English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio, translated from the author's LaTeX source. The Portuguese original is the authoritative version.

Question 1. *Show that if $\mathcal{M}$ is $\mathbb{R}^3$ minus a line, then $H^1(\mathcal{M})$ is not trivial.*

Solution: Up to diffeomorphism, there is only one way to remove a line from $\mathbb{R}^3$, since the positions of the points of any two lines differ only by a rotation and (possibly) a nonzero translation, and the composition of a rotation with a translation is a diffeomorphism. And since De Rham cohomology is invariant under diffeomorphisms, we may then without loss of generality assume that the removed line is the z-axis, that is, consider

$$\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3 \mid x \neq 0 \text{ or } y \neq 0\}$$

Define the following 1-form τ on $\mathcal{M}$

$$\tau = -\frac{y}{x^2 + y^2} \cdot \mathrm{d}x + \frac{x}{x^2 + y^2} \cdot \mathrm{d}y$$

A direct computation shows that τ is closed. But since the integral of τ over the circle

$$\mathbb{S}^1 = \{(\cos(t), \sin(t), 0) \mid t \in [0, 2\pi)\}$$

is equal to

$$\int_{\mathbb{S}^1} \tau = 2\pi$$

We conclude that τ is not exact (otherwise we would have, by Stokes' theorem, $2\pi = 0$, an absurdity). Therefore $0 \neq [\tau] \in H^1(\mathcal{M})$, *id est*, $H^1(\mathcal{M})$ is not trivial.

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Question 2. *If $\mathcal{M}$ is $\mathbb{R}^3$ minus a point, show that $H^3(\mathcal{M})$ is trivial without using topological arguments. Hint: try to adapt the argument used above for the computation of $H^2(\mathbb{R}^2 - (0,0))$.*

Solution: Up to diffeomorphism, there is only one way to remove a point from $\mathbb{R}^3$ (it suffices to consider translations). Therefore we may assume without loss of generality that the removed point is the origin. We will denote by $r : \mathbb{R}^3 \to \mathbb{R}$ the radius function, given by $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ for each $(x, y, z) \in \mathbb{R}^3$. Consider on $\mathcal{M}$ the 2-form τ determined by

$$\tau = \frac{x}{r^3(x, y, z)} \cdot \mathrm{d}y \wedge \mathrm{d}z - \frac{y}{r^3(x, y, z)} \cdot \mathrm{d}x \wedge \mathrm{d}z + \frac{z}{r^3(x, y, z)} \cdot \mathrm{d}x \wedge \mathrm{d}y$$

for each $(x, y, z) \in \mathcal{M}$. Since

$$\begin{aligned}
\mathrm{d}\tau &= \frac{\partial}{\partial x}\left(\frac{x}{r^3(x, y, z)}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{\partial}{\partial y}\left(\frac{-y}{r^3(x, y, z)}\right) \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z \\
&\quad + \frac{\partial}{\partial z}\left(\frac{z}{r^3(x, y, z)}\right) \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \frac{-2x^2 + y^2 + z^2}{r^5(x, y, z)} \cdot \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{x^2 - 2y^2 + z^2}{r^5(x, y, z)} \cdot \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\quad + \frac{x^2 + y^2 - 2z^2}{r^5(x, y, z)} \cdot \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= 0
\end{aligned}$$

we see that τ is closed. Now let ω be any 3-form on $\mathcal{M}$. Without loss of generality, we may assume that ω satisfies

$$\omega = \frac{f(x,y,z)}{r^2(x,y,z)} \cdot \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

for some smooth $f : \mathcal{M} \to \mathbb{R}$. We are now looking for a 2-form η on $\mathcal{M}$ such that $\omega = \mathrm{d}\eta$. Supposing that $\eta = g\tau$ for some smooth $g : \mathcal{M} \to \mathbb{R}$, we see that this condition is equivalent to

$$\begin{aligned} \frac{f(x,y,z)}{r^2(x,y,z)} \cdot \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \mathrm{d}\eta = \mathrm{d}g \wedge \tau = (g_x \mathrm{d}x + g_y \mathrm{d}y + g_z \mathrm{d}z) \wedge \left(\frac{x}{r^3} \mathrm{d}y \wedge \mathrm{d}z - \frac{y}{r^3} \mathrm{d}y \wedge \mathrm{d}z \right) \\ \iff \frac{x g_x + y g_y + z g_z}{r(x,y,z)} = f(x,y,z) \end{aligned} \tag{1}$$

for each $(x,y,z) \in \mathcal{M}$. Geometrically, this condition is equivalent to

$$\mathrm{d}g|_{(x,y,z)}(\mathbf{e}_r) = f(x,y,z), \ \forall (x,y,z) \in \mathcal{M}$$

where

$$\mathbf{e}_r(x,y,z) = r^{-1}(x,y,z) \cdot (x,y,z)$$

is the unit vector in the radial direction; that is, f is the radial derivative of g. We can then determine g via radial integrations, *id est*

$$g(x,y,z) = \int_1^{r(x,y,z)} f(s \cdot \mathbf{e}_r) \ \mathrm{d}s$$

We now need to verify that this definition of g satisfies equation (1). For this, we will need the following lemmas:

Lemma (L.1). *Let $F, b : \mathbb{R}^n \to \mathbb{R}$ be differentiable functions. Define the function $H : \mathbb{R}^n \to \mathbb{R}$ by*

$$H(t, x_2, \cdots, x_n) = \int_1^t F(y, x_2, \cdots, x_n) \ \mathrm{d}y$$

Then the partial derivatives of the function $G : \mathbb{R}^n \to \mathbb{R}$ defined by

$$G(x_1, \cdots, x_n) = H(b(x_1, \cdots, x_n), x_2, \cdots, x_n)$$

are given by

$$\frac{\partial G}{\partial x_i}(x_1, x_2, \cdots, x_n) = \frac{\partial b}{\partial x_i}(x_1, x_2, \cdots, x_n) F(b(x_1, x_2, \cdots, x_n), x_2, \cdots, x_n) + \int_1^{b(x_1, x_2, \cdots, x_n)} \frac{\partial F}{\partial x_i}(y, \cdots, x_n) \ \mathrm{d}y$$

for each $1 \leq i \leq n$.

