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An introduction to the Ricci flow and Einstein manifolds

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Abstract

We show that Einstein manifolds are self-similar solutions of the Ricci flow and present a natural generalization: Ricci solitons. Moreover, we prove a 1988 result due to W. Kuhnel on Einstein manifolds that are locally conformal to a manifold of constant curvature.

Introduction

Ever since, one of the quite active research areas in geometry has been understanding the relation between topology and curvature[1]. One of the most important problems of the last century has been precisely the topological classification of 3-manifolds. More specifically in this direction, Poincaré asked (see [3]) at the end of a 1904 paper (reformulated in current language): *if $\mathcal{M}$ is a closed 3-manifold with trivial fundamental group, is $\mathcal{M}$ diffeomorphic[2] to $\mathbb{S}^3$?*

The *Poincaré Conjecture* states that it is, and its proof involves precisely the study of curvature. In the 1980s, Thurston developed an approach (see [5] and the references therein) by considering 3-manifolds that admit Riemannian metrics of constant negative curvature -1. Although there are natural obstructions to the existence of such metrics on arbitrary 3-manifolds, Thurston formulated a more general conjecture which, roughly speaking, states that every closed 3-manifold can be decomposed in a canonical way into pieces such that each one has one of eight types of geometric structures. In dimension ≤ 3, the Ricci tensor determines (and is determined by) the curvature tensor, so that any Riemannian metric of constant Ricci curvature also has constant sectional curvature, and one can show that the universal cover of a closed manifold of constant positive curvature is diffeomorphic to the sphere (therefore every simply connected manifold of constant positive curvature is isometric to the sphere).

Thus, a good guess for a possible approach to the Poincaré conjecture and to the more general problem of Thurston's geometrization is to answer the following question: *under appropriate hypotheses on the fundamental group of a 3-manifold $\mathcal{M}$, how can one guarantee the existence of a metric of constant Ricci curvature on $\mathcal{M}$?* The key piece for producing such a metric is the

[1] This may be the sectional, Ricci or scalar curvature. Classical examples of what is meant here include the theorems of Killing-Hopf and Bonnet-Myers and the prescribed scalar curvature problem.

[2] One can show (see [5]) that, up to diffeomorphisms, every 3-manifold has a unique differentiable structure. Thus, if a 3-manifold is homotopy equivalent to $\mathbb{S}^3$, it is also diffeomorphic to $\mathbb{S}^3$.

Ricci flow equation, introduced by Richard Hamilton, which evolves an arbitrary initial metric g_0 on $\mathcal{M}$ as follows:

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}$$
$$g(0) = g_0,$$

where $\mathrm{Ric}_{g(t)}$ denotes the Ricci curvature of the metric $g(t)$. The idea here is to try to evolve the metric so that it evolves into one of Thurston's eight fundamental geometric structures. Hamilton's program, completed by Grigori Perelman, consists of using the Ricci flow to attack this conjecture. Through their works, it was expected that the Ricci flow could be used to *infer* the existence of a geometric decomposition by taking any initial Riemannian metric on a closed 3-manifold and proving sufficiently many analytic, geometric and topological results about the corresponding solutions of the Ricci flow with surgery. Indeed this goal was achieved, and understanding how requires precisely a very good understanding of, among several other subjects, the Ricci flow.

Einstein manifolds

Definition 1. *A Riemannian manifold $(\mathcal{M}^n, g)$ is said to be an Einstein manifold if, for any pair of vector fields X, Y on $\mathcal{M}$, we have $\mathrm{Ric}(X,Y) = \lambda g(X,Y)$, where $\lambda : \mathcal{M} \to \mathbb{R}$ is a real function.*

