Tensors on Riemannian manifolds

English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese handwritten notes by Matheus Horácio, transcribed from the scanned notes. The Portuguese original is the authoritative version.

$$f = f_i e^i \Rightarrow f(e_s) = f_i e^i(e_s) = f_s \qquad v = \sum_i v^i e_i \Rightarrow e^s(v) = \sum_i v^i \delta^s_i = v^s$$
$$f = \sum_s f(e_s) e^s \qquad v = \sum_s e^s(v) e_s$$

Let V be a finite-dimensional real vector space, V^* its dual and $\mathrm{End}(V)$ the vector space of endomorphisms of V. A multilinear map

$$T\colon \underbrace{V^* \times V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \to \mathbb{R}$$

is called a real tensor of type (r, s) on V. The vector space of all such maps will be denoted by $\mathcal{T}^r_s(V)$.

We will say that $T \in \mathcal{T}^r_s(V)$ is r-times contravariant and s-times covariant. This nomenclature will be justified shortly.

Note that $\mathcal{T}^1_0(V) = V^{**}$, which can be naturally identified with V via

$$V \ni v \mapsto (V^* \ni f \mapsto f(v)) \in V^{**}$$

Definition. Let $\mathcal{B} = \{e_i\}_{1 \le i \le m}$ and $\mathcal{B}^* = \{e^i\}_{1 \le i \le m}$ be dual bases of V and V^*, respectively (that is, $e^i(e_s) = \delta^i_s \ \forall i, s$). Given $T \in \mathcal{T}^r_s(V)$, the components of T in the basis $\mathcal{B}$ are the real numbers defined by

$$T^{i_1 \ldots i_r}{}_{s_1 \ldots s_s} := T(e^{i_1}, \ldots, e^{i_r}, e_{s_1}, \ldots, e_{s_s})$$

for any $1 \le i_1, \ldots, i_r, s_1, \ldots, s_s \le m$.

Proposition. The map

$$\begin{aligned} \Psi\colon \mathrm{End}(V) &\to \mathcal{T}^1_1(V) \\ T &\mapsto \underbrace{\Psi(T)}_{\cong T}\colon V^* \times V \to \mathbb{R} \\ &\qquad (f, v) \mapsto f(T(v)) \end{aligned}$$

is an isomorphism.

Proof. Linearity and the fact that Ψ is well defined are trivial. If $\Psi(T) = \Psi(\tilde{T})$, then $f(T(v)) = f(\tilde{T}(v))$ for any $f \in V^*$ and $v \in V$. In particular, taking $v = e_i$ and $f = e^s$ with $i \neq s$, we see that $e^s(T(e_i)) = e^s(\tilde{T}(e_i))$, whence $T = \tilde{T}$. Given $g \in \mathcal{T}^1_1(V)$, defining

$$T(v) = \sum_{i=1}^{m} g(e^i, v) e_i$$

we see that

$$[\Psi(T)](f,v) = f(T(v)) = f\Big(\sum_{i=1}^{m} g(e^i,v)e_i\Big) = \sum_{i=1}^{m} g(e^i,v)f(e_i)$$
$$= g(f,v) \implies \Psi(T) = g$$

so that T is surjective. □

Definition. Let V and W be finite-dimensional real vector spaces. A multilinear map

$$T\colon \underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \to W$$

is called a W-tensor of type (r,s) on V. The vector space of all such maps will be denoted by $\mathrm{Hom}_s^r(V,W)$.

In particular,

$$\mathrm{Hom}_s^r(V,\mathbb{R}) = \mathcal{T}_s^r(V)$$

Proposition. $\mathrm{Hom}_s^r(V,V)$ is isomorphic to $\mathcal{T}_{s+1}^r(V)$.

Proof. It is simple to verify that

$$\Psi\colon \mathrm{Hom}_s^r(V,V) \to \mathcal{T}_s^{r+1}(V)$$
$$T \mapsto \Psi(T)\colon (V^*)^{r+1} \times V^s \to \mathbb{R}$$
$$(w^1,\dots,w^r,w^{r+1},v_1,\dots,v_s)$$
$$\downarrow$$
$$\underbrace{[\Psi(T)]}_{\cong T}(w^1,\dots,w^{r+1},v_1,\dots,v_s) = w^{r+1}(\overbrace{T(w^1,\dots,w^r,v_1,\dots,v_s)}^{\in V})$$

is an isomorphism. □

In general no distinction is made between real tensors and vector tensors.
The existence of an inner product $\langle\,,\rangle$ on V allows even more identifications.

Definition. We define the flat map $\flat\colon V \to V^*$ by $\flat(v) = \langle v,\cdot\rangle$. $\quad (\flat(v))(w) = \langle v,w\rangle$

Proposition. The flat map is an isomorphism.

Proof. Linearity is clear. If $v \in \operatorname{Ker}\flat$, then

$$\langle v,w\rangle = 0 \ \forall w \in V \implies v = 0$$

Hence $\flat$ is an isomorphism.

Definition. Given $f \in V^*$, there exists a unique $f^\# \in V$ such that

$$f(v) = \langle f^\#, v\rangle \ \forall v \in V$$

The sharp map is then well defined:

$$\#\colon V^* \to V \qquad v = v^i e_i \qquad v^i \xrightarrow{\flat} v_i \qquad \text{“lowers the pitch”}$$
$$f \mapsto f^\# \qquad f = f_i e^i \qquad f_i \xrightarrow{\#} f^i \qquad \text{“raises the pitch”}$$

which is the inverse of flat.

