One chart to rule them all!

English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio, re-typeset from the published PDF because the LaTeX source was not available. The Portuguese original is the authoritative version.

It is common in the literature to use the fact that tensors are *coordinate-free* without formally making explicit what is meant by this. The main purpose of this text is to justify this phrase in detail. First, we will establish some important notation and preliminary concepts:

Notation (N.1). We will denote by $\mathbb{V}$ a fixed finite-dimensional real vector space, say $\dim(\mathbb{V}) = n$.

Notation (N.2). $(\mathcal{M}^n, g)$ denotes a fixed Riemannian manifold of dimension n, and $\mathscr{C}^\infty(\mathcal{M})$ denotes the space of all smooth real functions defined on $\mathcal{M}$.

Notation (N.3). Let $\mathcal{M}^n$ be a differentiable manifold, $\psi : U \subset \mathcal{M} \to V \subset \mathbb{R}^n$ a chart around an arbitrarily fixed point $p \in \mathcal{M}$ and $\left\{\dfrac{\partial}{\partial x^i} = \partial_i\right\}_{1\leq i\leq n} \subset \Gamma(TU)$ the local frame induced by ψ. We will denote by $\{\mathrm{d}x^i\}_{1\leq i\leq n} \subset \Gamma(T^*U)$ its associated coframe, defined by:

$$\begin{aligned} \mathrm{d}x^i : U &\to T^*U \\ p \mapsto \mathrm{d}x^i\Big|_p : T_p\mathcal{M} &\to \mathbb{R} \\ v &\mapsto \mathrm{d}x^i\Big|_p(v) = v^i. \end{aligned}$$

for each $v = \displaystyle\sum_{1\leq i\leq n} v^i \left.\frac{\partial}{\partial x^i}\right|_p \in T_p\mathcal{M}$. Equivalently,

$$\mathrm{d}x^i\Big|_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta^i_j \ \forall p \in U.$$

In the special case $\mathcal{M} = \mathbb{R}^n$, in which we have a single chart $\varphi = \mathrm{Id}$ and the (co)frames become global, we will denote them by

$$\left\{\mathrm{d}r^i\right\}_{1\leq i\leq n} \text{ and } \left\{\frac{\partial}{\partial r^i}\right\}_{1\leq i\leq n},$$

which can be naturally identified with the canonical basis of $\mathbb{R}^n$. Note that, by a very simple computation, the differential of the i-th coordinate function of the chart ψ (given by $r^i \circ \psi$, where $r^i : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the i-th coordinate of $\mathbb{R}^n$), coincides with $\mathrm{d}x^i$, *id. est*, $\mathrm{d}x^i = \mathrm{d}\left(r^i \circ \psi\right)$, which justifies our notation.

Definition (D.1). A real tensor of type (r, s) on $\mathbb{V}$ is a multilinear map $T : (\mathbb{V}^*)^r \times \mathbb{V}^s \to \mathbb{R}$. The set of real tensors of type (r, s) on $\mathbb{V}$ will be denoted by $\mathscr{T}^r_s(\mathbb{V})$.

Definition (D.2). Let $\beta = \{\mathbf{e}_i\}_{1\leq i\leq n}$ and $\beta^* = \{\mathbf{e}^i\}_{1\leq i\leq n}$ be dual bases of $\mathbb{V}$ and $\mathbb{V}^*$, respectively (that is, $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$ for each $1 \leq i, j \leq n$). Given $T \in \mathscr{T}^r_s(\mathbb{V})$, the components of T in the basis β are the real numbers defined by:

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s} \doteq T(\mathbf{e}^{i_1}, \cdots, \mathbf{e}^{i_r}, \mathbf{e}_{j_1}, \cdots, \mathbf{e}_{j_s}),$$

for any $1 \leq i_1, \cdots i_r, j_1, \cdots j_s \leq n$.

Remark (O.1). A mistaken interpretation of the phrase "tensors are coordinate-free" would be to say that, given any two bases $\{\mathbf{e}_i\}_{1\leq i\leq n}$ and $\{\widetilde{\mathbf{e}}_i\}_{1\leq i\leq n}$ and any $T\in\mathscr{T}^r_s(\mathbb{V})$, it holds that

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}=T^{\widetilde{i}_1\cdots\widetilde{i}_r}{}_{\widetilde{j}_1\cdots\widetilde{j}_s},$$

for any $1\leq i_1,\cdots,i_r,j_1,\cdots,j_s,\widetilde{i}_1,\cdots,\widetilde{i}_r,\widetilde{j}_1\cdots\widetilde{j}_r\leq n$, which is evidently absurd! In fact, what happens is that to prove that two tensors $T,Q\in\mathscr{T}^r_s(\mathbb{V})$ are equal, that is, that

$$T(\omega^1,\cdots,\omega^r,v_1,\cdots,v_s)=Q(\omega^1,\cdots,\omega^r,v_1,\cdots,v_s),$$

for every $(\omega^1,\cdots,\omega^r,v_1,\cdots,v_s)\in(\mathbb{V}^*)^r\times\mathbb{V}^s$, it suffices to prove that