Proof: First, it follows from the chain rule that

$$\frac{\partial G}{\partial x_i} = \frac{\partial b}{\partial x_i} \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x_i}$$

Now, by the fundamental theorem of calculus, we have

$$\frac{\partial H}{\partial t}(t, x_2, \cdots, x_n) = F(t, x_2, \cdots, x_n)$$

And by the Leibniz rule, we also have

$$\frac{\partial H}{\partial x_i}(t, x_2, \cdots, x_n) = \int_1^t \frac{\partial F}{\partial x_i}(y, \cdots, x_n) \ \mathrm{d}y$$

We then conclude that

$$\frac{\partial G}{\partial x_i}(x_1, x_2, \cdots, x_n) = \frac{\partial b}{\partial x_i}(x_1, x_2, \cdots, x_n) F(b(x_1, x_2, \cdots, x_n), x_2, \cdots, x_n) + \int_1^{b(x_1, x_2, \cdots, x_n)} \frac{\partial F}{\partial x_i}(y, \cdots, x_n) \ \mathrm{d}y$$

for each $1 \leq i \leq n$, as desired.

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Lemma (L.2). *Let $r : \mathbb{R}^n \to R$ be the radius function, defined by $r(p) = \|p\|$ for each $p \in \mathbb{R}^n$, and consider the normalization function $h : \mathbb{R}^n \to \mathbb{R}^n$ given by $h(p) = \dfrac{p}{\|p\|} = (h_1(p), \cdots, h_n(p)) \in \mathbb{R}^n$. Then h satisfies*

$$\sum_{1\leq i\leq n} x_i \left(h_j\right)_{x_i} = 0$$

for each fixed $1 \leq j \leq n$.

Proof: A direct computation shows that

$$\frac{\partial}{\partial x_i} h_j = \frac{\partial}{\partial x_i}\left(\frac{x_j}{r}\right) = \frac{\delta_{ij} r^2 - x_i x_j}{r^3}$$

Therefore

$$\begin{aligned}\sum_{1\leq i\leq n} x_i \left(h_j\right)_{x_i} &= \frac{x_j \cdot r^2 - x_j^3}{r^3} - \sum_{\substack{1\leq i\leq n\\ i\neq j}} \frac{x_i^2 \cdot x_j}{r^3} \\ &= \frac{x_j}{r^3} \sum_{\substack{1\leq i\leq n\\ i\neq j}} x_i^2 - \frac{x_j}{r^3} \sum_{\substack{1\leq i\leq n\\ i\neq j}} x_i^2 = 0\end{aligned}$$

as desired.

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Define $F : \mathbb{R}^4 \to \mathbb{R}$ by $F(s, x, y, z) \doteq f(s \cdot \mathbf{e}_r)$. It follows from lemma **(L.1)** that

$$\frac{\partial g}{\partial x_i}(x, y, z) = \frac{x_i}{r} \cdot \underbrace{F(r(x, y, z), x, y, z)}_{=f(x,y,z)} + \int_1^{r(x,y,z)} \frac{\partial F}{\partial x_i}(s, x, y, z)\, \mathrm{d}s$$

Now, by the chain rule, we have

$$\frac{\partial F}{\partial x_i}(s, x, y, z) = s \cdot \left(\sum_{1\leq j\leq 3} f_{x_j} \left(h_j\right)_i \right)$$

Hence

$$\frac{\partial g}{\partial x_i}(x, y, z) = \frac{x_i}{r} f + \int_1^{r(x,y,z)} s \cdot \left(\sum_{1\leq j\leq 3} f_{x_j} \left(h_j\right)_i \right) \mathrm{d}s$$

Thus, we have

$$x_i \cdot \frac{\partial g}{\partial x_i}(x, y, z) = \frac{x_i^2}{r} f + \int_1^{r(x,y,z)} s \cdot \left(\sum_{1\leq j\leq 3} x_i \cdot f_{x_j} \left(h_j\right)_i \right) \mathrm{d}s$$

It follows that

$$\sum_{1\leq i\leq 3} x_i \cdot \frac{\partial g}{\partial x_i}(x, y, z) = rf + s \int_1^{r(x,y,z)} \underbrace{\sum_{1\leq i,j\leq n} f_j \cdot x_i (h_j)_{x_i}}_{=0, \text{ by lemma } \textbf{(L.2)}} = rf$$

and therefore condition **(1)** is obviously satisfied.

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Remark (O.1). *Mutatis mutandis,* the argument presented above generalizes to show that every closed n-form on $\mathbb{R}^n$ is exact. Explicitly, in that case it will suffice to consider

$$\tau = \frac{\sum\limits_{1\leq i\leq n} x_i \star (\mathrm{d}x^i)}{(r(x_1,\cdots,x_n))^{\frac{n}{2}}}$$

and use the lemmas proved above.

