Warped products

English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio, translated from the author's LaTeX source. The Portuguese original is the authoritative version.

In general, the vast majority of computations in Riemannian Geometry boil down to proving tensorial identities. This is usually done in one of the following three ways:

- *the coordinate-free way*: in this case, we make explicit the action of each tensor on arbitrary fields on the manifold in question, checking that the results coincide.
- *using local coordinates*: in this case, we fix suitable local coordinates that make the computations easier (for example, normal coordinates) and take advantage of the multilinearity of tensors, which guarantees that the identity only needs to be checked on a basis (as detailed in [1]).
- *using the moving frame method*: in this case, we fix a local orthonormal frame (which induces a local orthonormal coframe) and compute the connection 1-forms and curvature 2-forms, respectively, which determine the connection and curvature of the manifold.

Depending on the situation, one of the three methods above may be significantly easier to apply than the other two. In this subsection, based on the reference [2], we will devote ourselves to exploring as much of the last method mentioned above as our applications require.

Let $(\mathcal{M}^n, g)$ be a Riemannian manifold and fix $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ a local g-orthonormal frame defined on an open set $U \subset \mathcal{M}$, with $\{\mathbf{e}^1, \dots, \mathbf{e}^n\}$ the associated coframe (determined by $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$). Since any field $X \in \Gamma(T\mathcal{M})$ is written locally as

$$\Gamma(TU) \ni X|_U = \sum_{1\leq j\leq n} X^j\, \mathbf{e}_j, \text{ where } \{X^j\}_{1\leq j\leq n} \subset \mathscr{C}^\infty(\mathcal{M})$$

then given $s \in \Gamma(TU)$ we can determine (using the linearity of ∇ in the first entry and the Leibniz rule) $\nabla_X s$ from knowledge of the fields $\{\nabla_X \mathbf{e}_j\}_{1\leq j\leq n}$. Being a field on U, $\nabla_X \mathbf{e}_j$ is a linear combination of the $\mathbf{e}_i$'s with coefficients ω^i_j depending on X, *id est:*

$$\nabla_X \mathbf{e}_j = \sum_{1\leq i\leq n} \omega^i_j(X)\, \mathbf{e}_i$$

The $\mathscr{C}^\infty(\mathcal{M})$-linearity of $\nabla_X \mathbf{e}_j$ in X guarantees the $\mathscr{C}^\infty(\mathcal{M})$-linearity of ω^i_j in X, so that $\omega^i_j \in \Gamma(T^*U)$ is a 1-form on U. The 1-forms ω^i_j on U are called the ***connection forms***, and the matrix $\omega = (\omega^i_j)_{1\leq i,j\leq n}$ is called the ***connection matrix***, of the connection ∇ with respect to the frame $\{\mathbf{e}_i\}_{1\leq i\leq n} \subset \Gamma(TU)$.

Analogously, given $X, Y \in \Gamma(TU)$, the field $\mathrm{Rm}(X,Y)\mathbf{e}_j$ is a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_n$, *id est:*

$$\mathrm{Rm}(X,Y)\mathbf{e}_j = \sum_{1\leq i\leq n} \Omega^i_j(X,Y)\, \mathbf{e}_i$$

Since

$$\mathrm{Rm}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

is alternating and $\mathscr{C}^\infty(\mathcal{M})$-bilinear, Ω^i_j is also alternating and $\mathscr{C}^\infty(\mathcal{M})$-bilinear. By the properties of Rm, Ω^i_j is therefore a 2-form on U. The 2-forms Ω^i_j are called the ***curvature forms***, and the matrix $\Omega = (\Omega^i_j)$ is called the curvature matrix, of the connection ∇ with respect to the frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ on U.

Remark (O.1). Note that Ω^i_j is determined by Rm, since

$$\begin{aligned}\Omega^i_j &= \frac{1}{2}\sum_{1\leq k,\ell\leq n}\Omega^i_j(\mathbf{e}_k,\mathbf{e}_\ell)\,\mathbf{e}^k\wedge\mathbf{e}^\ell\\ &= -\frac{1}{2}\sum_{1\leq k,\ell\leq n}\mathrm{Rm}(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_\ell)\,\mathbf{e}^k\wedge\mathbf{e}^\ell\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}\Omega^i_j(\mathbf{e}_k,\mathbf{e}_\ell) &= g\left(\sum_{1\leq t\leq n}\Omega^s_j(\mathbf{e}_k,\mathbf{e}_\ell)\mathbf{e}_s,\mathbf{e}_i\right)\\ &= g\left(\mathrm{Rm}(\mathbf{e}_k,\mathbf{e}_\ell)\mathbf{e}_j,\mathbf{e}_i\right)\\ &= -\mathrm{Rm}(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_\ell)\end{aligned}$$

Direct computations show the following results:

Theorem (T.1). *Let $(\mathcal{M}^n,g)$ be a Riemannian manifold and $U\subset\mathcal{M}$ an open set on which an orthonormal frame $\mathbf{e}_1,\dots,\mathbf{e}_n$ is defined, with associated coframe $\mathbf{e}^1,\dots,\mathbf{e}^n$. Then there exists an antisymmetric matrix $\left(\omega^i_j\right)_{1\leq i,j\leq n}$ of 1-forms such that*

$$\mathrm{d}\mathbf{e}^i+\sum_{1\leq j\leq n}\omega^i_j\wedge\mathbf{e}^j=0,\ \textit{for every } 1\leq i\leq n$$

Moreover, if $\left(\alpha^i_j\right)_{1\leq i,j\leq n}$ is any other antisymmetric matrix of 1-forms satisfying

$$\mathrm{d}\mathbf{e}^i+\sum_{1\leq j\leq n}\alpha^i_j\wedge\mathbf{e}^j=0,\ \textit{for every } 1\leq i\leq n \qquad (1)$$

then

$$\left(\alpha^i_j\right)_{1\leq i,j\leq n}=\left(\omega^i_j\right)_{1\leq i,j\leq n}$$

Theorem (T.2). *The curvature forms Ω^i_j are related to the connection forms ω^i_j by the following equation, commonly called* ***Cartan's second structure equation****:*

$$\Omega^i_j=\mathrm{d}\,\omega^i_j+\sum_{1\leq k\leq n}\omega^i_k\wedge\omega^k_j,\ \textit{for all } \{i,j\}\subset\{1,\dots,n\}$$

Proof: See [2].

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One of the most important and fruitful concepts of Riemannian geometry is the *warped product* of Riemannian manifolds, which generalizes the ordinary Riemannian product. This concept also has great relevance in physics, appearing in the theory of relativity as a model of the Cosmos itself or of black holes (see, for example, the Friedmann–Lemaître–Robertson–Walker cosmological model).

Definition (D.1). Let $(\mathcal{B},g_B)$ and $(\mathcal{M}_i,g_{\mathcal{M}_i})$ be Riemannian manifolds and $b_i:\mathcal{B}\to(0,\infty)$ smooth functions for each $i\in\{1,2,\cdots,k\}$. The multiply warped product of these manifolds is the product manifold $\mathcal{N}=\mathcal{B}\times\mathcal{M}_1\times\mathcal{M}_2\times\cdots\times\mathcal{M}_k$ endowed with the metric $g=g_{\mathcal{B}}\oplus b_1^2g_{\mathcal{M}_1}\oplus b_2^2g_{\mathcal{M}_2}\oplus\cdots\oplus b_k^2g_{\mathcal{M}_k}$ defined by

$$g=\sigma_0^*(g_{\mathcal{B}})\oplus(b_1\circ\pi)^2\,\sigma_1^*\left(g_{\mathcal{M}_1}\right)\oplus\cdots\oplus(b_k\circ\pi)^2\,\sigma_k^*\left(g_{\mathcal{M}_k}\right)$$

where $\sigma_0 : \mathcal{N} \to \mathcal{B}$ is the map defined by

$$\begin{aligned} \sigma_0 : \mathcal{N} &\to \mathcal{B} \\ (p, q_1, \cdots, q_k) &\mapsto p \in \mathcal{B} \end{aligned}$$

and for $i \geq 1$, $\sigma_i : \mathcal{N} \to \mathcal{M}_i$ is the map defined by

$$\begin{aligned} \sigma_i : \mathcal{N} &\to \mathcal{M}_i \\ (p, q_1, \cdots, q_i, \cdots, q_k) &\mapsto q_i \in \mathcal{M}_i \end{aligned}$$

Remark (O.2).

- It is common to denote the Riemannian manifold $(\mathcal{N} = \mathcal{B} \times \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_k, g = g_{\mathcal{B}} \oplus b_1^2 g_{\mathcal{M}_1} \oplus b_2^2 g_{\mathcal{M}_2} \oplus \cdots \oplus b_k^2 g_{\mathcal{M}_k})$ simply by $\mathcal{N} = \mathcal{B} \times_{h_1} \mathcal{M}_1^{d_1} \times \cdots \times_{h_k} \mathcal{M}_k^{d_k}$
- When $k = 1$, we have the usual simple warped product.
- If all $b_i \equiv 1$, we have a trivial Riemannian product.

Definition (D.2). If $\varphi : \mathcal{M} \to \mathcal{N}$ is a smooth map, we will say that two fields $X \in \Gamma(T\mathcal{M})$ and $Y \in \Gamma(T\mathcal{N})$ are φ-related (a fact we will denote by $X \overset{\varphi}{\sim} Y$) when $Y \circ \varphi = \mathrm{d}\varphi \circ X$.

Lemma (L.1). The tangent space $T_p\mathcal{M}$ at a point $p = (p_0, p_1, \cdots, p_k) \in \mathcal{M} = \mathcal{B} \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_k$ is canonically isomorphic (in the sense that there exists an isomorphism that does not depend on choices of bases) to the product $T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k$.