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}=Q^{i_1\cdots i_r}{}_{j_1\cdots j_s},$$

for any $1\leq i_1,\cdots,i_r,j_1,\cdots,j_s\leq n$, which is evidently true by the multilinearity of T and Q. Equivalently, to define a tensor $T\in\mathscr{T}^r_s(\mathbb{V})$, it suffices to declare its action on $\{\mathbf{e}^1,\cdots,\mathbf{e}^r,\mathbf{e}_1,\cdots,\mathbf{e}_s\}$, that is, it suffices to specify the n^{r+s} real numbers $\{T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\}$. When $r+s=n=3$, for example, we can think of T as a "cube" of dimensions n made of $3^3=27$ numbers. Therefore, in a certain sense tensors are generalizations of the usual "two-dimensional" matrices (equivalently, linear transformations) that we already know.

As is common with constructions involving only Linear Algebra, the considerations above can be "fiberized", in a sense that we will formalize next.

Definition (D.3). The (a,b)-tensor bundle over $\mathcal{M}$ is defined by

$$\mathscr{T}^a_b(T\mathcal{M})\doteq\bigcup_{p\in\mathcal{M}}\mathscr{T}^a_b(T_p\mathcal{M})=\{(p,T)\mid p\in\mathcal{M}\text{ and }T\in\mathscr{T}^a_b(T_p\mathcal{M})\}$$

Remark (O.2). Note that $\mathscr{T}^a_b(T\mathcal{M})$ is a differentiable manifold of dimension $n+n^{a+b}$. Its topology and atlas are induced analogously to those of the usual tangent bundle. That is, if

$$\mathcal{A}=\{\varphi_\alpha:U_\alpha\to\varphi_\alpha(U_\alpha)\subset\mathbb{R}^n\}_{\alpha\in I}$$

is a maximal atlas of $\mathcal{M}$ and

$$\begin{aligned}\pi:\mathscr{T}^a_b(T\mathcal{M})&\to\mathcal{M}\\(p,T)&\mapsto p\end{aligned}$$

denotes the natural projection, we define an atlas of $\mathscr{T}^a_b(T\mathcal{M})$ by

$$\tilde{\mathcal{A}}=\left\{\psi_\alpha:\mathscr{T}^a_b(U_\alpha)=\bigcup_{p\in U_\alpha}\mathscr{T}^a_b(T_p\mathcal{M})\to\varphi_\alpha(U_\alpha)\times\mathbb{R}^{n^{a+b}}\subset\mathbb{R}^{n+n^{a+b}}\right\}_{\alpha\in I}$$

where ψ_α is defined by

$$\begin{aligned}&\psi_\alpha\left(q,\sum_{\substack{1\leq i_1,\ldots,i_b\leq n\\1\leq j_1,\ldots,j_a\leq n}}T^{j_1\ldots j_a}_{i_1\ldots i_b}(q)\cdot\left.\frac{\partial}{\partial x^{j_1}_\alpha}\right|_q\otimes\left.\frac{\partial}{\partial x^{j_2}_\alpha}\right|_q\otimes\ldots\otimes\left.\frac{\partial}{\partial x^{j_a}_\alpha}\right|_q\otimes\left.\mathrm{d}x^{i_1}_\alpha\right|_q\otimes\left.\mathrm{d}x^{i_2}_\alpha\right|_q\otimes\ldots\otimes\left.\mathrm{d}x^{i_b}_\alpha\right|_q\right)\\&\quad=\left(\varphi_\alpha(q),\sum_{\substack{1\leq i_1,\ldots,i_b\leq n\\1\leq j_1,\ldots,j_a\leq n}}T^{j_1\ldots j_a}_{i_1\ldots i_b}(q)\cdot\left.\frac{\partial}{\partial r^{j_1}_\alpha}\right|_q\otimes\left.\frac{\partial}{\partial r^{j_2}_\alpha}\right|_q\otimes\ldots\otimes\left.\frac{\partial}{\partial r^{j_a}_\alpha}\right|_q\otimes\left.\mathrm{d}r^{i_1}_\alpha\right|_q\otimes\left.\mathrm{d}r^{i_2}_\alpha\right|_q\otimes\ldots\otimes\left.\mathrm{d}r^{i_b}_\alpha\right|_q\right)\end{aligned}$$

and we declare that a subset $O\subset\mathscr{T}^a_b(U_\alpha)$ is open in $\mathscr{T}^a_b(U_\alpha)$ if and only if $\varphi_\alpha(O)$ is open in $\varphi_\alpha(U_\alpha)\times\mathbb{R}^{n^{a+b}}$.