Question 3. *Show that if an exact 1-form defined on a manifold $\mathcal{M}$ without boundary is nonzero at every point of $\mathcal{M}$, then $\mathcal{M}$ is not compact.*

Solution: We will need the following lemma:

Lemma (L.3). *Let $\mathcal{M}$ be a compact manifold without boundary and $f : \mathcal{M} \to \mathbb{R}$ a smooth function. Then there exists a point $p \in \mathcal{M}$ such that $\mathrm{d}f_p \equiv 0$.*

Proof: Since $\mathcal{M}$ is compact and f is *a fortiori* continuous, there exists a maximum point $p \in \mathcal{M}$ of f, that is, a point $p \in \mathcal{M}$ such that $f(p) \geq f(q)$ for every $q \in \mathcal{M}$. Fix $v \in T_p\mathcal{M}$ arbitrarily and take a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. We then have

$$\begin{aligned}\mathrm{d}f_p(v) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f\circ\gamma)(t)\\ &= \lim_{t\to 0}\frac{(f\circ\gamma)(t) - (f\circ\gamma)(0)}{t}\end{aligned}$$

Since $\gamma(0) = p$ is by construction a maximum point of f, it follows that $(f\circ\gamma)(t) - (f\circ\gamma)(0) \leq 0$ for every $t \in (-\varepsilon, \varepsilon)$. Therefore

$$0 \geq \lim_{t\to 0^+}\frac{(f\circ\gamma)(t) - (f\circ\gamma)(0)}{t}, \text{ and also } 0 \leq \lim_{t\to 0^-}\frac{(f\circ\gamma)(t) - (f\circ\gamma)(0)}{t}$$

Since f is smooth, the one-sided limits above coincide, whence we conclude that

$$\mathrm{d}f_p(v) = \lim_{t\to 0}\frac{(f\circ\gamma)(t) - (f\circ\gamma)(0)}{t} = 0$$

The desired result then follows from the arbitrariness of v.

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We will now show the contrapositive of the question: that is, that every exact 1-form defined on a compact manifold without boundary vanishes at some point of $\mathcal{M}$. Indeed, if ω is an exact 1-form on a compact manifold in $\mathcal{M}$, then $\omega = \mathrm{d}f$ for some smooth $f : \mathcal{M} \to \mathbb{R}$, whence it follows by the previous lemma that ω vanishes at some point of $\mathcal{M}$.

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Remark (O.2). In the case where $\mathcal{M}$ has boundary, lemma **(L.3)** is false: for a counterexample, it suffices to consider the 1-form $\mathrm{d}x$ on the compact one-dimensional manifold $[0,1]$ (whose boundary is $\{0,1\}$). The proof of this lemma cannot be extended to the case with boundary because tangent vectors on the boundary may arise from curves defined only on intervals of the form $(-\varepsilon, 0]$ or $[0,\varepsilon)$, so that the argument of the one-sided limits coinciding can no longer be used (in that case $(f\circ\gamma)(t)$ will not make sense for one of the one-sided limits).

Question 4. *Show that the exterior product defined for differential forms on a manifold $\mathcal{M}$ of dimension n induces a product between cohomology classes: for $[\omega]\in H^k(\mathcal{M})$ and $[\tau]\in H^\ell(\mathcal{M})$ we define*

$$[\omega]\wedge[\tau]=[\omega\wedge\tau]$$

Using the product defined above, verify that the direct sum of the cohomology groups

$$H^*(\mathcal{M})=\bigoplus_{k=0}^{n} H^k(\mathcal{M})$$

admits a ring structure (in the algebraic sense).

Remark (O.3). In the original statement of the question there is a typo: we must take forms $[\omega]\in H^k(\mathcal{M})$ and $[\tau]\in H^\ell(\mathcal{M})$, not forms of the same degree.

Solution: All the algebraic properties that define a ring are obviously satisfied: $H^*(\mathcal{M})$ is clearly an abelian group under addition and a monoid under the multiplication operation $\wedge$, and $\wedge$ is distributive with respect to addition by definition. It remains only to verify that $\wedge$ induces a well-defined map (which, by a harmless abuse of notation, is also denoted by $\wedge$) on the equivalence classes of closed forms (where two equivalence classes are equal if their representatives differ by an exact form; we have already seen that this relation is well defined), that is, that the exterior product of closed forms is a closed form and that the map that $\wedge$ induces on the equivalence classes of closed forms does not depend on the chosen representatives. Indeed,

- if $[\omega]\in H^k(\mathcal{M})$ and $[\tau]\in H^\ell(\mathcal{M})$, then

  $$\mathrm{d}(\omega\wedge\tau)=(\mathrm{d}\omega)\wedge\tau+(-1)^k\cdot\omega\wedge\mathrm{d}\tau=0$$

  and therefore $\omega\wedge\tau$ is closed whenever ω and τ are closed.