Proof: Consider the maps

$$\begin{aligned} \iota_0 : \mathcal{B} &\to \mathcal{M} \\ \mathcal{B} \ni q_0 &\mapsto (q_0, p_1, \ldots, p_k) \in \mathcal{M} \end{aligned}$$

and, for $\alpha \geq 1$,

$$\begin{aligned} \iota_\alpha : \mathcal{N}_\alpha &\to \mathcal{M} \\ \mathcal{N}_\alpha \ni q_\alpha &\mapsto (p_0, p_1, \ldots, p_{\alpha-1}, q_\alpha, p_{\alpha+1}, \ldots, p_k) \end{aligned}$$

I claim that the map ψ defined by

$$\begin{aligned} \psi : T_{(p_0,p_1,\cdots,p_k)}\left(\mathcal{B} \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_k\right) &\to T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k \\ v &\mapsto \left(\mathrm{d}\left(\sigma_0\right)_p(v), \mathrm{d}\left(\sigma_1\right)_p(v), \mathrm{d}\left(\sigma_2\right)_p(v), \cdots, \mathrm{d}\left(\sigma_k\right)_p(v)\right) \end{aligned}$$

is an isomorphism, with inverse given by

$$\begin{aligned} \varphi : T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k &\to T_{(p_0,p_1,\cdots,p_k)}\left(\mathcal{B} \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_k\right) \\ (v_0, v_1, \ldots, v_k) &\mapsto \sum_{0 \leq \ell \leq k} \mathrm{d}\left(\iota_\ell\right)_{p_\ell}(v_\ell) \end{aligned}$$

First, note that

$$\sigma_i \circ \iota_\ell = \begin{cases} \text{the identity map } \mathrm{Id}_{\mathcal{N}_i} : \mathcal{N}_i \to \mathcal{N}_i, \text{ if } i = \ell \\ \text{the constant map } \mathcal{N}_\ell \ni q_\ell \mapsto p_i \in \mathcal{N}_i, \text{ if } i \neq \ell \end{cases}$$

And therefore, it follows from the chain rule that

$$\begin{aligned} \mathrm{d}\left(\sigma_i \circ \iota_\ell\right)_{p_\ell} &= \mathrm{d}\left(\sigma_i\right)_p \circ \mathrm{d}\left(\iota_\ell\right)_{p_\ell} \\ &= \begin{cases} \text{the identity map } \mathrm{Id}_{T_{p_i}\mathcal{N}_i} : T_{p_i}\mathcal{N}_i \to T_{p_i}\mathcal{N}_i, \text{ if } i = \ell \\ \text{the identically zero map } T_{p_\ell}\mathcal{N}_\ell \ni v_\ell \mapsto 0 \in T_{p_\ell}\mathcal{N}_\ell, \text{ if } i \neq \ell \end{cases} \end{aligned} \tag{2}$$

Therefore, for every $(v_0, v_1, \ldots, v_k) \in T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k$, we have

$$\begin{aligned}(\psi \circ \varphi)(v_0, v_1, \ldots, v_k) &= \left(\mathrm{d}(\sigma_0)_p\left(\sum_{0\leq\ell\leq k} \mathrm{d}(\iota_\ell)_{p_\ell}(v_\ell)\right), \ldots, \mathrm{d}(\sigma_k)_p\left(\sum_{0\leq\ell\leq k} \mathrm{d}(\iota_\ell)_{p_\ell}(v_\ell)\right)\right)\\ &= \left(\sum_{0\leq\ell\leq k} \mathrm{d}(\sigma_0)_p\left(\mathrm{d}(\iota_\ell)_{p_\ell}(v_\ell)\right), \ldots, \sum_{0\leq\ell\leq k} \mathrm{d}(\sigma_k)_p\left(\mathrm{d}(\iota_\ell)_{p_\ell}(v_\ell)\right)\right)\\ &= (v_0, v_1, \ldots, v_k)\end{aligned}$$

where in the last equality we used **(2)**. It follows that ψ is surjective. Since $T_{(p_0,p_1,\cdots,p_k)}(\mathcal{B} \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_k)$ and $T_{p_0}\mathcal{B} \times T_{p_1}\mathcal{N}_1 \times \cdots \times T_{p_k}\mathcal{N}_k$ have the same dimension, it follows that ψ is an isomorphism. Since a bijective map has a unique right inverse, it also follows that φ is the inverse of ψ.

■

Remark (O.3). For each $i \in \{0, 1, \cdots, k\}$, the set of fields on $\mathcal{M}$ that arise as lifts of fields on factors of the product will be denoted by

$$\mathcal{L}(\mathcal{N}_i) = \{X \in \Gamma(T\mathcal{M}) \mid \mathrm{d}\sigma_j(X) = 0, \text{ for every } j \neq i\}$$

Given $X_i \in \Gamma(T\mathcal{N}_i)$, it is easy to see that there exists a unique field $\mathrm{Lev}(X_i) \in \mathcal{L}(\mathcal{N}_i)$ satisfying $(\mathrm{d}\sigma_i \circ \mathrm{Lev})(X_i) = X_i$ (in fact, $\mathrm{Lev}(X_i) = \psi^{-1}((0, \cdots, X_i, 0, \cdots, 0))$). The field $\mathrm{Lev}(X_i)$ is called the lift of the field X_i. The abuse of notation of identifying a field or a 1-form with its lift is therefore harmless, and we will commit it several times from now on. Note that $\mathrm{Lev}(X_i)$ is the unique field σ_i-related to X_i and σ_j-related to $0 \in \Gamma(T\mathcal{N}_j)$ for all $j \neq i$.

Remark (O.4). It is now convenient to recall the Gauss equation. We say that a map between Riemannian manifolds $f : (\mathcal{M}^n, g_\mathcal{M}) \to \left(\widetilde{\mathcal{M}}^{n+p}, \widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}\right)$ is an isometric immersion when

$$f^*\left(\widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}\right) = g_\mathcal{M}$$

or, equivalently,

$$g_\mathcal{M}(p)(v, w) = \widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}(f(p))(\mathrm{d}f_p(v), \mathrm{d}f_p(w))$$

for all $p \in \mathcal{M}$ and $v, w \in T_p\mathcal{M}$. In this case, committing the usual abuse of identifying fields with their extensions, we have

$$\widetilde{\mathrm{Rm}}(X, Y, Z, W) = \mathrm{Rm}(X, Y, Z, W) - \widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}(\mathbf{II}(X, W), \mathbf{II}(Y, Z)) + \widetilde{g}_{\widetilde{\mathcal{M}}^{n+p}}(\mathbf{II}(X, Z), \mathbf{II}(Y, W)) \qquad \textbf{(3)}$$

for all $X, Y, Z, W \in \Gamma(T\mathcal{M})$. In this regard, it is important to stress that, once we fix

$$p = (p_0, p_1, \ldots, p_k) \in \mathcal{M} = \mathcal{B} \times \mathcal{N}_1 \times \ldots \times \mathcal{N}_k$$

the inclusion

$$\begin{aligned}\iota_\alpha : (\mathcal{N}_\alpha, g_{\mathcal{N}_\alpha}) &\hookrightarrow \mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1 \times \ldots \times_{h_k} \mathcal{N}_k\\ q_\alpha &\mapsto (p_0, \ldots, p_{\alpha-1}, q_\alpha, p_{\alpha+1}, \ldots, p_k)\end{aligned}$$

is *not* an isometric immersion. Indeed, fixing arbitrarily $q_\alpha \in \mathcal{N}_\alpha$ and denoting

$$P = (p_0, \ldots, p_{\alpha-1}, q_\alpha, p_{\alpha+1}, \ldots, p_k)$$

we have

$$\begin{aligned}\left[(\iota_\alpha)^*(g_{\mathcal{M}})\right](q_\alpha)(v,w) &= g_{\mathcal{M}}(P)\left(\mathrm{d}(\iota_\alpha)_{q_\alpha}(v), \mathrm{d}(\iota_\alpha)_{q_\alpha}(w)\right)\\ &= h_\alpha^2(p_0)\left[(\sigma_\alpha)^*(g_{\mathcal{N}_\alpha})\right]_P\left(\mathrm{d}(\iota_\alpha)_{q_\alpha}(v), \mathrm{d}(\iota_\alpha)_{q_\alpha}(w)\right)\\ &= h_\alpha^2(p_0)\,(g_{\mathcal{N}_\alpha})_{q_\alpha}\left(\mathrm{d}(\sigma_\alpha)_P\left(\mathrm{d}(\iota_\alpha)_{q_\alpha}(v)\right), \mathrm{d}(\sigma_\alpha)_{q_\alpha}\left(\mathrm{d}(\iota_\alpha)_{q_\alpha}(w)\right)\right)\\ &= h_\alpha^2(p_0)(q_\alpha)\,(g_{\mathcal{N}_\alpha})_{q_\alpha}(v,w)\end{aligned} \tag{4}$$

where in the last equality we used equation **(2)**. However, it is clear from the computations done in **(4)** that with the metric $(h_\alpha(p_0))^2\, g_{\mathcal{N}_\alpha}$, the inclusion

$$\begin{aligned}\iota_\alpha : \left(\mathcal{N}_\alpha, (h_\alpha(p_0))^2\, g_{\mathcal{N}_\alpha}\right) &\hookrightarrow \mathcal{M} = \mathcal{B}\times_{h_1}\mathcal{N}_1\times\ldots\times_{h_k}\mathcal{N}_k\\ q_\alpha &\mapsto (p_0,\ldots,p_{\alpha-1},q_\alpha,p_{\alpha+1},\ldots,p_k)\end{aligned}$$

is an isometric immersion, so that in this context we can use equation **(3)** (as we will do later).

Remark (O.5). Although equation **(3)** is usually used to determine the curvature of submanifolds, in some situations we can use it in the opposite direction: that is, through knowledge of the curvature of specific submanifolds, we can determine the curvature of the total manifold. An interesting exercise in Riemannian Geometry (which we will comment on in more detail in Example **(E.1)**) that uses this technique involves computing the curvature of the warped product $\mathcal{I}\times_\varphi \mathbb{Q}^n(\kappa)$, where $\mathcal{I}\subset\mathbb{R}$ is an interval, $\varphi:\mathcal{I}\to(0,\infty)$ is a smooth function, and we define

$$\mathbb{Q}^n(\kappa) \doteq \begin{cases}\mathbb{R}^n, & \text{if } \kappa = 0\\ \mathbb{S}^n, & \text{if } \kappa = 1\\ \mathbb{H}^n, & \text{if } \kappa = -1\end{cases}$$

Lemma (L.2). If $\varphi:\mathcal{M}\to\mathcal{N}$ is a smooth map, then

$$\mathrm{d}\varphi([X,Y]) = [\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)],\ \forall X,Y\in\Gamma(T\mathcal{M})$$

Proof: It follows from the definition of the Lie bracket and from the definition of the differential of a map. Indeed, for any $p\in\mathcal{M}$ and $g\in\mathscr{C}^\infty(\mathcal{N})$ we have

$$\begin{aligned}\left(\mathrm{d}\varphi_p([X,Y](p))\right)(g) &= [X,Y](p)(g\circ\varphi)\\ &= X_p(Y(g\circ\varphi)) - Y_p(X(g\circ\varphi))\\ &= X_p(\mathrm{d}\varphi(Y)(g)\circ\varphi) - Y_p(\mathrm{d}\varphi(X)(g)\circ\varphi)\\ &= \mathrm{d}\varphi_p(X_p)(\mathrm{d}\varphi(Y)(g)) - \mathrm{d}\varphi_p(Y_p)(\mathrm{d}\varphi(X)(g))\\ &= [\mathrm{d}\varphi(X),\mathrm{d}\varphi(Y)]_p(g)\end{aligned}$$

■

Lemma (L.3). If $\varphi:\mathcal{M}\to\mathcal{N}$ is a smooth map and $X_1,X_2\in\Gamma(T\mathcal{M}), Y_1,Y_2\in\Gamma(T\mathcal{N})$ are fields satisfying $X_i \overset{\varphi}{\sim} Y_i$ for each $1\leq i\leq 2$, then $[X_1,X_2]\overset{\varphi}{\sim}[Y_1,Y_2]$.