Let us see how these isomorphisms act in coordinates. Before that, we will introduce the following:

Einstein notation. To avoid writing things like

$$\sum_{1\leq i_1,\dots,i_r,s_1,\dots,s_s\leq m}$$

we work with the following convention:

- the final index of the summation and the summation symbol itself are dropped, with the understanding that they are obvious from the context (generally always the dimension of the manifold)
- we agree that whenever the same index appears in the same monomial expression up and down, we are in fact summing the expression over that index
- If an index appears in the denominator of an expression, it is always down.

Examples.

- $v = \sum_{i=1}^m v^i e_i$ or $f = \sum_{i=1}^m f_i e^i$ become $v = v^i e_i$, $f = f_i e^i$
- $\dfrac{\partial}{\partial x^i} = \sum_{s=1}^m \dfrac{\partial y^s}{\partial x^i}\dfrac{\partial}{\partial y^s}$ becomes $\dfrac{\partial}{\partial x^i} = \dfrac{\partial y^s}{\partial x^i}\dfrac{\partial}{\partial y^s}$

Remark. We must always be careful when detecting the "dummy" indices and the fixed ones. When making substitutions, for example, we have that if $p^i = a^i_s v^s$ and $v^s = b^s_i w^i$, it is not correct to write

$$p^i = a^i_s b^s_i w^i$$

What happens is that in fact, in $v^s = b^s_i w^i$ the index i is dummy, so we could write $v^s = b^s_k w^k$ and substitute:

$$p^i = a^i_s b^s_k w^k \qquad (g_{is} = \langle e_i, e_s\rangle)$$

Proposition. Let $\{e_i\}_{1\leq i\leq m}$ be a basis of V. If $v = v^i e_i \in V$ and $f = f_i e^i \in V^*$, then

$$\flat(v) = v_\flat = (v_\flat)_i e^i \quad \text{and} \quad \#(f) = f^\# = (f^\#)^i e_i$$

$$v_i = g_{is} v^s \quad \text{and} \quad f^i = g^{is} f_s \qquad (v_\flat)_i = v_i,\ (f^\#)^i = f^i$$

Proof. Writing $v_\flat = v_i e^i$, we have that

$$v_\flat(e_s) = v_i e^i(e_s) = v_i \delta^i_s = v_s \qquad \begin{array}{l}\Rightarrow v_\flat = v_\flat(e_s)e^s \\ \Rightarrow f = f(e_s)e^s\end{array}$$

But

$$v_\flat(e_s) = \langle v, e_s\rangle = \langle v^i e_i, e_s\rangle = v^i g_{is}$$

Hence

$$v_s = v^i g_{is} \iff v_i = v^s g_{si} = v^s g_{is} \qquad \begin{array}{c} v = e^s(v)e_s \\ \Rightarrow f^\# = e^s(f^\#)e_s\end{array}$$

Writing $f^\# = (f^\#)^i e_i$, we see that

$$e^s(f^\#) = (f^\#)^i e^s(e_i) = (f^\#)^i \delta^s_i = (f^\#)^s$$

But $e^s(f^\#) = \langle f^\#, e_s\rangle = f(e_s)$. And we have

$$f(e_s) = \langle f^\#, e_s\rangle = \langle (f^\#)^i e_i, e_s\rangle = f^i g_{is}$$

□

Corollary. $(e_i)_\flat = g_{is}e^s$, $(e^i)^\# = g^{is}e_s$

Proof. It follows from $e_i = \delta_i^s e_s$ and from the previous proposition that

$$(e_i)_\flat = g_{sk}\delta_i^k e^s = g_{si}e^s = g_{is}e^s \qquad e^1((x,y,z)) = x$$

Analogously, from $e^i = \delta_k^i e^k$, we have

$$(e^i)^\# = g^{ks}\delta_s^i e_k = g^{ik}e_k \qquad \begin{matrix}(e_1)_\flat = e^1 \quad (e^1)^\# = e_1 \\ e_1, e_2, e_3\end{matrix}$$

Proposition. The map

$$\begin{aligned}\#_1 \colon \mathcal{T}_0^2(V) &\to \mathcal{T}_1^1(V) \\ T &\mapsto T^{\#_1} \colon V^* \times V \to \mathbb{R} \\ &\qquad (f, v) \mapsto T(f^\#, v)\end{aligned}$$

is an isomorphism.

We will need the following lemma first:

Lemma. $\dim \mathcal{T}_1^1(V) = \dim \mathcal{T}_2^0(V) = m^2$

Proof. We will first define the following map:

$$\begin{aligned}\otimes \colon V^* \times V &\to \mathcal{T}_1^1(V) \\ (v, f) &\mapsto v \otimes f \colon V^* \times V \to \mathbb{R} \\ &\qquad (g, w) \mapsto g(v)f(w)\end{aligned}$$

which extends naturally to

$$\otimes \colon \mathcal{T}_s^r(V) \times \mathcal{T}_l^k(V) \to \mathcal{T}_{s+l}^{r+k}(V), \quad \text{by}$$

$$\begin{aligned}(T \otimes S)(w^1, \ldots, w^{r+k}, v_1, \ldots, v_{s+l}) = T(w^1, \ldots, w^r, v_1, \ldots, v_s) \\ \cdot S(w^{r+1}, \ldots, w^{r+k}, v_{s+1}, \ldots, v_{s+l})\end{aligned}$$