- if $\tau'=\tau+\mathrm{d}\sigma$, then

  $$\omega\wedge\tau'=\omega\wedge\tau+\omega\wedge\mathrm{d}\sigma$$

  But since

  $$\mathrm{d}(\omega\wedge\sigma)=(\mathrm{d}\omega)\wedge\sigma+(-1)^k\cdot\omega\wedge\mathrm{d}\sigma=(-1)^k\cdot\omega\wedge\mathrm{d}\sigma \text{ (since } \mathrm{d}\omega=0\text{, as } \omega \text{ is closed by hypothesis)}$$

  it follows that $\omega\wedge\sigma$ is exact: that is, $\omega\wedge\tau'$ and $\omega\wedge\tau$ differ by an exact form, and therefore $[\omega\wedge\tau']=[\omega\wedge\tau]$. *Mutatis mutandis,* we see that if ω' is another representative of ω, then $[\omega'\wedge\tau]=[\omega\wedge\tau]$. This concludes the verification that the map $\wedge$ is well defined on the equivalence classes.

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Question 5. *Compute the De Rham cohomology of the n-dimensional sphere $\mathbb{S}^n$.*

Solution: First, note that since $\mathbb{S}^n$ is connected, $H^0_{\mathrm{dR}}(\mathbb{S}^n) = \mathbb{R}$. In what follows, we will consider the open sets $U = \mathbb{S}^n \setminus \{p\}$, $V = \mathbb{S}^n \setminus \{q\}$, where $p, q \in \mathbb{S}^n$ are arbitrary points, which are diffeomorphic to $\mathbb{R}^n$ via stereographic projection. It is also easy to see that the intersection $U \cap V = \mathbb{S}^n \setminus \{p, q\}$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$ (again via stereographic projection), which has the same homotopy type as $\mathbb{S}^{n-1}$ $\left(\text{via the retraction } \mathbb{R}^n \setminus \{0\} \ni x \mapsto \frac{x}{\|x\|} \in \mathbb{S}^{n-1}\right)$. In the case $n = 1$, the following piece of the Mayer-Vietoris exact sequence

$$H^{-1}_{\mathrm{dR}}(U \cap V) \to H^0_{\mathrm{dR}}(\mathbb{S}^1) \to H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \to H^0_{\mathrm{dR}}(U \cap V) \to H^1_{\mathrm{dR}}(\mathbb{S}^1) \to H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V)$$

can be written as

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^2 \to H^1_{\mathrm{dR}}(\mathbb{S}^1) \to 0$$

It follows that

$$1 - 2 + 2 - \dim H^1_{\mathrm{dR}}(\mathbb{S}^1) = 0 \iff \dim H^1_{\mathrm{dR}}(\mathbb{S}^1) = 1$$

and therefore $H^1_{\mathrm{dR}}(\mathbb{S}^1) = \mathbb{R}$. In the case $n \geq 2$, using the fact that U and V are diffeomorphic to $\mathbb{R}^n$ and their intersection is connected, we see that the following piece of the Mayer-Vietoris exact sequence:

$$H^{-1}_{\mathrm{dR}}(U \cap V) \to H^0_{\mathrm{dR}}(\mathbb{S}^n) \to H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \to H^0_{\mathrm{dR}}(U \cap V) \to H^1_{\mathrm{dR}}(\mathbb{S}^n) \to H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V)$$

can be written as

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to H^1_{\mathrm{dR}}(\mathbb{S}^n) \to 0$$

Hence,

$$1 - 2 + 1 - \dim H^1_{\mathrm{dR}}(\mathbb{S}^n) = 0 \iff H^1_{\mathrm{dR}}(\mathbb{S}^n) = 0$$

When $n \geq 2$ and $k \geq 2$, we can use the following piece of the Mayer-Vietoris exact sequence

$$0 = H^{k-1}_{\mathrm{dR}}(U) \oplus H^{k-1}_{\mathrm{dR}}(V) \to H^{k-1}_{\mathrm{dR}}(U \cap V) \cong H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{n-1}) \to H^k_{\mathrm{dR}}(\mathbb{S}^n) \to H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) = 0$$

to conclude that

$$H^k_{\mathrm{dR}}(\mathbb{S}^n) \cong H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{n-1})$$

Therefore,

$$H^n_{\mathrm{dR}}(\mathbb{S}^n) \cong H^{n-1}_{\mathrm{dR}}(\mathbb{S}^{n-1}) \cong \cdots H^2_{\mathrm{dR}}(\mathbb{S}^2) \cong H^1_{\mathrm{dR}}(\mathbb{S}^1) \cong \mathbb{R}$$

And in general, when $\ell \geq 1$ and $n \geq 2$, we have in an entirely analogous way

$$H^{n-\ell}_{\mathrm{dR}}(\mathbb{S}^n) = H^{n-\ell-1}_{\mathrm{dR}}(\mathbb{S}^{n-1}) = \cdots = H^2_{\mathrm{dR}}(\mathbb{S}^{\ell+2}) = H^1_{\mathrm{dR}}(\mathbb{S}^{\ell+1}) = 0$$

It is then proved that:

$$H^k_{\mathrm{dR}}(\mathbb{S}^n) = \begin{cases} \mathbb{R}, & \text{if } k \in \{0, n\} \\ 0, & \text{otherwise} \end{cases}$$

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Question 6. *Compute the De Rham cohomology of a compact, connected, oriented surface S of genus g.*

Solution: We will denote a generic surface satisfying the statement of the question by Σ_g, and we will denote the complement of a point of Σ_g by $\widetilde{\Sigma}_g = \Sigma_g \setminus \{p\}$. I claim that

$$H^k_{\mathrm{dR}}(\widetilde{\Sigma}_g) = \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ \mathbb{R}^{2g}, & \text{if } k = 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$H^k_{\mathrm{dR}}(\Sigma_g) = \begin{cases} \mathbb{R}, & \text{if } k \in \{0, 2\} \\ \mathbb{R}^{2g}, & \text{if } k = 1 \\ 0, & \text{otherwise} \end{cases}$$