Proof: An immediate consequence of the previous lemma. Indeed, given any $p\in\mathcal{M}$ we have

$$\begin{aligned}\mathrm{d}\varphi_p([X_1,X_2](p)) &= [\mathrm{d}\varphi_p(X_1(p)), \mathrm{d}\varphi_p(X_2(p))]\\ &= [Y_1,Y_2](\varphi(p))\end{aligned}$$

■

Naturally, knowledge of the connections and curvatures of the factors of such a product determines the connection and curvature of the product itself. To determine these, we will use the moving frame method. First, we will establish some notation that is important for the proofs of the following lemmas (also stated in [3], with a sign error in item (8) of Proposition 2.4, corrected in item **(8)** of Lemma **(L.7)**).

Let $\mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiply warped product with metric

$$g = g_{\mathcal{B}} \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$$

and connection $\nabla^g = \nabla$. We will denote $\mathcal{N}_0^r \doteq \mathcal{B}$ and $d_i \doteq \dim(\mathcal{N}_i)$ for each $0 \leq i \leq k$. For each $0 \leq i \leq n$, we will fix local $g_{\mathcal{N}_i}$-orthonormal frames, which we will denote by $\{\mathbf{e}_{(i,j)}\}_{1\leq j\leq d_i}$. Their associated coframes will be denoted by $\{\Theta^{(i,j)}\}_{1\leq i,j\leq d_i}$ (and are determined by $\Theta^{(i,j)}(\mathbf{e}_{(i,s)}) = \delta_s^j$). The connection 1-forms and curvature 2-forms associated with these frames will be denoted by $\{\omega_q^{(i,j)}\}_{1\leq j,q\leq d_i}$ and $\{\Omega_q^{(i,j)}\}_{1\leq j,q\leq d_i}$, respectively. That being so, note that

$$\left\{\mathbf{e}_{(0,1)}, \ldots, \mathbf{e}_{(0,r)}, \frac{1}{h_1}\mathbf{e}_{(1,1)}, \ldots, \frac{1}{h_1}\mathbf{e}_{(1,d_1)}, \ldots, \frac{1}{h_k}\mathbf{e}_{(k,1)}, \ldots, \frac{1}{h_k}\mathbf{e}_{(k,d_k)}\right\}$$

is a local g-orthonormal frame. We will use the notation

$$\widetilde{\mathbf{e}}_{(i,j)} = \begin{cases} \mathbf{e}_{(0,j)}, & \text{if } i = 0 \\ \dfrac{1}{h_i}\mathbf{e}_{(i,j)}, & \text{if } i \neq 0 \end{cases}$$

so that

$$\{\widetilde{\mathbf{e}}_{(i,j)}\}_{0\leq i\leq k, 1\leq j\leq d_i}$$

is a local g-orthonormal frame, its associated local coframe being given by

$$\{\widetilde{\Theta}^{(i,j))}\}_{0\leq i\leq k, 1\leq j\leq d_i}$$

where

$$\widetilde{\Theta}^{(i,j)} = \begin{cases} \Theta^{(0,j)}, & \text{if } i = 0 \\ h_i\Theta^{(i,j)}, & \text{if } i \neq 0 \end{cases}$$

so that on an open set $U \subset \mathcal{M}$, every field $s \in \Gamma(TU)$ is written as

$$s|_U = \sum_{\substack{0\leq i\leq k \\ 1\leq j\leq d_i}} \widetilde{s}^{(i,j)}\,\widetilde{\mathbf{e}}_{(i,j)} = \sum_{\substack{0\leq i\leq k \\ 1\leq j\leq d_i}} \widetilde{\Theta}^{(i,j)}(s)\,\widetilde{\mathbf{e}}_{(i,j)}$$

Finally, the connection 1-forms and curvature 2-forms of $(\mathcal{M}, g)$ with respect to this frame will be denoted by $\{\widetilde{\omega}_{(p,m)}^{(i,j)}\}_{\substack{0\leq i,p\leq k \\ 1\leq j,m\leq d_i}}$ and $\{\widetilde{\Omega}_{(p,m)}^{(i,j)}\}_{\substack{0\leq i,p\leq k \\ 1\leq j,m\leq d_i}}$, so that

$$\nabla_s\left(\widetilde{\mathbf{e}}_{(p,m)}\right) = \sum_{\substack{0\leq i\leq k \\ 1\leq j\leq d_i}} \widetilde{\omega}_{(p,m)}^{(i,j)}(s)\,\widetilde{\mathbf{e}}_{(i,j)}$$

and

$$\mathrm{Rm}(s_1, s_2)\widetilde{\mathbf{e}}_{(p,m)} = \sum_{\substack{0\leq i\leq k \\ 1\leq j\leq d_i}} \widetilde{\Omega}_{(p,m)}^{(i,j)}(s_1, s_2)\,\widetilde{\mathbf{e}}_{(i,j)}$$

Lemma (L.4). Following the notation above, let $\mathcal{M} = \mathcal{B}^r \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiply warped product with metric $g = g_{\mathcal{B}} \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$ and connection $\nabla^g = \nabla$. The connection 1-forms $\{\widetilde{\omega}_{(p,m)}^{(i,j)}\}_{\substack{0\leq i,p\leq k\\ 1\leq j,m\leq d_i}}$ are given by

(a)

$$\widetilde{\omega}_{(0,\ell)}^{(0,j)} = \omega_\ell^{(0,j)}, \text{ for all } 1 \leq j, \ell \leq r$$

(b)

$$\widetilde{\omega}_{(\beta,\ell)}^{(0,j)} = -\frac{\mathbf{e}_{(0,j)}(h_\beta)}{h_\beta}\,\widetilde{\Theta}^{(\beta,\ell)}, \text{ for any } 1 \leq j \leq r \text{ and } \beta, \ell \geq 1$$

(c)

$$\widetilde{\omega}_{(0,a)}^{(\gamma,j)} = \frac{\mathbf{e}_{(0,a)}(h_\gamma)}{h_\gamma}\,\widetilde{\Theta}^{(\gamma,j)}, \text{ if } 1 \leq \gamma \leq k \text{ and } 1 \leq a, j \leq r$$

(d)

$$\widetilde{\omega}_{(\beta,p)}^{(\gamma,q)} = \delta_{\gamma\beta}\,\omega_p^{(\gamma,q)}, \text{ if } 1 \leq \gamma, \beta \leq k \text{ and } 1 \leq p, q \leq d_\gamma$$

Proof: The idea of the proof is classical: motivated by Theorem 1, we will define an antisymmetric matrix $\left(\alpha_{(p,m)}^{(i,j)}\right)_{\substack{0\leq i,p\leq k\\ 1\leq j,m\leq d_i}}$ satisfying

$$\begin{cases} \mathrm{d}\widetilde{\Theta}^{(0,j)} + \displaystyle\sum_{1\leq\ell\leq r} \alpha_{(0,\ell)}^{(0,j)} \wedge \widetilde{\Theta}^{(0,\ell)} + \sum_{\substack{1\leq\gamma\leq k\\ 1\leq\beta\leq d_\gamma}} \alpha_{(\gamma,\beta)}^{(0,j)} \wedge \widetilde{\Theta}^{(\gamma,\beta)} = 0 \\ \mathrm{d}\widetilde{\Theta}^{(\beta,\gamma)} + \displaystyle\sum_{1\leq\ell\leq r} \alpha_{(0,\ell)}^{(\beta,\gamma)} \wedge \widetilde{\Theta}^{(0,\ell)} + \sum_{\substack{1\leq\xi\leq k\\ 1\leq\eta\leq d_\xi}} \alpha_{(\xi,\eta)}^{(\beta,\gamma)} \wedge \widetilde{\Theta}^{(\xi,\eta)} = 0 \end{cases} \tag{5}$$

for all $1 \leq j \leq r$, $1 \leq \beta \leq k$, $1 \leq \gamma \leq d_\beta$. From the uniqueness stated in Theorem 1, it will follow that $\left(\alpha_{(p,m)}^{(i,j)}\right)_{\substack{0\leq i,p\leq k\\ 1\leq j,m\leq d_i}}$ is the matrix of 1-forms of the Levi-Civita connection of $\mathcal{M}$. Since

$$\begin{aligned} \mathrm{d}\widetilde{\Theta}^{(0,j)} &= \mathrm{d}\Theta^{(0,j)} \\ &= -\sum_{1\leq\ell\leq r} \omega_\ell^{(0,j)} \wedge \widetilde{\Theta}^{(0,\ell)} \end{aligned} \tag{6}$$

And also

$$\begin{aligned} \mathrm{d}\widetilde{\Theta}^{(\gamma,j)} &= \mathrm{d}\left(h_\gamma \Theta^{(\gamma,j)}\right) \\ &= \mathrm{d}h_\gamma \wedge \Theta^{(\gamma,j)} + h_\gamma\,\mathrm{d}\Theta^{(\gamma,j)} \\ &= \left(\sum_{1\leq a\leq r} \mathbf{e}_{(0,a)}\left(h_\gamma\right)\,\Theta^{(0,a)}\right) \wedge \Theta^{(\gamma,j)} - h_\gamma \sum_{1\leq\ell\leq d_\gamma} \omega_\ell^{(\gamma,j)} \wedge \Theta^{(\gamma,\ell)} \\ &= -\sum_{1\leq a\leq r} \left(\mathbf{e}_{(0,a)}\left(h_\gamma\right)\,\Theta^{(\gamma,j)}\right) \wedge \widetilde{\Theta}^{(0,a)} - \sum_{1\leq\ell\leq d_\gamma} \omega_\ell^{(\gamma,j)} \wedge \widetilde{\Theta}^{(\gamma,\ell)} \end{aligned} \tag{7}$$

We will define

$$\begin{aligned} \alpha_{(0,\ell)}^{(0,j)} &= \omega_\ell^{(0,j)} \\ \alpha_{(\gamma,\beta)}^{(0,j)} &= -\frac{\mathbf{e}_{(0,j)}(h_\gamma)}{h_\gamma}\,\widetilde{\Theta}^{(\gamma,\beta)} \\ \alpha_{(0,\ell)}^{(\beta,\gamma)} &= \frac{\mathbf{e}_{(0,\ell)}(h_\beta)}{h_\beta}\,\widetilde{\Theta}^{(\beta,\gamma)} \end{aligned}$$

$$\alpha_{(\xi,\eta)}^{(\beta,\gamma)} = 0, \text{ for any } \xi \geq 1 \text{ with } \xi \neq \beta$$

It is easy to see that the matrix α thus defined is antisymmetric. Moreover, it follows from equations **(6)** and **(7)** that α thus defined satisfies **(5)**, whence the desired result follows.