I claim that

$$\{e_i \otimes e^s \mid 1 \le i, s \le m\}$$

is a basis of $\mathcal{T}_1^1(V)$. Indeed, if

$$a_s^i\, e_i \otimes e^s = 0$$

then evaluating at (e^k, e_l), we have

$$\begin{aligned}(a_s^i\, e_i \otimes e^s)(e^k, e_l) = a_s^i (e_i \otimes e^s)(e^k, e_l) &= a_s^i\, e^k(e_i)e^s(e_l) \\ &= a_s^i \delta_i^k \delta_l^s = a_l^k = 0\end{aligned}$$

Hence it remains to show that this set spans $\mathcal{T}_1^1(V)$. Indeed, given $T \in \mathcal{T}_1^1(V)$, we have for any $f \in V^*$ and $v \in V$ that

$$\begin{aligned}T(f, v) = T(f(e_i)e^i, e^s(v)e_s) &= f(e_i)e^s(v)T_s^i \\ &= T_s^i(e_i \otimes e^s)(f, v) \\ &= (T_s^i\, e_i \otimes e^s)(f, v)\end{aligned}$$

$$\implies T = T_s^i\, e_i \otimes e^s$$

Finally, it is clear that

$$\{e^i \otimes e^s \mid 1 \le i, s \le m\}$$

is a basis of $\mathcal{T}_2^0(V)$. □

Proof of the proposition. By the lemma it suffices to show injectivity. Indeed, since

$$\begin{aligned}(T^{\#_1})^i_s = T^{\#_1}(e^i, e_s) = T((e^i)^\#, e_s) &= T(g^{ik}e_k, e_s)\\ &= g^{ik}T_{ks}\end{aligned}$$

Then $T^{\#_1} = 0 \implies g^{ik}T_{ks} = 0 \implies g^{ik}g_{il}T_{ks} = \delta^k_l T_{ks} = T_{ls}$

$$\implies T = 0.$$

□

Remark. Generally the isomorphisms are not made explicit. That is, instead of $(T^{\#_1})^i_s = g^{ik}T_{ks}$ we simply write

$$T^i_s = g^{ik}T_{ks}$$

Even though on the left-hand side we are not working with T itself, but with $T^{\#_1}$.

Remark. Analogously,

$$\begin{aligned}\flat_1 \colon \mathcal{T}^2_0(V) &\to \mathcal{T}^1_1(V)\\ T &\mapsto T_{\flat_1} \colon V^* \times V \to \mathbb{R}\\ &\qquad (f, v) \mapsto T(f, v_\flat)\end{aligned}$$

is an isomorphism.

Proposition (General case)**.** $\mathcal{T}^a_b(V) \cong \mathcal{T}^r_s(V)$ provided that $r + s = a + b$.

Proof. There are several possible isomorphisms. One of them is

$$\mathcal{T}^r_s(V) \xrightarrow{\Psi_1} \mathcal{T}^0_{r+s}(V) \xrightarrow{\mathrm{Id}} \mathcal{T}^0_{a+b}(V) \xrightarrow{\Psi_3} \mathcal{T}^a_b(V)$$

where $(\Psi_1(T))(v_1, \ldots, v_{r+s}) = T((v_1)_\flat, (v_2)_\flat, \ldots, (v_r)_\flat, v_{r+1}, \ldots, v_{r+s})$ and

$$(\Psi_3(T))(w^1, \ldots, w^a, v_1, \ldots, v_B) = T((w^1)^\#, (w^2)^\#, \ldots, (w^a)^\#, v_1, \ldots, v_B).$$

Concrete examples

1: Every field $X \in \mathfrak{X}(M)$ can be seen as a 1-form, given by $X_\flat = \langle X, \cdot \rangle$.
2: The Ricci tensor can be seen as an endomorphism, given by

$$\begin{aligned}\widetilde{Ric} \colon \Gamma(TM) &\to \Gamma(TM) \qquad\qquad (Ric(X, \cdot))(Y) = Ric(X, Y)\\ X &\mapsto \widetilde{Ric}(X) = \#(Ric(X, \cdot))\end{aligned}$$

which is determined by $\langle \widetilde{Ric}(X), Y \rangle = Ric(X, Y)$.
3: The curvature tensor can be seen in different ways:

$$\begin{aligned}R^{1,3} \colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM)\\ (X, Y, Z) &\mapsto R(X, Y)Z\end{aligned}$$

$$\begin{aligned}R^{0,4} \colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) &\to C^\infty(M)\\ (X, Y, Z, W) &\mapsto W_\flat(R^{1,3}(X, Y, Z))\end{aligned}$$

$$\begin{aligned}R^{0,2} \colon \Gamma(TM) \times \Gamma(TM) &\to \mathrm{End}(\Gamma(TM))\\ (X, Y) &\mapsto R^{0,2}(X, Y) \colon \Gamma(TM) \to \Gamma(TM)\\ &\qquad Z \mapsto R(X, Y)Z\end{aligned}$$

Lemma. There exists a unique linear map $tr_1^1 \colon \mathcal{T}_1^1(V) \to \mathbb{R}$ such that

$$tr_1^1(v \otimes f) = f(v) \ \forall f \in V^*, \ \forall v \in V$$

Proof. In terms of a basis $\{e_i\}_{1 \le i \le m}$, we must have for any $T \in \mathcal{T}_1^1(V)$ that

$$\begin{aligned} tr_1^1(T) = tr_1^1(T_s^i \, e_i \otimes e^s) = T_s^i \, e^s(e_i) &= T_s^i \delta_i^s \\ &= T_i^i \end{aligned}$$