We have already dealt with the cases $g \in \{0, 1, 2\}$ in class. Assuming the induction hypothesis that the claim holds for some integer $g \geq 2$, we will show that it also holds for $g + 1$. The case $k = 0$ is trivial by the connectedness of all the surfaces involved. Since $\Sigma_{g+1} \cong \widetilde{\Sigma}_g \cup \widetilde{\Sigma}_1$ (open sets whose intersection has the same homotopy type as $\mathbb{S}^1$), we have the following piece of the Mayer-Vietoris exact sequence:

$$0 \to H^1_{\mathrm{dR}}(\Sigma_{g+1}) \to H^1_{\mathrm{dR}}(\widetilde{\Sigma}_g) \oplus H^1_{\mathrm{dR}}(\widetilde{\Sigma}_1) \xrightarrow{\alpha} \mathbb{R} \to H^2_{\mathrm{dR}}(\Sigma_{g+1}) \to 0$$

which, by the induction hypothesis, is written as

$$0 \to H^1_{\mathrm{dR}}(\Sigma_{g+1}) \to \mathbb{R}^{2(g+1)} \xrightarrow{\alpha} \mathbb{R} \to H^2_{\mathrm{dR}}(\Sigma_{g+1}) \to 0$$

By lemma 28.3 of the book *An Introduction to Manifolds* by Loring Tu, the map $\alpha \equiv 0$. By the exactness of the Mayer-Vietoris sequence, we then conclude (it suffices to repeatedly use the rank-nullity theorem) that $H^1_{\mathrm{dR}}(\Sigma_{g+1}) = \mathbb{R}^{2(g+1)}$ and $H^2_{\mathrm{dR}}(\Sigma_{g+1}) = \mathbb{R}$.

It remains for us to prove that the claim is satisfied for the integer $g + 1$ in the case of $\widetilde{\Sigma}_{g+1}$. Covering Σ_{g+1} with the open sets $\widetilde{\Sigma}_{g+1}$ and a disk $\mathbb{D}_p$ containing p, we have the following piece of the Mayer-Vietoris exact sequence:

$$0 \to H^1_{\mathrm{dR}}(\Sigma_{g+1}) = \mathbb{R}^{2(g+1)} \to H^1_{\mathrm{dR}}(\widetilde{\Sigma}_{g+1}) \oplus H^1_{\mathrm{dR}}(\mathbb{D}_p) \xrightarrow{\alpha} \mathbb{R} \to \mathbb{R} \to H^2_{\mathrm{dR}}(\widetilde{\Sigma}_{g+1}) \to 0$$

Again, $\alpha \equiv 0$, whence we conclude by the exactness of the Mayer-Vietoris sequence that $H^1_{\mathrm{dR}}(\widetilde{\Sigma}_{g+1}) = \mathbb{R}^{2(g+1)}$ and $H^k_{\mathrm{dR}}(\widetilde{\Sigma}_{g+1}) = 0$ whenever $k \neq 0$. This concludes the proof by induction of our initial claim.

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Question 7. *Compute the de Rham cohomology of the real projective plane $\mathbb{RP}^2$.*

Solution: We will do the more general case and compute the De Rham cohomology of the real projective space $\mathbb{RP}^n$, by means of the following lemma

Lemma (L.4). *Let G be a finite group acting on a manifold $\mathcal{M}$ properly discontinuously (that is, there is a correspondence $\varphi : G \to \mathsf{Diff}(\mathcal{M})$, where $\mathsf{Diff}(\mathcal{M})$ denotes the set of all diffeomorphisms taking $\mathcal{M}$ to $\mathcal{M}$). The degree k De Rham cohomology of the quotient*

$$\mathcal{M}/G = \{[p] \mid p \in \mathcal{M}, \text{ where } [p] = \{\varphi_g(p) \mid g \in G\} \text{ for each } p \in \mathcal{M}\}$$

is the subset of $H^k_{dR}(\mathcal{M})$ fixed by the natural action of G, given by

$$H^k_{dR}(\mathcal{M}/G) \cong (H^k_{dR}(\mathcal{M}))^G \doteq \{[\omega] \mid \omega \in \Lambda^k(\mathcal{M}), (\varphi_g)^*(\omega) = \omega \ \forall g \in G\}$$

Proof: Consider the canonical projection onto the quotient, given by

$$\begin{aligned}\pi : \mathcal{M} &\to \mathcal{M}/G \\ p &\mapsto [p]\end{aligned}$$

Note that by construction $\pi \circ \varphi_g = \pi$ for every $g \in G$. Moreover, by the rank-nullity theorem, it suffices to show that the map

$$\pi^* : H^k_{\mathrm{dR}}(\mathcal{M}/G) \to H^k_{\mathrm{dR}}(\mathcal{M})$$

is injective and that its image is given by

$$\mathrm{img}(\pi^*) = \{[\omega] \mid \omega \in \Lambda^k(\mathcal{M}), (\varphi_g)^*(\omega) = \omega \ \forall g \in G\}$$

We will begin by characterizing the image of π^*. Suppose that $[\eta] \in \mathrm{img}(\pi^*)$. By definition, there then exists $\overline{\eta} \in \Lambda^k(\mathcal{M}/G)$ such that $\eta = \pi^*(\overline{\eta})$. Given $\varphi_g \in \mathsf{Diff}(\mathcal{M})$, it then holds that

$$\varphi_g^*(\eta) = \varphi_g^*\left[\pi^*(\overline{\eta})\right] = \left(\pi \circ \varphi_g\right)^*\left(\overline{\eta}\right) = \pi^*(\overline{\eta}) = \eta$$

and therefore a necessary condition for an element to be in the image of π^* is to be invariant under the action of G. We are interested in also proving the converse, that is, that if $(\varphi_g)^*\eta = \eta \ \forall g \in G$, then there exists $\overline{\eta} \in \Lambda^k(\mathcal{M}/G)$ such that $\eta = \pi^*(\overline{\eta})$.