■

Lemma (L.5). Let $\mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiply warped product with metric $g = g_{\mathcal{B}} \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$ and connection $\nabla^g = \nabla$. Suppose that $X, Y \in \mathcal{L}(\mathcal{B})$, $V \in \mathcal{L}(\mathcal{N}_\alpha)$ and $W \in \mathcal{L}(\mathcal{N}_\eta)$. Then:

(1) $\nabla_X Y = \nabla_X^{\mathcal{B}} Y$

(2) $\nabla_X V = \nabla_V X = \dfrac{X(h_\alpha)}{h_\alpha} V$

(3) $\nabla_V W = \begin{cases} 0, \text{ if } \alpha \neq \eta \\ \nabla_V^{\mathcal{N}_\eta} W - \dfrac{g(V,W)}{h_\eta} \text{grad}_{\mathcal{B}} h_\eta, \text{ if } \alpha = \eta \end{cases}$

Proof: We have

$$\begin{aligned}
\nabla_X Y &= \nabla_X \left(\sum_{1 \leq a \leq r} \widetilde{Y}^{(0,a)} \, \widetilde{\mathbf{e}}_{(0,a)} \right) \\
&= \sum_{1 \leq a \leq r} \left\{ X\left(\widetilde{Y}^{(0,a)}\right) \widetilde{\mathbf{e}}_{(0,a)} + \widetilde{Y}^{(0,a)} \, \nabla_X \left(\widetilde{\mathbf{e}}_{(0,a)} \right) \right\} \\
&= \sum_{1 \leq a \leq r} \left\{ X\left(Y^{(0,a)}\right) \mathbf{e}_{(0,a)} + Y^{(0,a)} \, \nabla_X \left(\widetilde{\mathbf{e}}_{(0,a)} \right) \right\}
\end{aligned}$$

Now, note that

$$\begin{aligned}
\nabla_X \left(\widetilde{\mathbf{e}}_{(0,a)} \right) &= \sum_{\substack{0 \leq i \leq k \\ 1 \leq j \leq d_i}} \widetilde{\omega}_{(0,a)}^{(i,j)}(X) \, \widetilde{\mathbf{e}}_{(i,j)} \\
&= \sum_{1 \leq j \leq r} \widetilde{\omega}_{(0,a)}^{(0,j)}(X) \, \widetilde{\mathbf{e}}_{(0,j)} + \sum_{\substack{1 \leq \gamma \leq k \\ 1 \leq j \leq d_\gamma}} \widetilde{\omega}_{(0,a)}^{(\gamma,j)}(X) \, \widetilde{\mathbf{e}}_{(\gamma,j)} \\
&= \sum_{1 \leq j \leq r} \omega_a^{(0,j)}(X) \, \mathbf{e}_{(0,j)} + \sum_{\substack{1 \leq \gamma \leq k \\ 1 \leq j \leq d_\gamma}} \mathbf{e}_{(0,a)}\left(h_\gamma\right) \underbrace{\Theta^{(\gamma,j)}(X)}_{=0, \text{ since } X \in \mathcal{L}(\mathcal{B})} \widetilde{\mathbf{e}}_{(\gamma,j)} \\
&= \nabla_X^{\mathcal{B}} \left(\mathbf{e}_{(0,a)} \right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla_X Y &= \sum_{1 \leq a \leq r} \left\{ X\left(Y^{(0,a)}\right) \mathbf{e}_{(0,a)} + Y^{(0,a)} \, \nabla_X \left(\mathbf{e}_{(0,a)} \right) \right\} \\
&= \nabla_X^{\mathcal{B}} \left(\sum_{1 \leq a \leq r} Y^{(0,a)} \, \mathbf{e}_{(0,a)} \right) \\
&= \nabla_X^{\mathcal{B}} Y
\end{aligned}$$

where in the third equality we used items **(a)** and **(c)** of Lemma **(L.4)**. Since $X\left(V^{(\alpha,\beta)}\right) = 0$, we also have

$$
\begin{aligned}
\nabla_X V &= \nabla_X \left(\sum_{1\leq\beta\leq d_\alpha} \widetilde{V}^{(\alpha,\beta)}\, \widetilde{\mathbf{e}}_{(\alpha,\beta)} \right) \\
&= \sum_{1\leq\beta\leq d_\alpha} \left\{ X\left(\widetilde{V}^{(\alpha,\beta)}\right) \widetilde{\mathbf{e}}_{(\alpha,\beta)} + \widetilde{V}^{(\alpha,\beta)}\, \nabla_X \left(\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right) \right\} \\
&= \sum_{1\leq\beta\leq d_\alpha} \left\{ \frac{X(h_\alpha)}{h_\alpha} V^{(\alpha,\beta)}\, \mathbf{e}_{(\alpha,\beta)} + \widetilde{V}^{(\alpha,\beta)} \left(\sum_{1\leq\ell\leq r} \underbrace{\widetilde{\omega}^{(0,\ell)}_{(\alpha,\beta)}(X)}_{=0,\text{ by (b) of (L.4)}}\, \widetilde{\mathbf{e}}_{(0,\ell)} + \sum_{\substack{1\leq\gamma\leq k \\ 1\leq s\leq d_\gamma}} \widetilde{\omega}^{(\gamma,s)}_{(\alpha,\beta)}(X)\, \widetilde{\mathbf{e}}_{(\gamma,s)} \right) \right\} \\
&= \frac{X(h_\alpha)}{h_\alpha} V + \sum_{1\leq s,\beta\leq d_\alpha} \underbrace{\omega^{(\alpha,s)}_{\beta}(X)}_{=0,\text{since } X\in\mathcal{L}(\mathcal{B})}\, \widetilde{\mathbf{e}}_{(\alpha,s)} \\
&= \frac{X(h_\alpha)}{h_\alpha} V
\end{aligned}
$$

where in the second-to-last equality we used item **(d)** of Lemma **(L.4)**.

Finally, using Lemma **(L.4)** again, we have

$$
\begin{aligned}
\nabla_V W &= \nabla_V \left(\sum_{1\leq\beta\leq d_\eta} \widetilde{W}^{(\eta,\beta)}\, \widetilde{\mathbf{e}}_{(\eta,\beta)} \right) \\
&= \sum_{1\leq\beta\leq d_\eta} V\left(\widetilde{W}^{(\eta,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\beta)} + \sum_{1\leq\beta\leq d_\eta} \widetilde{W}^{(\eta,\beta)}\, \nabla_V \left(\widetilde{\mathbf{e}}_{(\eta,\beta)}\right) \\
&= \sum_{1\leq\beta\leq d_\eta} V\left(\widetilde{W}^{(\eta,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\beta)} + \sum_{1\leq\beta\leq d_\eta} \widetilde{W}^{(\eta,\beta)} \left\{ \sum_{1\leq a\leq r} \widetilde{\omega}^{(0,a)}_{(\eta,\beta)}(V)\, \widetilde{\mathbf{e}}_{(0,a)} + \sum_{\substack{1\leq\xi\leq k \\ 1\leq\gamma\leq d_\xi}} \widetilde{\omega}^{(\xi,\gamma)}_{(\eta,\beta)}(V)\, \widetilde{\mathbf{e}}_{(\xi,\gamma)} \right\} \\
&= \sum_{1\leq\beta\leq d_\eta} V\left(\widetilde{W}^{(\eta,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\beta)} + \sum_{1\leq\beta\leq d_\eta} \widetilde{W}^{(\eta,\beta)} \left\{ -\sum_{1\leq a\leq r} \frac{\mathbf{e}_{(0,a)}(h_\eta)}{h_\eta}\, \widetilde{V}^{(\eta,\beta)}\, \widetilde{\mathbf{e}}_{(0,a)} + \sum_{1\leq\gamma\leq d_\eta} \omega^{(\eta,\gamma)}_{\beta}(V)\, \widetilde{\mathbf{e}}_{(\eta,\gamma)} \right\} \\
&= \sum_{1\leq\beta\leq d_\eta} \left\{ V\left(\widetilde{W}^{(\eta,\beta)}\right) \widetilde{\mathbf{e}}_{(\eta,\beta)} + \widetilde{W}^{(\eta,\beta)}\, \nabla_V \left(\widetilde{\mathbf{e}}_{(\eta,\beta)}\right) \right\} - \sum_{1\leq a\leq r} \frac{\mathbf{e}_{(0,a)}(h_\eta)}{h_\eta}\, g(V,W)\, \mathbf{e}_{(0,a)} \\
&= \nabla^{\mathcal{N}_\eta}_{\mathrm{d}\sigma_\eta(V)} W - \delta_{\alpha\eta}\, \frac{g(V,W)}{h_\eta} \mathrm{grad}_{\mathcal{B}} h_\eta \\
&= \delta_{\alpha\eta} \left(\nabla^{\mathcal{N}_\eta}_V W - \frac{g(V,W)}{h_\eta} \mathrm{grad}_{\mathcal{B}} h_\eta \right)
\end{aligned}
$$

■

Remark (O.6). The "natural" progression after having computed the connection 1-forms in Lemma **(L.4)** would be to use Theorem 2 to compute the curvature 2-forms. We will not do this because, in order to determine Rm, this path is much more complicated compared to simply using the definition of Rm in the computations, aided by Lemma **(L.4)**. We will do this in the proof of Lemma **(L.7)**.