Remark. If $\mathcal{M}^n$ is connected and Einstein with $n \geq 3$, then λ is constant on $\mathcal{M}$, and if $\lambda = 3$ then $\mathcal{M}^3$ has constant sectional curvature. Indeed, the second Bianchi identity is written as $\mathrm{d}\,\mathrm{Sc} = 2\,\mathrm{div}(\mathrm{Ric})$ (where Sc is the scalar curvature of $\mathcal{M}$); in our case $\mathrm{d}(\lambda \mathrm{tr}(g)) = n\mathrm{d}\lambda = 2\,\mathrm{div}(\lambda g)$, which becomes:

$$2g^{ij}\nabla_i(\lambda g_{jk})\,\mathrm{d}x^k = 2g^{ij}g_{jk}\nabla_i\lambda\,\mathrm{d}x^k = 2\,\mathrm{d}\lambda,$$

where we used Einstein notation. Therefore $n \geq 3 \implies \mathrm{d}\lambda \equiv 0$, and by the connectedness of $\mathcal{M}$, λ is then constant. For the second part, it suffices to take an arbitrary $p \in \mathcal{M}$, a geodesic frame $\{e_1, e_2, e_3\}$ at p, and then $\sum_{i=1}^{3} R(X, e_i, Y, e_i) = 2\lambda g(X,Y)$. Setting $X = Y = e_j$ with $j \in \{1,2,3\}$ and evaluating at p we obtain a system for λ that gives us $\lambda = R_{1212}(p) = R_{1313}(p) = R_{2323}(p)$, and the result then follows from Schur's theorem. Taking the trace of $\mathrm{Ric} = \lambda g$ and using the Bianchi identity it is also possible to show that Sc is constant.

Example. It is simple to show that any Riemannian manifold of constant curvature is Einstein.

We will now show that every Einstein manifold is a solution of the Ricci flow. Indeed, if $(\mathcal{M}, g)$ is Einstein, then $\mathrm{Ric}_{ij}(p,0) = \lambda g_{ij}(p,0)$ for all $p \in \mathcal{M}$ with λ constant. Setting $g_{ij}(x,t) = \rho^2(t) g_{ij}(x,0)$ we see that $\mathrm{Ric}_{ij}(p,t) = \mathrm{Ric}_{ij}(p,0) = \lambda g_{ij}(p,0)$ (it is simple to show that the Ricci tensor is preserved by changes of scale that do not depend on the point $p \in \mathcal{M}$). The Ricci flow equation is then written as:

$$\frac{\partial}{\partial t}\left(\rho^2(t) g_{ij}(p,0)\right) = -2\lambda g_{ij}(x,0),$$

which gives us the ODE

$$\frac{d\rho}{dt} = -\frac{\lambda}{\rho},$$

whose solution is given by

$$\rho^2(t) = 1 - 2\lambda t,$$

Thus, if $\lambda > 0$, the metric $g_{ij}(x,t)$ evolves homothetically until it shrinks to a point as $t \to T = \dfrac{1}{2\lambda}$, and if $\lambda < 0$ the metric expands homothetically at every instant. These two cases correspond, respectively, to scalar curvature that "blows up" (goes to ∞) and that goes to 0.

Ricci solitons

As said before, Hamilton's goal was to evolve a metric of a given arbitrary Riemannian manifold into a metric of "uniform" curvature, which is more tractable. There is a class of solutions that constitute an obstruction to this: the *solitons*. Geometrically (that is, up to isometries, which are distance-preserving diffeomorphisms), they evolve under the Ricci flow only by change of scale. Taking the quotient of the set of Riemannian metrics on a given differentiable manifold by the equivalence relation $g_1 \sim g_2 \iff g_1$ and g_2 differ only by homotheties and diffeomorphisms, we then see that solitons are fixed points of the Ricci flow equation.

Lemma 1. ([4]) On a Riemannian manifold $(\mathcal{M}^n, g)$, for an orthonormal frame $\{X_1, \cdots, X_n\}$, we have:

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i,$$

where ∇ denotes the Levi-Civita connection of g, $\mathcal{L}$ denotes the Lie derivative, and $\nabla_i X_j$ denotes the j-th component of $\nabla_{X_i} X$ in the basis $\{X_1, \cdots, X_n\}$.

Proof: Let ω be the 1-form dual to the field X, i.e, $\omega(Y) = \langle X, Y\rangle$. Using the product rule, the compatibility of the metric and the vanishing torsion of the Levi-Civita connection, we have:

$$\begin{aligned}
(\mathcal{L}_X g)(Y,Z) &= X(g(Y,Z)) - g\left(\mathcal{L}_X Y, Z\right) - g\left(Y, \mathcal{L}_X Z\right) \\
&= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle - \langle [X,Y], Z\rangle - \langle Y, [X,Z]\rangle \\
&= \langle \nabla_X Y - [X,Y], Z\rangle + \langle Y, \nabla_X Z - [X,Z]\rangle \\
&= \langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle \\
&= Y\langle X, Z\rangle - \langle X, \nabla_Y Z\rangle + Z\langle Y, X\rangle - \langle \nabla_Z Y, X\rangle \\
&= Y(\omega(Z)) - \omega\left(\nabla_Y Z\right) + Z(\omega(Y)) - \omega\left(\nabla_Z Y\right) \\
&= \left(\nabla_Y \omega\right)(Z) + \left(\nabla_Z \omega\right)(Y),
\end{aligned}$$

which is a coordinate-free version of the desired formula. Indeed, if $Y = X_i$ and $Z = X_j$, we have

$$\begin{aligned}
\left(\nabla_{X_i}\omega\right)(X_j) + \left(\nabla_{X_j}\omega\right)(X_i) &= X_i(\omega(X_j)) - \omega\left(\nabla_{X_i} X_j\right) + X_j(\omega(X_i)) - \omega\left(\nabla_{X_j} X_i\right) \\
&= X_i\langle X, X_j\rangle - \langle X, \nabla_{X_i} X_j\rangle + X_j\langle X_i, X\rangle - \left\langle \nabla_{X_j} X_i, X\right\rangle \\
&= \langle \nabla_{X_i} X, X_j\rangle + \left\langle X_i, \nabla_{X_j} X\right\rangle,
\end{aligned}$$

which is exactly what we wanted, since in the basis $\{X_1, \cdots, X_n\}$ the metric is given by $g_{ij} = \delta_{ij}$.

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Definition 2. *Let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow, and suppose that $\varphi_t : \mathcal{M}^n \to \mathcal{M}^n$ is a t-dependent family of diffeomorphisms (with $\varphi_0 = \mathrm{Id}_{\mathcal{M}}$) and that $\sigma(t)$ is a t-dependent scale factor (with $\sigma(0) = 1$). When:*

$$g(t) = \sigma(t)\varphi_t^*(g(0)),$$

we will say that the solution $(\mathcal{M}^n, g(t))$ is a Ricci soliton.

Remark. Taking the derivative of this expression and evaluating at $t = 0$ gives us:

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= \frac{d\sigma(t)}{dt}\varphi_t^*(g(0)) + \sigma(t)\frac{\partial}{\partial t}\varphi_t^* g(0)\\ -2\,\mathrm{Ric}_{g(0)} &= \sigma'(0)g(0) + \mathcal{L}_V g(0),\end{aligned}$$

where $V = \dfrac{d\varphi_t}{dt}$ by the definition of the Lie derivative. Thus, another equation that characterizes Ricci solitons is:

$$\mathrm{Ric} + \mathcal{L}_V g = \lambda g,$$

Setting $\lambda = \dfrac{\sigma'(0)}{2}$, we can use the result of the previous lemma to write this in coordinates as:

$$-2\,\mathrm{Ric}_{ij} = 2\lambda g_{ij} + \nabla_i V_j + \nabla_j V_i,$$

If it happens that V is the gradient of some smooth function f defined on $\mathcal{M}$, we will say that $(\mathcal{M}, g, V, \lambda)$ is a *gradient Ricci soliton with potential function* f, whose equation becomes:

$$\mathrm{Ric}_{ij} + \lambda g_{ij} + \nabla_i \nabla_j f = 0,$$

According to the equation above, a gradient Ricci soliton is said to be *shrinking* if $\lambda < 0$, *steady* if $\lambda = 0$, and *expanding* if $\lambda > 0$.

Example. Hamilton's cigar is the complete Riemannian surface $\left(\mathbb{R}^2, g_\Sigma\right)$, where

$$g_\Sigma = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{1 + x^2 + y^2},$$

and $\mathrm{d}x^2 = \mathrm{d}x \otimes \mathrm{d}x$. As a solution of the Ricci flow, its time-dependent version is:

$$g_\Sigma(t) = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{e^{4t} + x^2 + y^2},$$

With the change of variables $\tilde{x} = e^{-2t}x$ and $\tilde{y} = e^{-2t}y$, it is easy to show that $g_\Sigma(t) = \dfrac{\mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2}{e^{4t} + x^2 + y^2}$ is isometric to $g_\Sigma = g_\Sigma(0)$, that is, if we define the 1-parameter group of (conformal) diffeomorphisms $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ by setting $\varphi_t(x, y) = \left(e^{-2t}x, e^{-2t}y\right)$, then:

$$g_\Sigma(t) = \varphi_t^*(g_\Sigma(0)),$$

so that $g_\Sigma(t)$ is a steady Ricci soliton. It is also gradient, with potential function $f(x, y) = -\log\left(1 + x^2 + y^2\right)$.