Suppose then that $\eta \in \Lambda^k(\mathcal{M})$ satisfies $(\varphi_g)^*\eta = \eta \ \forall g \in G$. First, note that the fact that π is a local diffeomorphism guarantees that $\mathrm{d}\pi_p$ is an isomorphism. Thus, given $\overline{p} = [p] \in \mathcal{M}/G$ and $\overline{v_1}, \cdots, \overline{v_k} \in T_{\overline{p}}(\mathcal{M}/G)$, it follows that for each $1 \leq i \leq k$, there exists $v_i \in T_p\mathcal{M}$ such that $\mathrm{d}\pi_p(v_i) = \overline{v_i}$. We can then define

$$\overline{\eta}_{\overline{p}}(\overline{v_1}, \cdots, \overline{v_k}) = \eta_p(v_1, \cdots, v_k)$$

We now need to show that $\overline{\eta}$ is well defined. So let $q \in \mathcal{M}$ be such that $[q] = [p]$ and $u_1, \cdots, u_n \in T_q\mathcal{M}$ such that $\mathrm{d}\pi_q(u_i) = \overline{v_i}$. By definition, there exists $g \in G$ such that $\varphi_g(p) = q$. Now, since

$$\begin{aligned}\mathrm{d}\pi_q\left(\mathrm{d}\left(\varphi_g\right)_p(v_i)\right) &= \mathrm{d}\left(\pi \circ \varphi_g\right)_p(v_i) \\ &= \mathrm{d}\pi_p(v_i) \\ &= \overline{v_i}\end{aligned}$$

it follows from the fact that $\mathrm{d}\pi_q$ is an isomorphism that $u_i = \mathrm{d}\left(\varphi_g\right)_p(v_i)$. Therefore

$$\begin{aligned}\eta_q(u_1, \cdots, u_k) &= \eta_{\varphi_g(p)}(\mathrm{d}\left(\varphi_g\right)_p(v_1), \cdots, \mathrm{d}\left(\varphi_g\right)_p(v_k)) \\ &= \left[\varphi_g^*(\eta)\right]_p(v_1, \cdots, v_k) \\ &= \eta_p(v_1, \cdots, v_k)\end{aligned}$$

whence we conclude that $\overline{\eta}$ is indeed well defined. It is immediate from the construction of $\overline{\eta}$ that $\eta = \pi^*(\overline{\eta})$.

We will now show that π^* is injective. When $k = 0$, $H^0_{\mathrm{dR}}(\mathcal{M})$ consists simply of the constant functions on $\mathcal{M}$. Thus, if $[\overline{w}] \in H^0_{\mathrm{dR}}(\mathcal{M})$ satisfies $\pi^*(\overline{\omega}) = 0$, then $\overline{\omega}_{\overline{p}} = 0$ for every $\overline{p} \in \mathcal{M}/G$, and therefore $\overline{\omega}$ is identically zero. Let us now suppose then $1 \leq k \leq n = \dim(\mathcal{M}/G) = \dim(\mathcal{M})$. If $[\omega] \doteq \pi^*(\overline{\omega}) = 0$, then ω is exact, that is, there exists $\eta \in \Lambda^{k-1}(\mathcal{M})$ such that $\mathrm{d}\eta = \omega$. By the characterization of the image of π^* that we have just proved, it follows that $\mathrm{d}\eta$ is fixed by the natural action of G, that is, $(\varphi_g)^*(\mathrm{d}\eta) = \mathrm{d}\eta$ for every $g \in G$. We may suppose without loss of generality that η itself is also fixed by the natural action of G. Indeed, if this were not the case, we could work with the following form τ instead of η: define

$$\tau \doteq \frac{1}{|G|}\sum_{g \in G}(\varphi_g)^*(\eta)$$

We then have

$$\begin{aligned}\mathrm{d}\tau = \frac{1}{|G|}\sum_{g\in G}\mathrm{d}\left[(\varphi_g)^*(\eta)\right] &= \frac{1}{|G|}\sum_{g\in G}\varphi_g^*(\mathrm{d}\eta)\\ &= \frac{1}{|G|}\sum_{g\in G}\mathrm{d}\eta = \omega\\ &= \mathrm{d}\eta\end{aligned}$$

which justifies the option of working with τ instead of η. Moreover, $(\varphi_g^*)(\tau) = \tau$ for every $g \in G$ by construction. We will then suppose without loss of generality that η is preserved by the natural action of G. It then follows from the characterization of the image of π^* that we have just proved that $\eta = \pi^*(\overline{\eta})$ for some $\eta \in \Lambda^k(\mathcal{M}/G)$. Since π is a local diffeomorphism and forms are objects of a local nature, we may commit the abuse of notation of writing $\overline{\eta} = \left(\pi^{-1}\right)^*(\eta)$. Locally, we then have

$$\begin{aligned}\mathrm{d}\overline{\eta} &= \mathrm{d}\left(\left(\pi^{-1}\right)^*(\eta)\right)\\ &= \left(\pi^{-1}\right)^*(\mathrm{d}\eta)\\ &= \left(\pi^{-1}\right)^*(\omega) = \overline{\omega}\end{aligned}$$

and therefore $\mathrm{d}\overline{\eta} = \overline{\omega}$, so that $[\overline{\omega}] = 0$. We then conclude that π^* is injective, as desired.