Lemma (L.6). Let $\mathcal{M} = \mathcal{B} \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a multiply warped product with metric $g = g_{\mathcal{B}} \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$. Let us denote $\mathrm{grad}^{\mathcal{M}} = \mathrm{grad}$ and $\Delta^{\mathcal{M}} = \Delta$. If $\phi \in \mathscr{C}^\infty(\mathcal{B})$ and $\psi \in \mathscr{C}^\infty(\mathcal{N}_i)$, then

(1) $\mathrm{grad}(\phi \circ \pi) = \mathrm{grad}_{\mathcal{B}} \phi$

(2) $\operatorname{grad}(\psi \circ \sigma_i) = \dfrac{\operatorname{grad}_{\mathcal{N}_i}\psi}{h_i^2}$

(3) $\Delta(\phi \circ \pi) = \Delta_{\mathcal{B}}\phi + \displaystyle\sum_{1\leq i\leq k} d_i \frac{g_{\mathcal{B}}\left(\operatorname{grad}_{\mathcal{B}}\phi, \operatorname{grad}_{\mathcal{B}} h_i\right)}{h_i}$

(4) $\Delta(\psi \circ \sigma_i) = \dfrac{\Delta_{\mathcal{N}_i}\psi}{h_i^2}$

Proof: Fix $p = (p_0, q_1, \ldots, q_k) \in \mathcal{M}$. Since $\phi \circ \pi$ is constant on $\{p_0\} \times \mathcal{N}_1 \times \ldots \times \mathcal{N}_k$ for every $p_0 \in \mathcal{B}$, we have

$$\begin{aligned}
\operatorname{grad}\left(\phi \circ \pi\right)(p) &= \sum_{1\leq i\leq r} \mathbf{e}_{(0,i)}(p)\left(\phi \circ \pi\right)\mathbf{e}_{(0,i)}(p) \\
&= \sum_{1\leq i\leq r} \mathrm{d}\left(\phi \circ \pi\right)_p\left(\mathbf{e}_{(0,i)}(p)\right)\mathbf{e}_{(0,i)}(p) \\
&= \sum_{1\leq i\leq r} \mathrm{d}\phi_{p_0}\left(\mathbf{e}_{(0,i)}(p_0)\right)\mathbf{e}_{(0,i)}(p) \\
&= \sum_{1\leq i\leq r} \mathbf{e}_{(0,i)}(p_0)\left(\phi\right)\mathbf{e}_{(0,i)}(p) \\
&= \left(\operatorname{grad}_{\mathcal{B}}\phi\right)(p)
\end{aligned}$$

Analogously,

$$\begin{aligned}
\operatorname{grad}\left(\psi \circ \sigma_i\right)(p) &= \sum_{1\leq j\leq d_i} \widetilde{\mathbf{e}}_{(i,j)}(p)(\psi)\,\widetilde{\mathbf{e}}_{(i,j)}(p) \\
&= h_i^{-2}\left(\operatorname{grad}_{\mathcal{N}_i}\psi\right)(p)
\end{aligned}$$

From the definition of the Laplacian, from item **(2)** of Lemma **(L.5)** and from item **(1)** of the present lemma, it also follows that

$$\begin{aligned}
\Delta\left(\phi \circ \pi\right) &= \sum_{1\leq i\leq r} g\left(\nabla_{\mathbf{e}_{(0,i)}}\nabla^{\mathcal{B}}\phi, \mathbf{e}_{(0,i)}\right) + \sum_{\substack{1\leq \alpha\leq k \\ 1\leq \beta\leq d_\alpha}} g\left(\nabla_{\widetilde{\mathbf{e}}_{(\alpha,\beta)}}\nabla^{\mathcal{B}}\phi, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right) \\
&= \Delta_{\mathcal{B}}\phi + \sum_{\substack{1\leq \alpha\leq k \\ 1\leq \beta\leq d_\alpha}} g\left(\frac{\left(\nabla^{\mathcal{B}}\phi\right)(h_\alpha)}{h_\alpha}\widetilde{\mathbf{e}}_{(\alpha,\beta)}, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right) \\
&= \Delta_{\mathcal{B}}\phi + \sum_{1\leq \alpha\leq k} d_\alpha \frac{\left(\nabla^{\mathcal{B}}\phi\right)(h_\alpha)}{h_\alpha} \\
&= \Delta_{\mathcal{B}}\phi + \sum_{1\leq \alpha\leq k} d_\alpha \frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}}\phi, \nabla^{\mathcal{B}}h_\alpha\right)}{h_\alpha}
\end{aligned}$$

Finally,

$$\begin{aligned}
\Delta\left(\psi \circ \sigma_i\right) &= \sum_{1\leq j\leq r} g\left(\nabla_{\mathbf{e}_{(0,j)}}\nabla\psi, \mathbf{e}_{(0,j)}\right) + \sum_{\substack{1\leq \alpha\leq k \\ 1\leq \beta\leq d_\alpha}} g\left(\nabla_{\widetilde{\mathbf{e}}_{(\alpha,\beta)}}\nabla\psi, \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right) \\
&= \sum_{1\leq j\leq r} g\left(\nabla_{\mathbf{e}_{(0,j)}}\left(\frac{\nabla^{\mathcal{N}_i}\psi}{h_i^2}\right), \mathbf{e}_{(0,j)}\right) + \sum_{\substack{1\leq \alpha\leq k \\ 1\leq \beta\leq d_\alpha}} g\left(\nabla_{\widetilde{\mathbf{e}}_{(\alpha,\beta)}}\left(\frac{\nabla^{\mathcal{N}_i}\psi}{h_i^2}\right), \widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\leq j\leq r} g\left(-\frac{2\,\mathbf{e}_{(0,j)}(h_i)}{h_i^3}\,\nabla^{\mathcal{N}_i}\psi + \frac{1}{h_i^2}\,\nabla_{\mathbf{e}_{(0,j)}}\nabla^{\mathcal{N}_i}\psi, \mathbf{e}_{(0,j)}\right)\\
&+ \sum_{\substack{1\leq\alpha\leq k\\ 1\leq\beta\leq d_\alpha}} h_i^{-4}\, g\left(\nabla_{\mathbf{e}_{(\alpha,\beta)}}\nabla^{\mathcal{N}_i}\psi, \mathbf{e}_{(\alpha,\beta)}\right)\\
&= \sum_{\substack{1\leq\alpha\leq k\\ 1\leq\beta\leq d_\alpha}} h_i^{-2}\, g^{\mathcal{N}_i}\left(\nabla_{\mathbf{e}_{(\alpha,\beta)}}\nabla^{\mathcal{N}_i}\psi, \mathbf{e}_{(\alpha,\beta)}\right)\\
&= h_i^{-2}\,\Delta_{\mathcal{N}_i}\psi
\end{aligned}
$$

■

Lemma (L.7). Let $\mathcal{M} = \mathcal{B}\times_{h_1}\mathcal{N}_1^{d_1}\times\cdots\times_{h_k}\mathcal{N}_k^{d_k}$ be a multiply warped product with metric $g = g_{\mathcal{B}}\oplus h_1^2 g_{\mathcal{N}_1}\oplus\cdots\oplus h_k^2 g_{\mathcal{N}_k}$. Let us denote $\mathrm{Rm}^{\mathcal{M}} = \mathrm{Rm}$ and take $X,Y,Z\in\mathcal{L}(\mathcal{B})$, $V\in\mathcal{L}(\mathcal{N}_\alpha), W\in\mathcal{L}(\mathcal{N}_\eta)$ and $U\in\mathcal{L}(\mathcal{N}_\beta)$. Then

(1) $\mathrm{Rm}(X,Y)Z = \mathrm{Rm}_B(X,Y)Z$.

(2) $\mathrm{Rm}(V,X)Y = -\dfrac{(\nabla^2_{\mathcal{B}}h_\alpha)(X,Y)}{h_\alpha}V$.

(3) $\mathrm{Rm}(X,V)W = \mathrm{Rm}(V,W)X = \mathrm{Rm}(V,X)W = 0$ whenever $\alpha\notin\{\eta\}$.

(4) $\mathrm{Rm}(X,Y)V = 0$.

(5) $\mathrm{Rm}(V,W)X = 0$ whenever $\alpha = \eta$.

(6) $\mathrm{Rm}(V,W)U = 0$ whenever $\alpha\in\{\eta\}\neq\{\beta\}$.

(7) $\mathrm{Rm}(U,V)W = -g(V,W)\,\dfrac{g_{\mathcal{B}}(\mathrm{grad}_{\mathcal{B}}h_\alpha, \mathrm{grad}_{\mathcal{B}}h_\beta)}{h_\alpha h_\beta}\,U$ whenever $\alpha\in\{\eta\}\neq\{\beta\}$.

(8) $\mathrm{Rm}(X,V)W = -\dfrac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X(\mathrm{grad}_{\mathcal{B}}h_\alpha)$ whenever $\alpha\in\{\eta\}$.

(9) $\mathrm{Rm}(V,W)U = \mathrm{Rm}_{\mathcal{N}_\alpha}(V,W)U + \dfrac{\|\mathrm{grad}_{\mathcal{B}}h_\alpha\|^2}{(h_\alpha^2)}\cdot(g(V,U)W - g(W,U)V)$ whenever $\{\alpha,\eta\} = \{\beta\}$.

