Geodesic frames

English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio, translated from the author's LaTeX source. The Portuguese original is the authoritative version.

One of the most useful tools for computations in Riemannian geometry is the use of *geodesic frames*. Two questions that naturally arise for students are the following:

- how does one prove the existence of geodesic frames?
- why is there no loss of generality in computations when using geodesic frames?

The purpose of this text is to answer the first question. For a detailed answer to the second, see [1].

Definition (D.1). Geodesics starting at p whose images are contained in a normal neighborhood of p are called *radial geodesics*.

Theorem (T.1). *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold and $p \in \mathcal{M}$ an arbitrarily fixed point. Then there exist a neighborhood $U_p \ni p$ and a local frame $\{E_i\}_{1\leq i\leq n} \subset \Gamma(TU_p)$ satisfying*

$$g(E_i(q), E_j(q)) = \delta_{ij} \ \forall q \in U_p, \ \text{and} \ \nabla_{E_i(p)} E_j = 0,$$

for all $1 \leq i, j \leq n$.

Proof: Let U_p be a normal neighborhood of p and fix an orthonormal basis $\{b_i\}_{1\leq i\leq n}$ of $T_p\mathcal{M}$. This basis induces an isomorphism

$$\begin{aligned} B : \mathbb{R}^n &\to T_p\mathcal{M} \\ (x^1, \cdots, x^n) &\mapsto \sum_{1\leq i\leq n} x^i \, b_i. \end{aligned}$$

Now consider the chart $\varphi = \left(\exp_p \circ B\right)^{-1}$ around p and let $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}_{1\leq i\leq n}$ be its associated coordinate basis. Since

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p &= \mathrm{d}(\exp_p \circ B) \left(\left.\frac{\partial}{\partial r^i}\right|_0 \right) \\ &= \underbrace{\mathrm{d}(\exp_p)_0}_{=\mathrm{Id}_{T_p\mathcal{M}}} \circ \underbrace{\mathrm{d}B_0}_{=B} \left(\left.\frac{\partial}{\partial r^i}\right|_0 \right) \\ &= B \left(\left.\frac{\partial}{\partial r^i}\right|_0 \right) \\ &= b_i, \end{aligned}$$

it follows that $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}_{1\leq i\leq n}$ is an orthonormal basis of $T_p\mathcal{M}$, so that we can therefore define $E_i(p) = \left.\frac{\partial}{\partial x^i}\right|_p$ for each $1 \leq i \leq n$. To extend this basis to a local orthonormal frame, it suffices to recall the fact that

parallel transport is an isometry and to perform parallel transport along radial geodesics. More precisely, for each $q \in U_p$, there exists a unique radial geodesic $\gamma_{p,q} : [0,1] \to \mathcal{M}$ joining p and q, and we can then define

$$E_i(q) = P_{\gamma_{p,q},0,1}(E_i(p)).$$

With respect to the chart φ, it is clear that the coordinate representation of the unique geodesic $\gamma_v : I \to \mathcal{M}$ starting at p with initial velocity $v \in T_p\mathcal{M}$ is given by $t \mapsto tv$, and therefore the geodesic equation reads

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\gamma_v(t)\right)^k + \sum_{1\leq i,j\leq n} \Gamma_{ij}^k\left(\gamma(t)\right) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_v(t)\right)^i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_v(t)\right)^j = 0, \ \forall t \in I \text{ and } \forall 1 \leq k \leq n.$$

In particular, we have

$$\sum_{1\leq i,j\leq n} \Gamma_{ij}^k(p)\, v^i v^j = 0, \ \forall v = \sum_{1\leq i\leq n} v^i b_i \in T_p\mathcal{M}, \text{ for every } 1 \leq k \leq n.$$

Taking $v = \partial_a$ for any fixed $1 \leq a \leq n$, we conclude that $\Gamma_{aa}^k(p) = 0$ for all $1 \leq a, k \leq n$. Finally, taking $v \in \{\partial_a + \partial_b, \partial_b - \partial_a\}$ for any fixed indices $1 \leq a, b \leq n$ and subtracting the resulting equations, we obtain $\Gamma_{ab}^k(p) = 0$ for all $1 \leq a, b, k \leq n$, whence it follows that $(\nabla_{E_i} E_j)(p) = 0$ for all $1 \leq i, j \leq n$.

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Remark (O.1). It is easy to verify that in normal coordinates, all the first partial derivatives of g_{ij} vanish at p. This shows that there are no geometric invariants of order < 2. As seen in [4], this fact can also be seen from the expansion of the metric in normal coordinates, given by

$$g_{ij}(t) = \delta_{ij} + \frac{1}{3} \sum_{1\leq k,\ell\leq n} \mathrm{Rm}_{ik\ell j}\, x^k\, x^\ell + O\left(|t|^3\right).$$

Remark (O.2). By the tensoriality of ∇ in the first entry, it is also clear that the geodesic frame of lemma 1 satisfies

$$\nabla_v E_i = 0,$$

for every $v \in T_p\mathcal{M}$.

References

[1] **Horácio, M.A.R.M.** *One chart to rule them all!* Text found on personal webpage. Available at: **https://sagangromov.github.io/assets/pdf/OneChart.pdf**. Accessed on February 15, 2023.

[2] **Horácio, M.A.R.M.** Estimativas de curvatura para sólitons de Ricci gradiente quadridimensionais. Master's dissertation, Universidade de Brasília, (2023).

[3] **Lee, John M.** Introduction to smooth manifolds. Second edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013. xvi+708 pp. ISBN: 978-1-4419-9981-8.

[4] **Lee, John M.** Introduction to Riemannian manifolds. Second edition of [MR1468735]. Graduate Texts in Mathematics, 176. Springer, Cham, 2018. xiii+437 pp. ISBN: 978-3-319-91754-2; 978-3-319-91755-9.