Defining then $tr(T) = T_i^i$, it remains to show that this definition does not depend on the chosen basis. Indeed, if $\{\tilde{e}_i\}_{1 \le i \le m}$ is another basis, then recalling that all $v \in V$ and $f \in V^*$ are written as

$$v = e^i(v) e_i, \quad f = f(e_i) e^i \qquad \tilde{T}_i^i = \sum_i T(\tilde{e}^i, \tilde{e}_i)$$

We have that

$$\begin{aligned} T_i^i = T(e^i, e_i) = T(e^i(\tilde{e}_s)\tilde{e}^s, \tilde{e}^k(e_i)\tilde{e}_k) &= e^i(\tilde{e}_s)\tilde{e}^k(e_i) T_{\tilde{k}}^{\tilde{s}} \\ &= T_{\tilde{k}}^{\tilde{s}} \, \tilde{e}^k(e^i(\tilde{e}_s) e_i) = T_{\tilde{k}}^{\tilde{s}} \, \tilde{e}^k(\tilde{e}_s) \\ &= T_{\tilde{k}}^{\tilde{s}} \, \delta_{\tilde{s}}^{\tilde{k}} = T_{\tilde{s}}^{\tilde{s}} = \sum_{s=1}^{m} T(\tilde{e}^s, \tilde{e}_s) \end{aligned}$$

as desired.

Definition. The contraction in the a-th contravariant entry and b-th covariant entry is the map $tr_b^a \colon \mathcal{T}_s^r(V) \to \mathcal{T}_{s-1}^{r-1}(V)$ given by

$$\begin{aligned} [tr_b^a(T)](f^1, \dots, f^{r-1}, v_1, \dots, v_{s-1}) := tr_1^1\big(T(f^1, \dots, f^{a-1}, \bullet, f^a, \dots, f^{r-1}, \\ v_1, \dots, v_{b-1}, \bullet, v_b, \dots, v_s)\big) \end{aligned}$$

where

$$\begin{aligned} &T(f^1, \dots, f^{a-1}, \bullet, f^a, \dots, f^{r-1}, v_1, \dots, v_{b-1}, \bullet, v_b, \dots, v_s) \colon V^* \times V \to \mathbb{R} \\ &(g, w) \mapsto T(f^1, \dots, \underset{-}{^{a-1}}, g, f^a, v_1, \dots, v_{b-1}, w, v_b, \dots, v_s) \end{aligned}$$

We also define

$$\begin{aligned} tr_{1,2} \colon \mathcal{T}_2^0(V) &\to \mathbb{R}, & tr^{1,2} \colon \mathcal{T}_0^2(V) &\to \mathbb{R} \\ T &\mapsto tr_1^1(T^{\#_1}) & T &\mapsto tr_1^1(T_{\flat_1}) \end{aligned}$$

In these cases, we have

$$\begin{aligned} T \in \mathcal{T}_2^0(V) \implies tr_{1,2}(T) &= (T^{\#_1})_i^i \\ &= T((e^i)^\#, e_i) = T(g^{si} e_s, e_i) \\ &= g^{si} T_{si} = g^{is} T_{is} \\ T \in \mathcal{T}_0^2(V) \implies tr_{1,2}(T) &= (T_{\flat_1})_i^i = T(e^i, (e_i)_\flat) \\ &= T(e^i, g_{is} e^s) = g_{is} T^{is} \end{aligned}$$

We also define

$$\begin{aligned} tr_{a,b} \colon \mathcal{T}_s^r(V) \to \mathcal{T}_{s-2}^r(V) \qquad & 1 \le a, b \le s-2 \\ T \mapsto tr_{a,b}(T) \colon (V^*)^r \times V^{s-2} \to \mathbb{R} & \end{aligned}$$

$$[tr_{a,b}(T)](f^1,\dots,f^r,v_1,\dots,v_{s-2}) = tr_{1,2}\big(T(f^1,\dots,f^r,v_1,\dots,v_{a-1},\bullet,v_a,\dots,v_{b-1},\bullet,\dots,v_{s-2})\big)$$

$$tr^{a,b}\colon \mathcal{T}^r_s(V) \to \mathcal{T}^{r-2}_s(V)$$
$$[tr^{a,b}(T)](f^1,\dots,f^{r-2},v_1,\dots,v_s) = tr^{1,2}(T(f^1,\dots,\bullet,\dots,\bullet,f^{r-2},v_1,\dots,v_s))$$

In coordinates, we have

$$[tr_{a,b}(T)]^{i_1\dots i_r}{}_{s_1\dots s_{s-2}} = g^{kl}\,T^{i_1\dots i_r}{}_{s_1\dots k\dots l\dots s_{s-2}}$$
$$[tr_{a,b}(T)]^{i_1\dots i_{r-2}}{}_{s_1\dots s_s} = g_{kl}\,T^{i_1\dots k\dots l\dots i_r}{}_{s_1\dots s_s}$$