Example. One can verify that the product $(0, \infty) \times \mathbb{S}^{n-1}$, with ϕ and with the metric given by $g = \mathrm{d}r^2 + (\phi(r))^2 g_{\mathbb{S}^{n-1}}$ and the potential function f satisfying the system:

$$f'' = (n-1)\frac{\phi''}{\phi}, \qquad \phi\phi' f' = -(n-2)\left(1 - (\phi')^2\right) + \phi\phi'',$$

defines a complete, steady, rotationally symmetric gradient Ricci soliton with positive curvature operator on $\mathbb{R}^n$. Up to homothety, it is the only one satisfying such conditions on $\mathbb{R}^n$ for $n \geq 3$.

The examples above show that there exist Ricci solitons that are not Einstein, and therefore Ricci solitons really are natural generalizations of Einstein manifolds.

Definition 3. *Two Riemannian metrics $g, \widetilde{g}$ on a manifold $\mathcal{M}$ are said to be conformal if there exists a positive differentiable function $f : \mathcal{M} \to \mathbb{R}$ such that for every $p \in \mathcal{M}$ and all $u, v \in T_p\mathcal{M}$ we have $g_p(u, v) = f(p)\widetilde{g}_p(u, v)$. A Riemannian manifold $(\mathcal{M}, g)$ is said to be locally conformally flat if for each point $p \in M$ there exist a neighborhood U_p and a differentiable function f defined on M such that (U_p, fg) is flat (that is, the curvature of (U_p, fg) vanishes on U_p).*

The two examples above are locally conformally flat gradient Ricci solitons. The study of locally conformally flat gradient Ricci solitons has attracted a lot of interest in recent years, and a complete classification was only reached recently, due to the works of Ni, Wallach, Xiaodong Cao, Biao Wan, Zhow Zhang, R. Bryant, López and García Rio.

It is natural to ask what happens when we have locally conformally flat Einstein manifolds, or, more generally, Einstein manifolds locally conformal to a Riemannian manifold of constant curvature. We will now show the last result of this work, due to W. Kuhnel, from 1988: *Einstein manifolds locally conformal to some manifold of constant sectional curvature have constant sectional curvature.*

Theorem 2. *Let $(\mathcal{M}^n, \widetilde{g})$ be an Einstein manifold and suppose that $\mathcal{M}$ is conformal to a manifold of constant sectional curvature, i.e, there exists $\psi : \mathcal{M} \to \mathbb{R}$ such that $(\mathcal{M}, \psi^2\widetilde{g})$ has constant sectional curvature K. Then $(\mathcal{M}^n, \widetilde{g})$ has constant sectional curvature.*

We will need a few lemmas first.

Lemma 3. If $g, \widetilde{g}$ are two conformal Riemannian metrics on $\mathcal{M}$, i.e $\widetilde{g} = e^{-2\varphi}g = \psi^{-2}g$ with $\varphi, \psi : \mathcal{M} \to \mathbb{R}$, then the following relations hold for g and $\widetilde{g}$:

1. $\widetilde{\nabla}_X Y = \nabla_X Y - (X\varphi)Y - (Y\varphi)X + \langle X, Y\rangle \operatorname{grad}\varphi$
2. $$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z - \langle \nabla_X \operatorname{grad}\varphi, Z\rangle\, Y + \langle \nabla_Y \operatorname{grad}\varphi, Z\rangle\, X \\ -\langle X, Z\rangle \nabla_Y \operatorname{grad}\varphi + \langle Y, Z\rangle \nabla_X \operatorname{grad}\varphi + (Y\varphi)(Z\varphi)X \\ -(X\varphi)(Z\varphi)Y - \langle \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle \cdot (\langle Y, Z\rangle X - \langle X, Z\rangle Y) \\ +\{(X\varphi)\langle Y, Z\rangle - (Y\varphi)\langle X, Z\rangle\} \cdot \operatorname{grad}\varphi\end{aligned}$$
3. $$\widetilde{\operatorname{Ric}} = \operatorname{Ric} + \left(\Delta\varphi - (n-2)\|\operatorname{grad}\varphi\|^2\right) g + (n-2)e^{-\varphi}\nabla^2\left(e^{\varphi}\right)$$

