbb{R}^1$ as t instead of x^1, and we denote the derivative of γ at a point t by $\gamma'(t)$. Writing $\gamma = (\gamma^1, \cdots, \gamma^n)$, the vector $\gamma'(t)$ is exactly the row vector $((\gamma^1)'(t), \cdots, (\gamma^n)'(t))$. Our goal now is to extend this to manifolds.

Definition 23. *A curve in a differentiable manifold $\mathcal{M}$ is a smooth map $\gamma : (a, b) \to \mathcal{M}$, where we think of (a, b) as a one-dimensional smooth manifold. Fix $t \in (a, b)$. There are, a priori, two different ways of defining an element $\gamma'(t) \in T_{\gamma(t)}\mathcal{M}$, which we will call the velocity vector of γ at time t.*

1. *First, we can define a derivation on $\mathcal{C}^\infty(\mathcal{M})$ at $\gamma(t)$ by setting*

$$\gamma'(t)(f) \doteq (f \circ \gamma)'(t)$$

   for each $f \in \mathcal{C}^\infty(\mathcal{M})$.

2. *Alternatively, if we think of γ as a smooth map between manifolds, we can define a tangent vector $\gamma'(t)$ at $\gamma(t)$ via the derivative $\mathrm{d}\gamma_t$:*

$$\gamma'(t) = \mathrm{d}\gamma_t \left(\left. \frac{\partial}{\partial t} \right|_t \right) \in T_{\gamma(t)}\mathcal{M}$$

To see that these two definitions coincide, let $\sigma : U \to O$ be a chart around $\gamma(t)$ and let us denote by (x^i) the coordinates of σ. Let $\gamma^i \doteq x^i \circ \gamma$, so that γ^i is a curve in $\mathbb{R}$. Applying theorem 2.9 to the first definition, we see that

$$\gamma'(t) = (\gamma^i)'(t) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)},$$

since $\gamma'(t)(x^i) = (x^i \circ \gamma)'(t) = (\gamma^i)'(t)$. But applying lemma 2.11 to the second definition, we see that this definition gives the same formula above for $\gamma'(t)$.

Lemma 2.13. *Let $\mathcal{M}$ be a differentiable manifold and let $\gamma, \delta : (-\varepsilon, \varepsilon) \to \mathcal{M}$ be two smooth curves such that $\gamma(0) = \delta(0)$. Then $\gamma'(0)$ and $\delta'(0)$ coincide as elements of $T_{\gamma(0)}\mathcal{M}$ if, and only if, for some (and therefore for any) chart $\sigma : U \to O$ defined on a neighborhood of $\gamma(0)$, we have*

$$(\sigma \circ \gamma)'(0) = (\sigma \circ \delta)'(0).$$

Proof: Let us denote by (x^i) the coordinates of σ. The stated condition is equivalent to asking that $(\gamma^i)'(0) = (\delta^i)'(0)$ for each i, where $\gamma^i = x^i \circ \gamma$ and $\delta^i = x^i \circ \delta$. The desired result then follows from the formula obtained for $\gamma'(t)$ just above, since $\left\{ \left. \frac{\partial}{\partial x^i} \right|_{\gamma(0)} \right\}_{1 \leq i \leq n}$ is a basis of $T_{\gamma(0)}\mathcal{M}$.

■

Exercise 55. *Prove that the tangent vector $\gamma'(t)$ defined in item 1 of definition 22 is indeed a derivation. Moreover, justify the "(...) for any" stated in the lemma above.*

Less evident but still true is the fact that every tangent vector can be written as the velocity vector of a curve:

Proposition 2.14. *Let $\mathcal{M}$ be a differentiable manifold of dimension n, $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$. Then there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\gamma'(0) = v$.*

Proof: Let us take a chart $\sigma : U \to O \subset \mathbb{R}^n$ centered at x. As always, let us denote the coordinates of σ by (x^i), and let

$$v = a^i \left.\frac{\partial}{\partial x^i}\right|_x \in T_x\mathcal{M}$$

be arbitrary, where the a^i are real numbers. Since O is open, there exists $\varepsilon > 0$ such that the vector $(ta^1, \cdots, ta^n)$ remains in O whenever $|t| < \varepsilon$. Then, defining the curve γ by

$$\begin{aligned} \gamma : (-\varepsilon, \varepsilon) &\to \mathcal{M} \\ t &\mapsto \gamma(t) \doteq \sigma^{-1}(ta^1, ta^2, \cdots, ta^n) \end{aligned}$$

it is clear that γ is well defined, is smooth and, by what we have already done, satisfies $\gamma(0) = x, \gamma'(0) = v$.