Example. The Ricci tensor can be defined by

$$Ric(x,y) = tr(R(\cdot,x)y) = tr^1_1(R^1_1(\cdot,x)y)$$

where $[R^1_1(\cdot,x)y](f,v) = f(R(v,x)y)$. Thus

$$\begin{aligned} Ric_{kl} = [R^1_1(\cdot,e_k)e_l]^i_i = e^i(R_{ikl}) &= e^i(e^s(R_{ikl})e_s) \\ &= e^s(R_{ikl})\delta^i_s = R^s_{ikl}\delta^i_s = R^s_{skl} \end{aligned}$$

Example. We can define

$$scal = tr_{1,2}(Ric) = g^{is}(Ric)_{is} = g^{is}R_{is}$$

Lemma. Let (M,g) be a Riemannian manifold and ∇ its Levi-Civita connection determined by g. Then there exists a unique connection (which by abuse of notation will also be denoted by ∇)

$$\nabla\colon \Gamma(TM)\times \underbrace{\mathcal{T}^r_s(M)}_{:=\bigcup_{p\in M}\mathcal{T}^r_s(T_pM)} \to \mathcal{T}^r_s(M)$$

such that:

i) On $\mathfrak{X}(M)$, ∇ coincides with the original connection

ii) On $C^\infty(M)$, $\nabla_X f = X(f)$

iii) ∇ satisfies the following product rule:

$$\nabla_X(F\otimes G) = (\nabla_X F)\otimes G + F\otimes(\nabla_X G)$$

iv) ∇ commutes with all traces: if tr denotes the trace with respect to any pair of indices, then

$$\nabla_X(tr\,F) = tr(\nabla_X F)$$

These four properties completely determine ∇, so that for any $T\in\mathcal{T}^r_s(M)$, $X_i\in\mathfrak{X}(M)$ and $w^s\in\mathcal{T}^1_0(M)$, we have

$$\begin{aligned} (\nabla_X T)(w^1,\dots,w^r,X_1,\dots,X_s) = X(T(w^1,\dots,w^r,X_1,\dots,X_s)) \\ - \sum_{i=1}^r T(w^1,\dots,\nabla_X w^i,\dots,w^r,X_1,\dots,X_s) \\ - \sum_{i=1}^s T(w^1,\dots,w^r,X_1,\dots,\nabla_X X_i,\dots,X_s) \end{aligned}$$

where for every $w\in\mathcal{T}^1_0(M)=\Lambda^1(M)$, we have

$$(\nabla_X w)(Y) + w(\nabla_X Y) = \nabla_X[w(Y)]$$

Corollary. Given $T \in \mathcal{T}^l_k(M)$, we have

$$\begin{aligned}(\nabla_m T)^{s_1\dots s_l}{}_{i_1\dots i_k} &:= \nabla_m T^{s_1\dots s_l}{}_{i_1\dots i_k} \\ &= \partial_m(T^{s_1\dots s_l}{}_{i_2\dots i_k}) + \sum_{s=1}^{l} T^{s_1\dots s_{s-1} q s_{s+1}\dots s_l}{}_{i_1\dots i_k}\Gamma^{s_s}_{mq} \\ &\quad - \sum_{t=1}^{k} T^{s_1\dots s_l}{}_{i_1\dots i_{r-1} q i_{r+1}\dots i_k}\Gamma^{q}_{mi_r}\end{aligned}$$

In particular, in normal coordinates centered at p, we have

$$(\nabla_m T)^{s_1\dots s_l}{}_{i_1\dots i_k} = [\partial_m(T^{s_1\dots s_l}{}_{i_1\dots i_k})](p)$$

Proof. By definition,

$$\begin{aligned}(\nabla_m T)^{s_1\dots s_l}{}_{i_1\dots i_k} &= (\nabla_{\partial_m} T)(dx^{s_1},\dots,dx^{s_l},\partial_{i_1},\dots,\partial_{i_k}) \\ &= \partial_m(T^{s_1\dots s_l}{}_{i_1\dots i_k}) - \sum_{t=1}^{k} T(dx^{s_1},\dots,dx^{s_l},\partial_{i_1},\dots,\nabla_{\partial_m}\partial_{i_t},\dots,\partial_{i_k}) \\ &\quad - \sum_{s=1}^{l} T(dx^{s_1},\dots,\nabla_{\partial_m} dx^{s_s},\dots,dx^{s_l},\partial_{i_1},\dots,\partial_{i_k})\end{aligned}$$

Noting that

$$\begin{aligned}(\nabla_{\partial_m} dx^{s_s})_a + dx^{s_s}(\nabla_{\partial_m}\partial_a) &= \nabla_{\partial_m}[dx^{s_s}(e_a)] = \nabla_{\partial_m}(\delta^{s_s}_a) = 0 \\ \implies (\nabla_{\partial_m} dx^{s_s})_a &= -dx^{s_s}(\nabla_{\partial_m}\partial_a) = -dx^{s_s}(\Gamma^{\beta}_{ma}\partial_\beta) \\ &= -\Gamma^{s_s}_{ma}\end{aligned}$$

We then have that

$$\nabla_{\partial_m} dx^{s_s} = (\nabla_{\partial_m} dx^{s_s})_a dx^a = -\Gamma^{s_s}_{ma} dx^a$$

And therefore

$$\begin{aligned}(\nabla_m T)^{s_1\dots s_l}{}_{i_1\dots i_k} &= \partial_m(T^{s_1\dots s_l}{}_{i_1\dots i_k}) - \sum_{t=1}^{k} T(dx^{s_1},\dots,dx^{s_l},\partial_{i_1},\dots,\Gamma^{p}_{mi_t}\partial_p,\dots,\partial_{i_k}) \\ &\quad - \sum_{s=1}^{l} T(dx^{s_1},\dots,-\Gamma^{s_s}_{ma}dx^a,\dots,dx^{s_l},\partial_{i_1},\dots,\partial_{i_k}) \\ &= \partial_m(T^{s_1\dots s_l}{}_{i_1\dots i_k}) - \sum_{t=1}^{k} T^{s_1\dots s_l}{}_{i_1\dots p\dots i_k}\Gamma^{p}_{mi_t} + \sum_{s=1}^{l} T^{s_1\dots a\dots s_s}{}_{i_1\dots i_k}\Gamma^{s_s}_{ma}\end{aligned}$$

as desired.