Proof: Item **(1)** is a direct consequence of item **(1)** of Lemma **(L.5)**. To prove item **(2)**, first note that $[X,V] = 0$ as an immediate consequence of Lemma **(L.3)**, so that

$$
\begin{aligned}
\mathrm{Rm}(V,X)Y &= \nabla_V\nabla_X Y - \nabla_X\nabla_V Y\\
&= \nabla_V\nabla^{\mathcal{B}}_X Y - \nabla_X\left(\frac{Y(h_\alpha)}{h_\alpha}V\right)\\
&= \frac{\left(\nabla^{\mathcal{B}}_X Y\right)(h_\alpha)}{h_\alpha}V - X\left(\frac{Y(h_\alpha)}{h_\alpha}\right)V - \frac{Y(h_\alpha)}{h_\alpha}\frac{X(h_\alpha)}{h_\alpha}V\\
&= \frac{\nabla_{\nabla^{\mathcal{B}}_X Y}(h_\alpha)}{h_\alpha}V - \frac{X(Y(h_\alpha))}{h_\alpha}V + \frac{X(h_\alpha)}{h_\alpha^2}Y(h_\alpha)V - \frac{X(h_\alpha)Y(h_\alpha)}{h_\alpha^2}V\\
&= -\frac{(\mathrm{Hess}_g(h_\alpha))(X,Y)}{h_\alpha}V
\end{aligned}
$$

Item **(3)** follows as a direct consequence of item **(3)** of Lemma **(L.5)**. Now, it follows from item **(2)** of Lemma **(L.5)** that

$$\begin{aligned}
\nabla_X\nabla_Y V - \nabla_Y\nabla_X V &= \nabla_X\left(\frac{Y(h_\alpha)}{h_\alpha}V\right) - \nabla_Y\left(\frac{X(h_\alpha)}{h_\alpha}V\right)\\
&= X\left(\frac{Y(h_\alpha)}{h_\alpha}\right)V + \frac{Y(h_\alpha)}{h_\alpha}\nabla_X V - Y\left(\frac{X(h_\alpha)}{h_\alpha}\right)V - \frac{X(h_\alpha)}{h_\alpha}\nabla_Y V\\
&= \frac{X(Y(h_\alpha))}{h_\alpha}V - \frac{X(h_\alpha)}{(h_\alpha)^2}Y(h_\alpha)\,V + \frac{Y(h_\alpha)}{h_\alpha}\frac{X(h_\alpha)}{h_\alpha}V\\
&- \frac{Y(X(h_\alpha))}{h_\alpha}V + \frac{X(h_\alpha)}{(h_\alpha)^2}Y(h_\alpha)\,V - \frac{X(h_\alpha)}{h_\alpha}\frac{Y(h_\alpha)}{h_\alpha}V\\
&= \frac{[X,Y](h_\alpha)}{h_\alpha}V\\
&= \nabla_{[X,Y]}V
\end{aligned}$$

and therefore $\mathrm{Rm}(X,Y)V = 0$, as claimed in item **(4)**. Item **(5)** is a simple consequence of item **(2)** of Lemma **(L.5)**. Indeed,

$$\begin{aligned}
\mathrm{Rm}(V,W)X &= \nabla_V\nabla_W X - \nabla_W\nabla_V X - \nabla_{[V,W]}X\\
&= \nabla_V\left(\frac{X(h_\alpha)}{h_\alpha}W\right) - \nabla_W\left(\frac{X(h_\alpha)}{h_\alpha}V\right) - \frac{X(h_\alpha)}{h_\alpha}[V,W]\\
&= \frac{X(h_\alpha)}{h_\alpha}\left(\nabla_V W - \nabla_W V - [V,W]\right)\\
&= 0
\end{aligned}$$

Item **(6)** is a direct consequence of items **(3)** and **(1)** of Lemma **(L.5)**. To prove item **(7)**, note that, again as a consequence of Lemma **(L.3)**, we have $[U,V]=0$, so that using Lemma **(L.5)**, we obtain:

$$\begin{aligned}
\mathrm{Rm}(U,V)W &= \nabla_U\nabla_V W - \nabla_V\underbrace{\nabla_U W}_{=0}\\
&= \nabla_U\left(\nabla^{\mathcal{N}_\alpha}_V W - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}h_\alpha\right)\\
&= -\frac{g(V,W)}{h_\alpha}\nabla_U\nabla^{\mathcal{B}}h_\alpha\\
&= -\frac{g(V,W)}{h_\alpha}\frac{\left(\nabla^{\mathcal{B}}h_\alpha\right)(h_\beta)}{h_\beta}U\\
&= -g(V,W)\,\frac{g_{\mathcal{B}}(\mathrm{grad}_{\mathcal{B}}h_\alpha,\mathrm{grad}_{\mathcal{B}}h_\beta)}{h_\alpha h_\beta}U
\end{aligned}$$

We will now prove item **(8)**. First, we have

$$\begin{aligned}
\nabla_X\nabla_V W &= \nabla_X\left(\nabla^{\mathcal{N}_\alpha}_V W - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}h_\alpha\right)\\
&= \frac{X(h_\alpha)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - X\left(\frac{g(V,W)}{h_\alpha}\right)\nabla^{\mathcal{B}}h_\alpha - \frac{g(V,W)}{h_\alpha}\nabla_X\nabla^{\mathcal{B}}h_\alpha\\
&= \frac{X(h_\alpha)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - \frac{(g(\nabla_X V,W)+g(V,\nabla_X W))}{h_\alpha}\nabla^{\mathcal{B}}h_\alpha + \frac{X(h_\alpha)}{(h_\alpha)^2}g(V,W)\nabla^{\mathcal{B}}h_\alpha
\end{aligned}$$

$$
\begin{aligned}
&- \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X\nabla^{\mathcal{B}}h_\alpha \\
&= \frac{X\left(h_\alpha\right)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - 2\frac{X\left(h_\alpha\right)}{\left(h_\alpha\right)^2}g(V,W)\nabla^{\mathcal{B}}h_\alpha + \frac{X\left(h_\alpha\right)}{\left(h_\alpha\right)^2}g(V,W)\nabla^{\mathcal{B}}h_\alpha - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X\nabla^{\mathcal{B}}h_\alpha \\
&= \frac{X\left(h_\alpha\right)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - \frac{X\left(h_\alpha\right)}{\left(h_\alpha\right)^2}g(V,W)\nabla^{\mathcal{B}}h_\alpha - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X\nabla^{\mathcal{B}}h_\alpha
\end{aligned}
$$

And also:

$$
\begin{aligned}
\nabla_V\nabla_X W &= \nabla_V\left(\frac{X\left(h_\alpha\right)}{h_\alpha}W\right) \\
&= \frac{X\left(h_\alpha\right)}{h_\alpha}\left(\nabla^{\mathcal{N}_\alpha}_V W - \frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}h_\alpha\right) \\
&= \frac{X\left(h_\alpha\right)}{h_\alpha}\nabla^{\mathcal{N}_\alpha}_V W - \frac{X\left(h_\alpha\right)}{\left(h_\alpha\right)^2}g(V,W)\nabla^{\mathcal{B}}h_\alpha
\end{aligned}
$$

Since $[X,V]=0$, it then follows that

$$
\mathrm{Rm}(X,V)W = -\frac{g(V,W)}{h_\alpha}\nabla^{\mathcal{B}}_X(\mathrm{grad}_{\mathcal{B}}h_\alpha)
$$

Finally, to prove **(9)**, first note that $\mathrm{Rm}(V,W)U \in \mathcal{L}(\mathcal{N}_\alpha)$, since given $X \in \mathcal{L}(\mathcal{B})$ and $L \in \mathcal{L}(\mathcal{N}_\xi)$ with $\xi \neq \alpha$, we have

$$
\begin{aligned}
g\left(\mathrm{Rm}(V,W)U,X\right) &= -g\left(\mathrm{Rm}(V,W)X,U\right) = 0, \text{ by item } \textbf{(5)} \\
g\left(\mathrm{Rm}(V,W)U,L\right) &= -g\left(\mathrm{Rm}(V,W)L,U\right) = 0, \text{ by item } \textbf{(6)}
\end{aligned}
$$

It then follows from the Gauss equation and from item **(3)** of Lemma **(L.5)** that given any $C \in \mathcal{L}(\mathcal{N}_\alpha)$, we have

$$
\begin{aligned}
g\left(\mathrm{Rm}(V,W)U,C\right) &= \left(h_\alpha\right)^2\,\mathrm{Rm}^{\mathcal{N}_\alpha}(V,W,U,C) - g\left(\mathbf{II}(V,C),\mathbf{II}(W,U)\right) + g\left(\mathbf{II}(V,U),\mathbf{II}(W,C)\right) \\
&= \left(h_\alpha\right)^2\,\mathrm{Rm}^{\mathcal{N}_\alpha}(V,W,U,C) - \frac{g(V,C)\,g(W,U)}{\left(h_\alpha\right)^2}\|\nabla^{\mathcal{B}}h_\alpha\|^2 + \frac{g(V,U)\,g(W,C)}{\left(h_\alpha\right)^2}\|\nabla^{\mathcal{B}}h_\alpha\|^2
\end{aligned}
$$

Equivalently, since $\mathrm{Rm}(V,W)U \in \mathcal{L}(\mathcal{N}_\alpha)$,

$$
\flat^g\left(\mathrm{Rm}(V,W)U\right) = \flat^g\left(\mathrm{Rm}^{\mathcal{N}_\alpha}(V,W)U + \frac{\|\nabla^{\mathcal{B}}h_\alpha\|^2}{\left(h_\alpha\right)^2}\cdot\left\{g(V,U)\,W - g(W,U)\,V\right\}\right)
$$

Since $\flat^g$ is an isomorphism on each fiber of $T\mathcal{M}$, it follows, as we wanted to show, that

$$
\mathrm{Rm}(V,W)U = \mathrm{Rm}^{\mathcal{N}_\alpha}(V,W)U + \frac{\|\nabla^{\mathcal{B}}h_\alpha\|^2}{\left(h_\alpha\right)^2}\cdot\left\{g(V,U)\,W - g(W,U)\,V\right\}
$$

■

Lemma (L.8). Let $\mathcal{M} = \mathcal{B}\times_{h_1}\mathcal{N}_1^{d_1}\times\cdots\times_{h_k}\mathcal{N}_k^{d_k}$ be a multiply warped product with metric $g = g_{\mathcal{B}}\oplus h_1^2 g_{\mathcal{N}_1}\oplus\cdots\oplus h_k^2 g_{\mathcal{N}_k}$. Suppose that $X,Y,Z \in \mathcal{L}(\mathcal{B})$, $V \in \mathcal{L}(\mathcal{N}_\alpha)$ and $W \in \mathcal{L}(\mathcal{N}_\eta)$. Then:

(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}_{\mathcal{B}}(X,Y) - \sum_{1\leq\mu\leq k}\frac{d_\mu}{h_\mu}\nabla^2_{\mathcal{B}}(h_\mu)(X,Y)$.

(2) $\mathrm{Ric}(X,V)=0$.

(3) $\mathrm{Ric}(V,W)=0$ if $\alpha\neq\eta$.