■

Proposition 2.15. *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a smooth map between two differentiable manifolds, and let $\gamma : (a, b) \to \mathcal{M}$ be a smooth curve in $\mathcal{M}$. Then:*

$$\mathrm{d}\varphi_{\gamma(t)}(\gamma'(t)) = (\varphi \circ \gamma)'(t)$$

Proof: Using the first item of definition 22: take $f \in \mathcal{C}^\infty(\mathcal{N})$. Then by the definition of $\mathrm{d}\varphi_x$ we have

$$\mathrm{d}\varphi_{\gamma(t)}[\gamma'(t)](f) = \gamma'(t)(f \circ \varphi) = (f \circ \varphi \circ \gamma)'(t) = (\varphi \circ \gamma)'(t)(f)$$

■

Exercise 56. *Prove the previous proposition using the second item of definition 22.*

Exercise 57. *Let $f : \mathcal{M} \to \mathbb{R}$ be smooth and $p \in \mathcal{M}$ such that $\mathrm{d}f_p = 0$. Define the operator $H(f)_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ in charts (x, U) with $p \in U$ by requiring that:*

$$H(f)_p\left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p\right) = \frac{\partial^2 (f \circ x^{-1})}{\partial r^i \partial r^j}(x(p))$$

and extending by bilinearity and symmetry. Show that $H(f)_p$ does not depend on the chart chosen and give an explicit expression for $H(f)_p(v, w)$, where $v, w \in T_p\mathcal{M}$ are arbitrary.

Exercise 58. *Let $v \in \mathbb{S}^n$ and define the height function with respect to v on the sphere as $h = h_v : \mathbb{S}^n \to \mathbb{R}$ given by $x \mapsto \langle x, v\rangle$. Show that h is smooth and compute its critical points.*

We will now take a brief look at the dual space of the tangent space.

Definition 24. *Let $\mathcal{M}$ be a differentiable manifold of dimension n and let $x \in \mathcal{M}$. We denote the dual space $\mathcal{L}(T_x\mathcal{M}, \mathbb{R})$ by $T_x^*\mathcal{M}$ and call it the cotangent space of $\mathcal{M}$ at x.*

Hence $T_x^*\mathcal{M}$ is another vector space of dimension n. Since elements of $T_x\mathcal{M}$ are linear derivations "eating" functions, we can interpret elements of $T_x^*\mathcal{M}$ as functions "eating" linear derivations.

Example. Let $\mathcal{M}$ be a differentiable manifold of dimension n and let $x \in \mathcal{M}$, U a neighborhood of x and $f \in \mathcal{C}^\infty(U)$. Then f defines an element $\mathrm{d}f_x \in T_x^*\mathcal{M}$ by

$$v \in T_x\mathcal{M} \mapsto \mathrm{d}f_x(v) \doteq v(f)$$

for each $v \in T_x\mathcal{M}$. $\mathrm{d}f_x$ is called the differential of f at x.

Remark. In the way seen above, $\mathrm{d}f_x$ is a linear map from $T_x\mathcal{M}$ to $\mathbb{R}$. In contrast, the derivative $\mathrm{d}f_x$ that we defined in the context of manifolds now (which we will denote right below by $\mathrm{D}f_x$ to avoid further confusion) is a linear function $T_x\mathcal{M} \to T_{f(x)}\mathbb{R}$, but once we make the obvious identification $T_{f(x)}\mathbb{R} \cong \mathbb{R}$, these two coincide and that is why we do not distinguish them. Note that

$$\mathbb{R} \ni \mathrm{d}f_x[v] = \mathrm{D}f_x(v)\left.\frac{\partial}{\partial t}\right|_{f(x)} \in T_{f(x)}\mathbb{R}$$

This makes perfect sense, since under the isomorphism mentioned earlier, at any $p \in \mathbb{R}$ the tangent vector

$$\left.\frac{\partial}{\partial t}\right|_p$$

is identified with the real number 1. It is customary to abuse notation and say that $T_p\mathbb{R} = \langle 1 \rangle$, that is, that $\{1\}$ is a basis of $T_p\mathbb{R} \cong \mathbb{R}$.

2.5 Vector fields

A vector field X on a differentiable manifold $\mathcal{M}$ is a correspondence that associates to each $p \in \mathcal{M}$ a tangent vector $X(p) \in T_p\mathcal{M}$. Considering a parametrization $\phi : U \subset \mathcal{M} \to \mathbb{R}^n$, we can write

$$X(p) = a^i(p)\left.\frac{\partial}{\partial x^i}\right|_p$$

where each $a^i : U \to \mathbb{R}$ is a real function defined on U. We say that the field X is differentiable at $p \in \mathcal{M}$ when in some parametrization (and therefore in any) its components a^i are differentiable. When X is differentiable at every point, we simply say that X is a differentiable vector field.