Definition. We define for $T \in \mathcal{T}^r_s(M)$ its total covariant derivative

$$\begin{aligned}\nabla T\colon (\Gamma(T^*M))^r \times (\Gamma(TM))^{s+1} &\to C^\infty(M) \\ (w^1,\dots,w^r,Y_1,\dots,Y_s,X) &\mapsto (\nabla_X T)(w^1,\dots,w^r,Y_1,\dots,Y_s)\end{aligned}$$

Example. Given $f \in C^\infty(M)$, we define $\operatorname{Hess} f = \nabla^2 f$, which is a $(0,2)$ tensor and acts by

$$\begin{aligned}\nabla^2 f\colon \Gamma(TM)\times\Gamma(TM) &\to C^\infty(M) \\ (X,Y) &\mapsto (\nabla^2 f)(X,Y) = [\nabla(\nabla f)](X,Y)\end{aligned}$$

We have

$$\begin{aligned}[\nabla(\nabla f)](X,Y) &= \nabla_Y[(\nabla f)](X)\\ &= \nabla_Y[\nabla_X f] - (\nabla f)(\nabla_Y X)\\ &= \nabla_Y[\nabla_X f] - \nabla_{\nabla_Y X} f = Y(X(f)) - \nabla_{\nabla_Y X} f\end{aligned}$$

Note that

$$(\nabla^2 f)(X,Y) - (\nabla^2 f)(Y,X) = -[X,Y](f) - \nabla_{\underbrace{\nabla_Y X - \nabla_X Y}_{\to [X,Y]}} f$$

$$\begin{aligned}= -[X,Y](f) - \nabla_{-[X,Y]} f &= -([X,Y](f) - \nabla_{[X,Y]} f)\\ &= -([X,Y](f) - [X,Y](f)) = 0\end{aligned}$$

And therefore Hess f is symmetric.

Example. We define $\Delta f = tr_{1,2}(\nabla^2 f)$. In local coordinates, we then have

$$\Delta f = g^{is}(\nabla^2 f)_{is} = g^{is}\nabla_i \nabla_s f = \nabla^{(e_s)^\#}(\nabla_{e_s} f) = \nabla^s \nabla_s f$$

In normal coordinates, we have

$$\Delta f = g^{is}(\partial_i \partial_s)(f) \qquad g^{is}\nabla_{e_i} = \nabla_{\underbrace{g^{is} e_i}_{=(e_s)^\#}}$$

Changes of coordinates:

Let $T \in \mathcal{T}^r_s(V)$ and consider bases $\{e_i\}_{1\le i\le m}$, $\{\tilde{e}_i\}_{1\le i\le m}$ of V related by $\tilde{e}_s = a^i_s e_i$, $e_s = b^i_s \tilde{e}_i$. Then

$$\tilde{e}^s = \tilde{e}^s(e_i)e^i = \tilde{e}^s(b^k_i \tilde{e}_k)e^i = b^i_s \delta^s_k e^i = b^i_k e^k$$

And therefore

$$\begin{aligned}\tilde{T}^{i_1 \dots i_r}{}_{s_1 \dots s_s} &= T(\tilde{e}^{i_1}, \dots, \tilde{e}^{i_r}, \tilde{e}_{s_1}, \dots, \tilde{e}_{s_s})\\ &= T(b^{i_1}_{k_1} e^{k_1}, \dots, b^{i_r}_{k_r} e^r, a^{l_1}_{s_1} e_{l_1}, \dots, a^{l_s}_{s_s} e_{l_s})\\ &= b^{i_1}_{k_1} \cdots b^{i_r}_{k_r} a^{l_1}_{s_1} \cdots a^{l_s}_{s_s} T^{k_1 \dots k_r}{}_{l_1 \dots l_s}\end{aligned}$$

Note also that (a^i_s) and (b^i_s) are inverses. Indeed,

$$(AB)_{sk} = a^i_s b^k_i \qquad (AB)_{sk} = a^i_s b^k_i$$

And we have

$$\begin{aligned}e_s = b^i_s \tilde{e}_i = b^i_s a^k_i e_k &\implies (BA)_{sk} = \delta_{sk} \ (*)\\ \tilde{e}_s = a^i_s e_i = a^i_s b^k_i \tilde{e}_k &\implies (AB)_{sk} = \delta_{sk}\end{aligned}$$

Thus, one way of thinking about this transformation law is that for each covariant index, a term of A contributes ("co" corresponds to A itself), and for each contravariant index, a term of A^{-1} contributes ("contra"-variant).