(4)

$$\mathrm{Ric}(V,W)=\mathrm{Ric}^{\mathcal{N}_\alpha}(V,W)-\left(\begin{array}{l}\dfrac{\Delta_{\mathcal{B}}h_\alpha}{h_\alpha}+(d_\alpha-1)\,\dfrac{\|\nabla^{\mathcal{B}}h_\alpha\|^2}{(h_\alpha)^2}\\[2ex] +\displaystyle\sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha}} d_\gamma\,\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}}h_\gamma,\nabla^{\mathcal{B}}h_\alpha\right)}{h_\gamma\,h_\alpha}\end{array}\right)g(V,W)\,,\ \text{if } \alpha=\eta.$$

Proof: First, we have

$$\begin{aligned}\mathrm{Ric}(X,Y)&=\sum_{1\leq i\leq r}\mathrm{Rm}\left(\mathbf{e}_{(0,i)},X,Y,\mathbf{e}_{(0,i)}\right)+\sum_{\substack{1\leq\mu\leq k\\ 1\leq\beta\leq d_\mu}}\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\mu,\beta)},X,Y,\widetilde{\mathbf{e}}_{(\mu,\beta)}\right)\\&=\mathrm{Ric}^{\mathcal{B}}(X,Y)+\sum_{\substack{1\leq\mu\leq k\\ 1\leq\beta\leq d_\mu}}g\left(-\frac{\left(\nabla^2_{\mathcal{B}}h_\mu\right)(X,Y)}{h_\mu}\,\widetilde{\mathbf{e}}_{(\mu,\beta)},\widetilde{\mathbf{e}}_{(\mu,\beta)}\right)\\&=\mathrm{Ric}^{\mathcal{B}}(X,Y)-\sum_{1\leq\mu\leq k}\frac{d_\mu}{h_\mu}\left(\nabla^2_{\mathcal{B}}h_\mu\right)(X,Y)\end{aligned}$$

By the $\mathscr{C}^\infty(\mathcal{M})$-bilinearity of Ric, to prove the remaining items we may assume without loss of generality that $V=\widetilde{\mathbf{e}}_{(\alpha,\beta)}$ and $W=\widetilde{\mathbf{e}}_{(\eta,\xi)}$. We have

$$\begin{aligned}\mathrm{Ric}(X,V)&=\sum_{1\leq i\leq r}g\left(\mathrm{Rm}\left(\mathbf{e}_{(0,i)},X\right)V,\mathbf{e}_{(0,i)}\right)+\sum_{\substack{1\leq\gamma\leq k\\ 1\leq\ell\leq d_\gamma}}g\left(\mathrm{Rm}\left(\mathbf{e}_{(\gamma,\ell)},X\right)V,\mathbf{e}_{(\gamma,\ell)}\right)\\&=-\sum_{1\leq i\leq r}\underbrace{g\left(\mathrm{Rm}\left(\mathbf{e}_{(0,i)},X\right)\mathbf{e}_{(0,i)},V\right)}_{=0,\text{ by }\mathbf{(1)}\text{ of }\mathbf{(L.7)}}-\sum_{\substack{1\leq\gamma\leq k\\ 1\leq\ell\leq d_\gamma}}\underbrace{g\left(\mathrm{Rm}\left(X,\mathbf{e}_{(\gamma,\ell)}\right)\mathbf{e}_{(\alpha,\beta)},\mathbf{e}_{(\gamma,\ell)}\right)}_{=0,\text{ by }\mathbf{(3)}\text{ and }\mathbf{(8)}\text{ of }\mathbf{(L.7)}}\\&=0\end{aligned}$$

If $\alpha\neq\eta$, we have:

$$\mathrm{Ric}\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)},\widetilde{\mathbf{e}}_{(\eta,\xi)}\right)=\sum_{1\leq i\leq r}\underbrace{g\left(\mathrm{Rm}\left(\mathbf{e}_{(0,i)},\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\eta,\xi)},\mathbf{e}_{(0,i)}\right)}_{=0,\text{ by }\mathbf{(3)}\text{ of }\mathbf{(L.7)}}+\sum_{\substack{1\leq\gamma\leq k\\ 1\leq p\leq d_\gamma}}g\left(\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\gamma,p)},\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\eta,\xi)},\widetilde{\mathbf{e}}_{(\gamma,p)}\right)$$

It is obvious that the terms of the last sum on the right-hand side above all vanish. Indeed, if $\gamma=\alpha$ this is absolutely trivial, and when γ,α and η are all distinct, item **(3)** of Lemma **(L.5)** guarantees that

$$\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\gamma,p)},\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\eta,\xi)}=0$$

Now, when $\alpha=\eta$, we have

$$\begin{aligned}\mathrm{Ric}\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)},\widetilde{\mathbf{e}}_{(\alpha,\xi)}\right)&=\sum_{1\leq i\leq r}g\left(\mathrm{Rm}\left(\mathbf{e}_{(0,i)},\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)},\mathbf{e}_{(0,i)}\right)+\sum_{\substack{1\leq\gamma\leq k\\ 1\leq p\leq d_\gamma}}g\left(\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\gamma,p)},\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)},\widetilde{\mathbf{e}}_{(\gamma,p)}\right)\\&=\sum_{1\leq i\leq r}g\left(\frac{-\delta_{\beta\xi}}{h_\alpha}\,\nabla^{\mathcal{B}}_{\mathbf{e}_{(0,i)}}\nabla^{\mathcal{B}}h_\alpha,\mathbf{e}_{(0,i)}\right)+\sum_{1\leq p\leq d_\alpha}g\left(\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\alpha,p)},\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)},\widetilde{\mathbf{e}}_{(\alpha,p)}\right)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha\\ 1\leq p\leq d_\gamma}} g\left(\mathrm{Rm}\left(\widetilde{\mathbf{e}}_{(\gamma,p)},\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)},\widetilde{\mathbf{e}}_{(\gamma,p)}\right)\\
&= -\frac{\delta_{\beta\xi}}{h_\alpha}\,\Delta_{\mathcal{B}}h_\alpha\\
&+ \sum_{1\leq p\leq d_\alpha} g\left(\mathrm{Rm}^{\mathcal{N}_\alpha}\left(\widetilde{\mathbf{e}}_{(\alpha,p)},\widetilde{\mathbf{e}}_{(\alpha,\beta)}\right)\widetilde{\mathbf{e}}_{(\alpha,\xi)} + \frac{\|\nabla^{\mathcal{B}}h_\alpha\|^2}{(h_\alpha)^2}\left\{\delta_{p\xi}\,\widetilde{\mathbf{e}}_{(\alpha,\beta)} - \delta_{\beta\xi}\,\widetilde{\mathbf{e}}_{(\alpha,p)}\right\},\widetilde{\mathbf{e}}_{(\alpha,p)}\right)\\
&- \sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha\\ 1\leq p\leq d_\gamma}} g\left(\delta_{\beta\xi}\,\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}}h_\gamma,\nabla^{\mathcal{B}}h_\alpha\right)}{h_\gamma\,h_\alpha}\widetilde{\mathbf{e}}_{(\gamma,p)},\widetilde{\mathbf{e}}_{(\gamma,p)}\right)\\
&= -\frac{\delta_{\beta\xi}}{h_\alpha}\,\Delta_{\mathcal{B}}h_\alpha + \mathrm{Ric}^{\mathcal{N}_\alpha}\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)},\widetilde{\mathbf{e}}_{(\alpha,\xi)}\right) + \frac{\|\nabla^{\mathcal{B}}h_\alpha\|^2}{(h_\alpha)^2}\,\delta_{\beta\xi} - \frac{\|\nabla^{\mathcal{B}}h_\alpha\|^2}{(h_\alpha)^2}\,\delta_{\beta\xi}\,d_\alpha\\
&- \delta_{\beta\xi}\sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha}} d_\gamma\,\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}}h_\gamma,\nabla^{\mathcal{B}}h_\alpha\right)}{h_\gamma\,h_\alpha}
\end{aligned}
$$

$$
= \mathrm{Ric}^{\mathcal{N}_\alpha}\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)},\widetilde{\mathbf{e}}_{(\alpha,\xi)}\right) - \left(\begin{array}{l}\dfrac{\Delta_{\mathcal{B}}h_\alpha}{h_\alpha} + (d_\alpha-1)\,\dfrac{\|\nabla^{\mathcal{B}}h_\alpha\|^2}{(h_\alpha)^2}\\ \qquad + \displaystyle\sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha}} d_\gamma\,\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}}h_\gamma,\nabla^{\mathcal{B}}h_\alpha\right)}{h_\gamma\,h_\alpha}\end{array}\right) g\left(\widetilde{\mathbf{e}}_{(\alpha,\beta)},\widetilde{\mathbf{e}}_{(\alpha,\xi)}\right)
$$

■

Lemma (L.9). Let $\mathcal{M} = B\times_{h_1}\mathcal{N}_1^{d_1}\times\cdots\times_{h_k}\mathcal{N}_k^{d_k}$ be a multiply warped product with metric $g = g_B\oplus h_1^2 g_{\mathcal{N}_1}\oplus\cdots\oplus h_k^2 g_{\mathcal{N}_k}$. Then the scalar curvature of $\mathcal{M}$ is given by

$$
\begin{aligned}
R = R_B - 2\sum_{1\leq i\leq k} d_i\frac{\Delta_B h_i}{h_i} + \sum_{1\leq i\leq k}\frac{R_{\mathcal{N}_i}}{h_i^2} - \sum_{1\leq i\leq k} d_i(d_i-1)\frac{\|\mathrm{grad}_B h_i\|_B^2}{h_i^2}\\
- \sum_{1\leq i\leq k}\sum_{\substack{1\leq\ell\leq k\\ \ell\neq i}} d_i d_\ell\frac{g_B(\mathrm{grad}_B h_i,\mathrm{grad}_B h_\ell)}{h_i h_\ell}
\end{aligned}
\tag{8}
$$