Definition (D.4). We will say that a section $T \in \Gamma\left(\mathscr{T}_b^a(T\mathcal{M})\right)$ is a tensor field on $\mathcal{M}$.

Remark (O.3). Note that it is common to call a tensor field simply a tensor.

A fortiori, when we have two tensors (a priori not necessarily equal) defined on a Riemannian manifold, the fact that tensors can (by definition) be evaluated pointwise guarantees that, to prove the equality between them, it suffices to arbitrarily fix a point $p \in \mathcal{M}^n$, a convenient local frame $\{E_i\}_{1\leq i\leq n}$ (commonly a geodesic frame or a frame induced by a local chart) around p, and to prove that the equalities between components are satisfied. Again, this does *not* mean that the components are independent of the fixed frame: we merely use the fact of multilinearity to guarantee that the definition of a tensor by its components is well defined (in contrast with not necessarily multilinear maps, which are not determined by their values on bases). Explicitly, any section of the bundle $T \in \Gamma\left(\mathscr{T}_b^a(T\mathcal{M})\right)$ (commonly called simply a tensor of type (a,b) on $\mathcal{M}$) is determined by knowledge of all its n^{a+b} components $\left\{T^{i_1\cdots i_a}{}_{j_1\cdots j_b}\right\} \subset \mathscr{C}^\infty(\mathcal{M})$, given explicitly by

$$\left\{T\left(\mathbf{e}^{(\alpha,i_1)},\ldots,\mathbf{e}^{(\alpha,i_a)},\mathbf{e}_{(\alpha,j_1)},\ldots,\mathbf{e}_{(\alpha,j_b)}\right)\right\}_{\alpha\in I} \subset \mathscr{C}^\infty(\mathcal{M}),$$

where

$$\mathbf{e}_{(\alpha,j_k)}(p) = \mathrm{d}\left(\varphi_\alpha^{-1}\right)_{\varphi_\alpha(p)}\left(\left.\frac{\partial}{\partial r^{j_k}}\right|_{\varphi(p)}\right),$$

and $\left\{\mathbf{e}^{(\alpha,i_\ell)}\right\}_{1\leq\ell\leq n}$ is the associated dual basis $\left\{\mathbf{e}^{(\alpha,i_\ell)}\right\}_{1\leq\ell\leq n}$ to the basis $\left\{\mathbf{e}_{(\alpha,j_k)}\right\}_{1\leq k\leq n}$, determined by

$$\mathbf{e}^{(\alpha,i_\ell)}\left(\mathbf{e}_{(\alpha,j_k)}\right) = \delta^{i_\ell}_{j_k}.$$

The converse direction, that is, when certain real functions indexed over an atlas of $\mathcal{M}$ are in fact tensor components, is a bit more complicated, so we will not elaborate further on it, but it is worth observing that this problem appears when trying, for example, to "globalize" the moving frame method (for more details, see [5]).

In short: two tensorial geometric quantities that coincide in *one* given coordinate system (resp. *one* given frame) coincide in *any* coordinate system (resp. *any* frame).

References

[1] **Terek, I.** A mini-course on tensors. Text found on a personal webpage. Available at: https://www.asc.ohio-state.edu/terekcouto.1/texts/tensors.pdf . Accessed on February 12, 2023.

[2] **Josué, R.** Notas de aula de geometria Riemanniana. Text found on a personal webpage. Available at: http://150.164.25.15/~rodney/notas_de_aula/geometria_riemanniana.pdf. Accessed on February 12, 2023.

[3] **Lee, John M.** Introduction to smooth manifolds. Second edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013. xvi+708 pp. ISBN: 978-1-4419-9981-8.

[4] **Lee, John M.** Introduction to Riemannian manifolds. Second edition of [MR1468735]. Graduate Texts in Mathematics, 176. Springer, Cham, 2018. xiii+437 pp. ISBN: 978-3-319-91754-2; 978-3-319-91755-9.

[5] **Tu, Loring W.** Differential geometry. Connections, curvature, and characteristic classes. Graduate Texts in Mathematics, 275. *Springer, Cham,* 2017. xvi+346 pp. ISBN: 978-3-319-55082-4; 978-3-319-55084-8.

[6] **Horácio, M.A.R.M.** Estimativas de curvatura para sólitons de Ricci gradiente quadridimensionais. Master's dissertation, Universidade de Brasília, (2023).