Example. Let (x,U), (y,V) be charts around a $p \in M$. Then every $v \in T_pM$ is written as $x^i = r^i \circ x$, $y^i = r^i \circ y$

$$v = v(x^i)\frac{\partial}{\partial x^i} \qquad \left(\text{since } \frac{\partial x^s}{\partial x^i} = \delta_{is}\right)$$

Hence

$$\left.\frac{\partial}{\partial y^s}\right|_p = \left.\frac{\partial}{\partial y^s}\right|_p (x^i) \left.\frac{\partial}{\partial x^i}\right|_p \implies \frac{\partial}{\partial y^s} = \frac{\partial x^i}{\partial y^s}\frac{\partial}{\partial x^i}$$

And we have for $T \in \mathcal{T}^r_s(M)$ that $\left(A = \left(\frac{\partial x^i}{\partial y^s}\right),\ B = \left(\frac{\partial y^i}{\partial x^s}\right)\right)$

$$\tilde{T}^{i_1\dots i_r}{}_{s_1\dots s_s} = \frac{\partial y^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial y^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial y^{s_1}}\cdots\frac{\partial x^{l_s}}{\partial y^{s_s}} T^{k_1\dots k_r}{}_{s_1\dots s_s}$$

Hence, $\tilde{T}^{i_1\dots i_r}{}_{s_1\dots s_s} \in C^\infty(M) \iff T^{k_1\dots k_r}{}_{s_1\dots s_s} \in C^\infty(M)$.

Extending the metric to tensors: Let $F, G \in \mathcal{T}^k_l(M)$. We define

$$\langle F, G\rangle = g_{i_1r_1}\cdots g_{i_kr_k} g^{s_1\mathbf{s}_1}\cdots g^{s_l\mathbf{s}_l} F^{i_1\dots i_k}{}_{s_1\dots s_l} G^{r_1\dots r_k}{}_{\mathbf{s}_1\dots\mathbf{s}_l}$$

In particular, if $k = 0$, we have $(Ric)^i_s = R^i_s = Ric((e^i)^\#, e_s)$

$$\langle F, G\rangle = F_{s_1\dots s_l} \underbrace{G^{s_1\dots s_l}}_{=G((e^{s_1})^\#,\dots,(e^{s_l})^\#)}$$

Definition. For a $T \in \mathcal{T}^1_k(M)$, we define

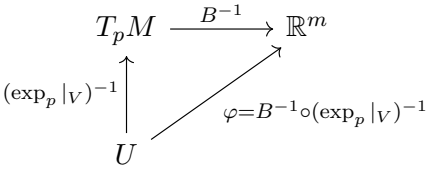
$$\begin{aligned}(\operatorname{div} T) \in \mathcal{T}^0_k(M),\ (\operatorname{div} T)(X_1,\dots,X_k) &= tr(X \mapsto (\nabla_X T)(X_1,\dots,X_k)) \\ &= \sum \langle (\nabla_{\partial_i} T)(X_1,\dots,X_k), \partial_i\rangle\end{aligned}$$

$tr(T) = \sum_i \langle T(e_i), e_i\rangle$

Second (contracted) Bianchi identity: $\operatorname{div} Ric = \frac{1}{2}\nabla scal$
Proof: At the end.

Normal coordinates:

$V = \exp_p(U)$



$$g_{is}(p) = \delta_{is}, \quad \Gamma^k_{is}(p) = 0 \quad \forall\, 1 \le i, s, k \le m$$
$$g_{is}(t) = \delta_{is} + \frac{1}{3}R_{ikls}(p)x^k x^l + O(|t^3|)$$

$(0,4)$ antisymmetric tensors $\iff$ Operators on $\Lambda^2(M)$

Let w and $\tilde{w}$ be 2-forms on a Riemannian manifold M and $P \in \mathcal{T}^0_4(M)$ a $(0,4)$ tensor such that

$$P_{iskl} = -P_{sikl} = -P_{islk}$$

We can write $e^i = dx^i$ and determine the local expressions of w and $\tilde{w}$. We have

$$\begin{aligned} w = \sum_{i<s} w_{is}\, dx^i \wedge dx^s &= \sum_{i<s} \frac{(w_{is} - w_{si})}{2}\, dx^i \wedge dx^s \\ &= \frac{1}{2}\sum_{i<s} w_{is}\, dx^i \wedge dx^s - \frac{1}{2}\sum_{s>i} w_{is}\, dx^s \wedge dx^i \\ &= \frac{1}{2}\sum_{i<s} w_{is}\, dx^i \wedge dx^s + \frac{1}{2}\sum_{s>i} w_{is}\, dx^s \wedge dx^i \\ &= \frac{1}{2}\sum_{1\le i,s\le m} w_{is}\, dx^i \wedge dx^s = \frac{1}{2} w_{is}\, dx^i \wedge dx^s \end{aligned}$$

We can then define an operator $\tilde{P}\colon \Lambda^2(M)\times\Lambda^2(M) \to C^\infty(M)$ by setting $\tilde{P}(dx^i\wedge dx^s, dx^k\wedge dx^l) = P_{islk}$ and extending by linearity. The relation

$$\langle \tilde{\tilde{P}}(X\wedge Y), V\wedge W\rangle = \tilde{P}(X\wedge Y, V\wedge W)$$

then determines an operator $\tilde{\tilde{P}}\colon \Lambda^2(M) \to \Lambda^2(M)$. Indeed $\tilde{\tilde{P}}$ is well defined, since

$$\begin{aligned} \langle \tilde{\tilde{P}}(Y\wedge X), V\wedge W\rangle = \tilde{P}(Y\wedge X, V\wedge W) &= -\tilde{P}(X\wedge Y, V\wedge W) \\ &= -\langle \tilde{\tilde{P}}(X\wedge Y), V\wedge W\rangle \end{aligned}$$