Proof: We have

$$
\begin{aligned}
R &= \sum_{1\leq i\leq r}\mathrm{Ric}\left(\mathbf{e}_{(0,i)},\mathbf{e}_{(0,i)}\right) + \sum_{\substack{1\leq\alpha\leq k\\ 1\leq\varphi\leq d_\alpha}}\mathrm{Ric}\left(\widetilde{\mathbf{e}}_{(\alpha,\varphi)},\widetilde{\mathbf{e}}_{(\alpha,\varphi)}\right)\\
&= R_{\mathcal{B}} - \sum_{\substack{1\leq\mu\leq k\\ 1\leq i\leq r}}\frac{d_\mu}{h_\mu}\left(\nabla^2_{\mathcal{B}}h_\mu\right)\left(\mathbf{e}_{(0,i)},\mathbf{e}_{(0,i)}\right)\\
&+ \sum_{\substack{1\leq\alpha\leq k\\ 1\leq\varphi\leq d_\alpha}}\mathrm{Ric}^{\mathcal{N}_\alpha}\left(\widetilde{\mathbf{e}}_{(\alpha,\varphi)},\widetilde{\mathbf{e}}_{(\alpha,\varphi)}\right) - \sum_{\substack{1\leq\alpha\leq k\\ 1\leq\varphi\leq d_\alpha}}\left(\begin{array}{l}\dfrac{\Delta_{\mathcal{B}}h_\alpha}{h_\alpha} + (d_\alpha-1)\,\dfrac{\|\nabla^{\mathcal{B}}h_\alpha\|^2}{(h_\alpha)^2}\\ \qquad + \displaystyle\sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha}} d_\gamma\,\frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}}h_\gamma,\nabla^{\mathcal{B}}h_\alpha\right)}{h_\gamma\,h_\alpha}\end{array}\right) g\left(\widetilde{\mathbf{e}}_{(\alpha,\varphi)},\widetilde{\mathbf{e}}_{(\alpha,\varphi)}\right)
\end{aligned}
$$

$$= R_{\mathcal{B}} - 2\sum_{1\leq\mu\leq k} d_\mu \frac{\Delta_{\mathcal{B}} h_\mu}{h_\mu} + \sum_{1\leq\mu\leq k} \frac{R_{\mathcal{N}_\mu}}{(h_\mu)^2} - \sum_{1\leq\mu\leq k} d_\mu\,(d_\mu - 1)\,\frac{\|\mathrm{grad}_{\mathcal{B}} h_\mu\|^2}{(h_\mu)^2}$$
$$- \sum_{1\leq\alpha\leq k}\sum_{\substack{1\leq\gamma\leq k\\ \gamma\neq\alpha}} d_\alpha\, d_\gamma \frac{g_{\mathcal{B}}\left(\nabla^{\mathcal{B}} h_\gamma, \nabla^{\mathcal{B}} h_\alpha\right)}{h_\gamma\, h_\alpha}$$

■

Example (E.1). For this example, we first recall that a Riemannian manifold $(\mathcal{M}, g)$ has constant sectional curvature equal to $S \in \mathbb{R}$ if and only if its curvature tensor is given by

$$\mathrm{Rm} = S\ (g \owedge g)$$

(where $\owedge$ denotes the Kulkarni-Nomizu product), or, equivalently, if and only if

$$\mathrm{Rm}(X, Y, Z, W) = S\ (g(X, W)\, g(Y, Z) - g(X, Z)\, g(Y, W))$$

for all $X, Y, Z, W \in \Gamma(T\mathcal{M})$. Now, denoting

$$\mathbb{Q}^n(\kappa) \doteq \begin{cases} \mathbb{R}^n, & \text{if } \kappa = 0 \\ \mathbb{S}^n, & \text{if } \kappa = 1 \\ \mathbb{H}^n, & \text{if } \kappa = -1 \end{cases}$$

We have (as proved in [4]) that the Riemann tensor of $\mathcal{I} \times_h \mathbb{Q}^n(\kappa)$ (where $\mathcal{I} \subset \mathbb{R}$ is an open interval and h is a smooth positive real function defined on $\mathcal{I}$) is given by

$$\mathrm{Rm}(X, Y, Z, W) = \left(\frac{(h')^2 - \kappa}{h^2} \circ \pi_{\mathcal{I}}\right)(g(X, Z)\, g(Y, W) - g(X, W)\, g(Y, Z))$$
$$+ \left(\frac{h\, h'' - (h')^2 + \kappa}{h^2} \circ \pi_{\mathcal{I}}\right) \begin{pmatrix} g(X, Z)\, g\,(Y, \partial_t)\ g\,(W, \partial_t) - g(Y, Z)\, g\,(X, \partial_t)\ g\,(W, \partial_t) \\ \\ - g(X, W)\, g\,(Y, \partial_t)\ g\,(Z, \partial_t) + g(Y, W)\, g\,(X, \partial_t)\ g\,(Z, \partial_t) \end{pmatrix}$$

In particular, $\mathcal{I} \times_h \mathbb{Q}^n(\kappa)$ has constant sectional curvature equal to $C \in \mathbb{R}$ if and only if

$$-C = \frac{(h')^2 - \kappa}{h^2} \text{ is constant and } \frac{h''}{h} = \frac{(h')^2 - \kappa}{h^2}$$

Consequently,

$$\mathbb{S}^n \text{ is locally isometric to } \mathcal{I} \times_{\sin(t)} \mathbb{S}^{n-1}$$
$$\mathbb{H}^n \text{ is locally isometric to } \mathcal{I} \times_{e^t} \mathbb{R}^{n-1} \text{ and to } \mathcal{I} \times_{\sinh(t)} \mathbb{S}^{n-1}$$
$$\mathbb{R}^n \text{ is locally isometric to } \mathcal{I} \times_t \mathbb{S}^{n-1}$$

By the following lemma, the Riemannian manifold of Example **(E.2)** is locally conformally flat.

Lemma (L.10). Let $\mathcal{M} = \mathcal{I} \times_h \mathcal{N}$ be a warped product. Then

$$\mathcal{M} \text{ is locally conformally flat } \iff (\mathcal{N}, g_{\mathcal{N}}) \text{ has constant sectional curvature}$$

Proof: See [5].

■

Example (E.2) (Bryant soliton). Let $g_{\mathbb{S}^{n-1}}$ be the usual metric on the $(n-1)$-dimensional unit round sphere. We will look for steady Ricci solitons that arise as warped products on $(0,\infty)\times\mathbb{S}^{n-1}$. Let us then consider metrics of the form

$$g = \mathrm{d}r^2 + (\phi(r))^2 g_{\mathbb{S}^{n-1}}$$

Taking a local orthonormal frame on $\mathbb{S}^{n-1}$, say

$$\left\{\frac{\partial}{\partial\theta^i}\right\}_{1\leq i\leq n-1}$$

we see that $\left\{\frac{\partial}{\partial r}, \frac{1}{\phi}\frac{\partial}{\partial\theta^i}\right\}_{1\leq i\leq n-1}$ is a local orthonormal frame on $(0,\infty)\times_\phi \mathbb{S}^{n-1}$. By item **(8)** of Lemma **(L.7)**, we see that the radial curvature of a plane passing through the radial vector $\frac{\partial}{\partial r}$ is given by

$$\begin{aligned}
K_{\mathrm{rad}} = K\left(\frac{\partial}{\partial r}, \frac{1}{\phi}\frac{\partial}{\partial\theta^i}\right) &= g\left(\mathrm{Rm}\left(\frac{\partial}{\partial r}, \frac{1}{\phi}\frac{\partial}{\partial\theta^i}\right)\frac{1}{\phi}\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial r}\right)\\
&= \frac{1}{\phi^2}\cdot g(\mathrm{Rm}(\partial_r,\partial_{\theta^i})\partial_{\theta^i},\partial_r)\\
&= -\frac{1}{\phi^2}\cdot\frac{\phi^2}{\phi}\cdot\phi''\\
&= -\frac{\phi''}{\phi}
\end{aligned}$$

Analogously, using item item **(9)** of Lemma **(L.7)**, we see that the sectional curvature of a plane P_{sph} perpendicular to $\frac{\partial}{\partial r}$ is given by

$$\begin{aligned}
K_{\mathrm{sph}} = K\left(\frac{1}{\phi}\partial_{\theta^i}, \frac{1}{\phi}\partial_{\theta^j}\right) &= \frac{1}{\phi^4}\cdot g\left(\mathrm{Rm}(\partial_{\theta^i},\partial_{\theta^j})\partial_{\theta^j},\partial_{\theta^i}\right)\\
&= \frac{1}{\phi^4}\cdot\phi^2\cdot g_{\mathbb{S}^{n-1}}\left(\left[\pi^*_{\mathbb{S}^{n-1}}\mathrm{Rm}\right](\partial_{\theta^i},\partial_{\theta^j})\partial_{\theta^j}\right),\partial_{\theta^i})\\
&= \frac{1}{\phi^2}\cdot\left(1-\frac{(\phi')^2}{\phi^2}\cdot\phi^2\right)\\
&= \frac{1-(\phi')^2}{\phi^2}
\end{aligned}$$

Therefore

$$\mathrm{Ric}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) = \sum_{1\leq i\leq n-1} K_{\mathrm{rad}} = -(n-1)\frac{\phi''}{\phi}$$

and

$$\begin{aligned}
\mathrm{Ric}\left(\frac{\partial}{\partial\theta^i},\frac{\partial}{\partial\theta^j}\right) &= \phi^2\cdot\mathrm{Ric}\left(\frac{1}{\phi}\cdot\frac{\partial}{\partial\theta^i},\frac{1}{\phi}\cdot\frac{\partial}{\partial\theta^j}\right)\\
&= \phi^2\cdot\left(\left[\sum_{\substack{1\leq i\leq n-2\\ i\leq j}} K_{\mathrm{sph}}\right] + K_{\mathrm{rad}}\right)\\
&= (n-2)\left(1-(\phi')^2\right) - \phi\phi''
\end{aligned}$$

We then conclude that

$$\mathrm{Ric}(g) = -(n-1)\frac{\phi''}{\phi}\cdot \mathrm{d}r^2 + \left((n-2)\left(1-(\phi')^2\right) - \phi\phi''\right) g_{\mathbb{S}^{n-1}}$$

Now, as a direct consequence of Lemma **(L.6)**, we also have

$$\nabla^2 f = f''(r)\cdot \mathrm{d}r^2 + \phi\phi' f' \cdot g_{\mathbb{S}^{n-1}}$$

We see that the equation for steady Ricci solitons $\mathrm{Ric}(g) + \nabla^2 f = 0$ is in this case equivalent to the following system of second-order ODEs:

$$\begin{cases} f'' = (n-1)\dfrac{\phi''}{\phi} \\ \phi\phi' f' = -(n-2)\left(1-(\phi')^2\right) + \phi\phi'' \end{cases}$$

By studying this system in detail, one can prove the existence of a complete, rotationally symmetric steady gradient Ricci soliton, which is unique up to homotheties.

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