Proof: The first equation follows by applying the Koszul formula for the Levi-Civita connections induced by g and $\widetilde{g}$. The second equation follows by applying the first to the terms of the form $\nabla_X\nabla_Y Z$ and $\widetilde{\nabla}_X\widetilde{\nabla}_Y Z$ and grouping the terms with the second derivatives of φ and its Hessian. The last equation follows from the second by taking the trace with respect to an orthonormal basis.

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Lemma 4. Let $(\mathcal{M}, g)$ be an Einstein manifold and $\widetilde{g} = \psi^{-2}g$. The following are equivalent:

1. $(\mathcal{M}, \widetilde{g})$ is an Einstein manifold
2. There exists a function $\lambda : \mathcal{M} \to \mathbb{R}$ with $\nabla_g^2\psi = \lambda g$
3. $\nabla_g^2\psi = \dfrac{\Delta\psi}{n} g$

Proof: It follows by applying the last part of the previous lemma and using $\operatorname{tr} \circ \nabla^2 = \Delta$.

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To prove theorem 2, it then remains to show that part 3 of lemma 4 implies that $(\mathcal{M}, \tilde{g})$ has constant sectional curvature. Indeed, letting $\widetilde{K}$ be the sectional curvature of $(\mathcal{M}^n, \tilde{g})$ and σ a 2-subspace of $T_p\mathcal{M}$, taking X, Y a g-orthonormal basis of σ, and setting $\widetilde{X} = \psi X$ and $\widetilde{Y} = \psi Y$, we have that $\widetilde{X}$ and $\widetilde{Y}$ are $\tilde{g}$-orthonormal and a basis of σ. Using the lemmas already proved, we have:

$$\begin{aligned}
\psi^{-2}\widetilde{K} &= K + \nabla^2\varphi(Y,Y) + \nabla^2\varphi(X,X) + (Y\varphi)^2 + (X\varphi)^2 - \|\operatorname{grad}\varphi\|^2 \\
&= K + \frac{1}{\psi}\nabla^2\psi(X,X) + \frac{1}{\psi}\nabla^2\psi(Y,Y) - \|\operatorname{grad}\varphi\|^2 \\
&= K + \frac{1}{\psi}\frac{\Delta\psi}{n} + \frac{1}{\psi}\frac{\Delta\psi}{n} - \|\operatorname{grad}\varphi\|^2 \\
&= K + \frac{2}{n}\frac{\Delta\psi}{\psi} - \|\operatorname{grad}\varphi\|^2,
\end{aligned}$$

and the desired result follows from Schur's Theorem.

Conclusion

We have seen that the Ricci flow plays an essential role in the resolution of Thurston's geometrization conjecture, and, in particular, of the Poincaré conjecture (see [6]), being a beautiful bridge between analysis, geometry and topology. Although the Ricci flow has gained more attention precisely because it has this immense power, it is interesting in its own right and still studied today. Ricci solitons and Einstein manifolds are particular solutions of the flow, and since Einstein manifolds are precisely the Riemannian manifolds of constant Ricci curvature, it is natural to ask whether they provide a good generalization of the concept of constant curvature in dimension ≥ 3. It is known that asking for constant sectional curvature is too much (Killing-Hopf theorem), since geometrically there are only 3 of them (simply connected), and only one once the sign is fixed. On the other hand, asking for constant scalar curvature is too weak, since by the positive resolution of the Yamabe problem, given any compact Riemannian manifold of dimension ≥ 3 with a metric g, there always exists g' conformal to g such that g' has constant scalar curvature. One can then ask what conditions must be imposed on a Riemannian manifold $(\mathcal{M}^n, g)$ for $\mathcal{M}$ to admit a metric of constant Ricci curvature. In principle we could try to construct locally conformally flat metrics, but theorem 2 guarantees that Einstein manifolds with such metrics have constant sectional curvature and therefore this strategy fails. This is an open problem, something that shows that Ricci curvature is not yet completely understood.

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