Note that

$$e^p \wedge e^q = \frac{1}{2}(e^p\wedge e^q)_{is}\, e^i \wedge e^s = \frac{1}{2} \underbrace{\delta^{pq}_{is}}\, e^i \wedge e^s$$

$$= \begin{cases} 1 & \text{if } (p,q) = (i,s) \\ -1, & \text{if } (p,q) = (s,i) \\ 0, & \text{otherwise} \end{cases}$$

By definition, we also have

$$\begin{aligned} (\tilde{\tilde{P}}w) &= (\tilde{\tilde{P}}w)_{is}\, e^i \wedge e^s \cdot \tfrac{1}{2} \\ \implies \langle \tilde{\tilde{P}}w, e^p\wedge e^q\rangle &= (\tilde{\tilde{P}}w)_{is}\, \delta^{is}_{pq} \\ \| & \\ \langle P(\tfrac{1}{2}w_{kl}\, e^k\wedge e^l), e^p\wedge e^q\rangle &= \sum_{k,l} \tfrac{1}{2} w_{kl} P_{klqp} \end{aligned}$$

Setting $(p,q) = (i,s)$, we see that

$$\sum_{k,l} \tfrac{1}{2} w_{kl} P_{klsi} = \sum_{k,l} \tfrac{1}{2} w_{kl} P_{islk} = (Pw)_{is}$$

And therefore

$$\boxed{Pw = \frac{1}{2}\sum_{i,s}(Pw)_{is}\, e^i\wedge e^s = \frac{1}{4}\sum_{i,s,k,l} P_{islk} w_{kl}\, e^i \wedge e^s} \qquad (*)$$

$(*)$ completely determines $P\colon \Lambda^2(M) \to \Lambda^2(M)$. We start with a $(0,4)$ tensor and obtain an operator on $\Lambda^2(M)$. Conversely, we could have started with an operator $P\colon \Lambda^2(M) \to \Lambda^2(M)$ and obtained a $(0,4)$-tensor by the relation

$$P_{pqrs} = \langle P(e^p\wedge e^q), e^s\wedge e^r\rangle$$

$$\nabla X = 0$$

Proof. Fix $v \in T_pM$, and let γ be the unique geodesic such that $\gamma(0) = p$, $\gamma'(0) = v$. $\exists! V \in \mathfrak{X}^\gamma(M)$ such that V is parallel along γ, i.e., $(\nabla_X V) \circ \gamma \equiv 0 \ \forall X \in \mathfrak{X}(M)$. In particular, we have that $\nabla_v X = 0 \ \forall X \in \mathfrak{X}(M)$. Take also an orthonormal frame $\{E_i\}_{1 \le i \le m}$ at $p \in M$ such that $(\nabla_{E_i} X)(p) = 0 \ \forall X \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
(dScal(V))(p) &= (V(S))(p) = V\Big(\sum_i \langle Ric(E_i), E_i\rangle\Big) \qquad S = Scal \\
&= V\Big(\sum_{i,s} \langle R(E_i, E_s)E_s, E_i\rangle\Big) \overset{\nabla V = 0 \text{ along } \gamma}{=} \sum_{i,s} \langle \nabla_V [R(E_i, E_s)E_s], E_i\rangle \\
&= \sum_{i,s} \langle (\nabla_V R)(E_i, E_s)E_s, E_i\rangle \overset{\text{Bianchi}}{=} -\sum \langle (\nabla_{E_s} R)(V, E_i)E_s, E_i\rangle \\
&\qquad\qquad - \sum \langle (\nabla_{E_i} R)(E_s, V)E_s, E_i\rangle \\
&= -\sum (\nabla_{E_s} R)(V, E_i, E_s, E_i) - \sum (\nabla_{E_i} R)(E_s, V, E_s, E_i) \\
&= \sum (\nabla_{E_s} R)(E_s, E_i, E_i, V) + \underbrace{\sum (\nabla_{E_i} R)(E_i, E_s, E_s, V)}_{= \sum (\nabla_{E_s} R)(E_s, E_i, E_i, V)} \\
&= 2\sum (\nabla_{E_s} R)(E_s, E_i, E_i, V) \\
&= 2\sum \nabla_{E_s}(R(E_s, E_i, E_i, V)) = 2\sum \nabla_{E_s}(\quad \underbrace{\langle Ric(E_s), V\rangle}_{= Ric(E_s, V) = Ric(V, E_s) \\ = \langle Ric(V), E_s\rangle} \quad) \\
&\qquad\qquad \textstyle\sum R(E_s, E_i)E_i = Ric(E_s) \\
&= 2\sum \nabla_{E_s}(\langle Ric(V), E_s\rangle) = 2\sum \langle (\nabla_{E_s} Ric)(V), E_s\rangle \\
&= 2(\operatorname{div} Ric)(V)(p)
\end{aligned}$$

$$\begin{gathered}
dScal = \tfrac{1}{2} \operatorname{div} Ric \\
(dScal)(v) = \tfrac{1}{2} (\operatorname{div} Ric)(v_p)
\end{gathered}$$