■

Note that $\mathbb{RP}^n = \mathbb{S}^n/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\mathrm{Id}_{\mathbb{S}^n}, A\}$, A being the antipodal map. Since the antipodal map preserves orientation only in the case where n is odd, we conclude that the odd-dimensional projective spaces have the same De Rham cohomology as the corresponding odd-dimensional spheres. In the case where n is even, the antipodal map reverses orientation and therefore acts as multiplication by -1, whence we then conclude that $H^k_{\mathrm{dR}}(\mathbb{RP}^{2m}) = 0$ for every $k \neq 0$. In summary,

$$H^k_{\mathrm{dR}}(\mathbb{RP}^n) = \begin{cases}\mathbb{R}, & \text{if } n \text{ is odd and } k \in \{0, n\}\\ 0, & \text{otherwise}\end{cases}$$

In particular, $H^0_{\mathrm{dR}}(\mathbb{RP}^2) = \mathbb{R}$ and $H^k_{\mathrm{dR}}(\mathbb{RP}^2) = 0$ for every $k \neq 0$.

■

Question 8. *Compute the De Rham cohomology of the plane $\mathbb{R}^2$ minus a finite number of points.*

Solution: We will denote by $\mathbb{R}^n_\ell$ the complement of ℓ points of $\mathbb{R}^n$. I claim that

$$H^k_{\mathrm{dR}}(\mathbb{R}^n_\ell) = \begin{cases}\mathbb{R}, & \text{if } k = 0\\ \mathbb{R}^\ell, & \text{if } k = n-1\\ 0, & \text{otherwise}\end{cases}$$

We have already seen that $\mathbb{R}^n_1$ and $\mathbb{S}^{n-1}$ have the same homotopy type, so we have already dealt with the case $\ell = 1$. We will finish the proof by induction by showing that if the claim holds for some integer $\ell \geq 1$, it also holds for $\ell+1$. Let us suppose then that the claim is satisfied for an integer $\ell \geq 1$. Since $\mathbb{R}^n = \mathbb{R}^n_\ell \cup \mathbb{R}^n_1$ (where

the first open set of the union above is the complement of ℓ points of $\mathbb{R}^n$ and the second is the complement of a point distinct from the previous ℓ points), we have the following piece of the Mayer-Vietoris exact sequence

$$H^k_{\mathrm{dR}}(\mathbb{R}^n) \to H^k_{\mathrm{dR}}(\mathbb{R}^n_\ell) \oplus H^k_{\mathrm{dR}}(\mathbb{R}^n_1) \to H^k_{\mathrm{dR}}(\mathbb{R}^n_{\ell+1}) \to H^{k+1}_{\mathrm{dR}}(\mathbb{R}^n)$$

If $k \notin \{0, n-1\}$, we then have the exact sequence

$$0 \to 0 \to H^k_{\mathrm{dR}}(\mathbb{R}^n_{\ell+1}) \to 0$$

whence we conclude that $H^k_{\mathrm{dR}}(\mathbb{R}^n_{\ell+1}) = 0$ whenever $k \notin \{0, n-1\}$. Since $\mathbb{R}^n_{\ell+1}$ is connected, we also have $H^0_{\mathrm{dR}}(\mathbb{R}^n_{\ell+1}) = \mathbb{R}$, so that it remains for us to deal with the case $k = n-1$. In that case, we have the exact sequence

$$H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n) = 0 \to H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n_\ell) \oplus H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n_1) \to H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n_{\ell+1}) \to H^n_{\mathrm{dR}}(\mathbb{R}^n) = 0$$

which, by the induction hypothesis, is written as

$$0 \to \mathbb{R}^{\ell+1} \to H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n_{\ell+1}) \to 0$$

Therefore $H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n_{\ell+1}) = \mathbb{R}^{\ell+1}$, which concludes the proof by induction of the claim.

■

Question 9. *Show that the tangent bundle of the sphere $\mathbb{S}^3$ is trivial.*

Remark (O.4). We saw in class that if $\pi : E \to \mathcal{M}$ is a vector bundle of rank n that admits a global frame (also called a global basis of sections) $\{s_i\}_{1\leq i\leq n}$, then E is a trivial bundle. In this case, we will say that the frame $\{s_i\}_{1\leq i\leq n}$ *witnesses* the triviality of E.