Exercise 59. *Show that there is no natural notion of what a "constant" vector field would be on differentiable manifolds in general. More specifically, show that if X is a vector field that can be expressed in local coordinates $(x^1, \cdots, x^n)$ around a $p \in \mathcal{M}$ by*

$$X(p) = X^k(p) \left.\frac{\partial}{\partial x^k}\right|_p$$

then the equality $\mathrm{d}X^j_p \equiv 0$ is not invariant under change of coordinates. Hint: you must prove that

$$\mathrm{d}\tilde{X}^i_p(v) = \frac{\partial \tilde{x}^i}{\partial x^j}(p) \cdot \mathrm{d}X^j_p(v) + \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k}(p) \cdot v^k X^j(p)$$

holds for every $v \in T_p\mathcal{M}$.

3 Epilogue: the Poincaré conjecture

3.1 Riemannian metrics

A Riemannian metric on a differentiable manifold $\mathcal{M}$ is a correspondence that associates to each point of $\mathcal{M}$ an inner product $\langle , \rangle_p$ (that is, a symmetric, positive definite and bilinear form) on the tangent space $T_p\mathcal{M}$, which varies differentiably in the following sense: if $\phi : U \to \mathbb{R}^n$ is a coordinate system around p with $\phi(p) = q$, then

$$\left\langle \left.\frac{\partial}{\partial x^i}\right|_p , \left.\frac{\partial}{\partial x^j}\right|_p \right\rangle_p = g_{ij}(p)$$

is a differentiable function on U. Since charts are diffeomorphisms and products of diffeomorphisms are diffeomorphisms, it is simple to show (using the change-of-basis rule for tangent vectors seen earlier) that this definition does not depend on the choice of coordinate system. When there is no possibility of confusion it is common to drop the index p in the function $\langle , \rangle_p$. We will soon prove that every differentiable manifold has (at least) one Riemannian metric.

Definition 25. *Let $\mathcal{M}$ be a differentiable manifold. A partition of unity is a collection $\{\lambda_i \mid i \in I\}$ of smooth functions $\lambda_i : \mathcal{M} \to \mathbb{R}$ satisfying*

1. *$0 \leq \lambda_i \leq 1$ for every $i \in I$.*
2. *The collection $\{\mathrm{supp}(\lambda_i)\}_{i \in I}$ is locally finite, that is, given any $x \in \mathcal{M}$ there are at most finitely many $i \in I$ such that $x \in \mathrm{supp}(\lambda_i)$.*
3. *For every $x \in \mathcal{M}$ we have*

$$\sum_{i \in I} \lambda_i(x) = 1$$

(note that by the previous item this sum has only finitely many nonzero terms for each x).

We say that a partition of unity $\{\lambda_i\}_{i \in I}$ is subordinate to an open cover $\{U_i\}_{i \in I}$ if $\mathrm{supp}(\lambda_i) \subset U_i$ for each $i \in I$.

Lemma 3.1. *Let $\mathcal{M}$ be a differentiable manifold. Then any open cover of $\mathcal{M}$ has a partition of unity subordinate to it.*

Proof: See [4].

■

Theorem 3.2. *Every differentiable manifold has a Riemannian metric.*

Proof: Let $\mathcal{M}$ be a differentiable manifold, $\{\varphi_i : U_i \to \varphi(U_i)\}$ an atlas for $\mathcal{M}$ and $\{f_i\}$ a partition of unity of $\mathcal{M}$ subordinate to the cover $\{U_i\}$. Each chart φ_i naturally induces a metric on U_i: given $p \in \mathcal{M}$ and tangent vectors $v, w \in T_p\mathcal{M}$, they are written in coordinates with respect to the basis associated with the chart φ_i as

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p \text{ and } w = w^j \left.\frac{\partial}{\partial x^j}\right|_p$$

And we then define the inner product of v and w by

$$\langle v, w\rangle_p^i = v^k w^\ell \delta_{k\ell}$$

Which then gives us a metric on U_i satisfying $g^i_{k\ell} = \delta_{k\ell}$. To globalize this metric, we use the partition of unity and define

$$\langle v, w\rangle_p = \sum_i f_i(p)\langle v, w\rangle_p^i$$

which is well defined, clearly symmetric, is differentiable since this sum is finite in a neighborhood of each point, and defines an inner product on $T_p\mathcal{M}$ because it is a finite linear combination of inner products.

■

The Poincaré conjecture is that every simply connected, closed (compact and without boundary) manifold $\mathcal{M}^n$ is homeomorphic to $\mathbb{S}^n$. In dimension ≤ 2 it is relatively easy to prove. In dimension ≥ 4, it takes much more work, but the solution was known well before the case $n = 3$ was solved, which incredibly happened with techniques making heavy use of Riemannian Geometry (and, of course, of several other techniques from analysis as well).

References

[1] Dan Pinponi, *Why is a topology made up of 'open' sets?*

[2] Marcel Berger, *A panoramic view of Riemannian Geometry.*

[3] Roger Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe.*

[4] John Lee, *An Introduction to Smooth Manifolds.*

[5] Will Merry, *Lecture Notes on Differential Geometry*.