Solution: The tangent space at a point $p \in \mathbb{S}^3$ consists of all the initial tangent vectors of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{S}^3$ that start at p, that is

$$T_p\mathbb{S}^3 = \{\gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \ \to \mathbb{S}^3 \text{ is smooth and satisfies } \gamma(0) = p, \text{ with } \varepsilon > 0 \text{ arbitrary}\}$$

Any such curve obviously satisfies $\|\gamma\|^2 = \langle \gamma, \gamma\rangle = 1$. Differentiating this equality and evaluating it at $t = 0$, we conclude that $\langle \gamma'(0), p\rangle = 0$. This shows that $T_p\mathbb{S}^3 \subset p^\perp = \{v \in \mathbb{R}^4 \mid \langle v, p\rangle = 0\}$. Now, if $0 \neq v \in \mathbb{R}^4$ satisfies $\langle v, p\rangle = 0$, then defining the following curve (where we may take, for example, $\varepsilon = 1$)

$$(-\varepsilon, \varepsilon) \ni t \mapsto \gamma_v(t) \doteq \cos(t)p + \frac{\sin(t)}{\|v\|}v$$

we see that γ_v satisfies $\mathrm{Im}(\gamma_v) \subset \mathbb{S}^3$ (since p and v are orthogonal by hypothesis and the trigonometric identity $\cos^2(t) + \sin^2(t) = 1$ holds), $\gamma_v(0) = p$ and $\gamma_v'(0) = \dfrac{v}{\|v\|} \in T_p\mathbb{S}^3$. Since $T_p\mathbb{S}^3$ is a vector space, we also have that $v \in T_p\mathbb{S}^3$. By the arbitrariness of v, it follows that $p^\perp \subset T_p\mathbb{S}^3$. Thus, we then conclude that

$$T_p\mathbb{S}^3 = p^\perp = \{v \in \mathbb{R}^4 \mid \langle p, v\rangle = 0\}$$

Taking inspiration now from the fact that the field $\mathbb{S}^1 \ni (x, y) \mapsto (-y, x)$ is a global section of $\mathbb{S}^1$ (which, by virtue of the one-dimensionality of $\mathbb{S}^1$, witnesses the triviality of the bundle $T\mathbb{S}^1$), we can consider the field

$$s_1 : \mathbb{S}^3 \to T\mathbb{S}^3$$
$$p = (x, y, z, w) \mapsto (p, -y, x, w, -z)$$

And to produce two more sections that witness the triviality of the tangent bundle of $\mathbb{S}^3$, we can apply appropriate rotation matrices to s_1 and obtain two more mutually orthogonal fields that, together, form (pointwise) a basis of the orthogonal complement in $\mathbb{S}^3$ of the space spanned by s_1. Namely, we can also consider the fields

$$s_2 : \mathbb{S}^3 \to T\mathbb{S}^3$$
$$p = (x, y, z, w) \mapsto (p, z, w, -x, -y)$$

and

$$s_3 : \mathbb{S}^3 \to T\mathbb{S}^3$$
$$p = (x, y, z, w) \mapsto (p, w, -z, y, -x)$$

Since s_1, s_2 and s_3 are mutually orthogonal, these sections are linearly independent (it is trivial to verify that indeed $s_i(p) \in \{p\} \times T_p\mathbb{S}^3 \cong T_p\mathbb{S}^3 = p^\perp$ for every $p \in \mathcal{M}$ and $1 \leq i \leq 3$.). And since

$$\dim(T_p\mathbb{S}^3) = 3 \ \forall p \in \mathcal{M}$$

it follows that s_1, s_2 and s_3 are a global basis of sections of $T\mathbb{S}^3$. As we saw in class, this guarantees the triviality of $T\mathbb{S}^3$.

■

Question 10. *Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be differentiable. Fix orientations for the spheres and choose a volume form θ on $\mathbb{S}^2$ such that*

$$\int_{\mathbb{S}^2} \theta = 1$$

Assuming that the de Rham cohomology groups of the spheres are known, show that:

- *there exists $\eta \in \Lambda^1(\mathbb{S}^3)$ such that $\mathrm{d}\eta = f^*(\theta)$.*
- *Defining*

  $$H(f) = \int_{\mathbb{S}^3} \eta \wedge \mathrm{d}\eta$$

  the value of $H(f)$ does not depend on the choice of θ and of η.
- *Compute the value of $H(h)$, where h is the map that defines the Hopf fibration.*

Remark (O.5). We will denote the usual coordinates on $\mathbb{R}^4$ and $\mathbb{R}^3$ by (x, y, z, t) and (x, y, z), respectively (which constitutes a harmless abuse of notation). The map h that defines the Hopf fibration is given by

$$h : \mathbb{S}^3 \to \mathbb{S}^2$$
$$\mathbb{S}^3 \ni p = (x, y, z, t) \mapsto h(p) = (2(xz + yt), 2(yz - xt), x^2 + y^2 - z^2 - t^2) \in \mathbb{S}^2$$

Solution: Consider the following "canonical" volume 2-form on the sphere $\mathbb{S}^2$:

$$\omega = x \cdot \mathrm{d}y \wedge \mathrm{d}z - y \cdot \mathrm{d}x \wedge \mathrm{d}z + z \cdot \mathrm{d}x \wedge \mathrm{d}y$$

We know from calculus that

$$\int_{\mathbb{S}^2} \omega = 4\pi$$

Then defining $\theta \doteq \dfrac{1}{4\pi}\omega$, we have

$$\int_{\mathbb{S}^2} \theta = 1$$

And we may take the orientations of the spheres in question to be the ones corresponding to their "canonical" volume forms. That being so

- Note that $\mathrm{d}\theta = 0$ for dimensional reasons (since $\mathrm{d}\theta$ is a 3-form on $\mathbb{S}^2$), hence θ is a closed form. Since the pullback commutes with the exterior derivative, $f^*(\theta)$ is also a closed 2-form on $\mathbb{S}^3$. Now, since $H^2_{\mathrm{dR}}(\mathbb{S}^3) = 0$ (which happens if, and only if, every closed 2-form on $\mathbb{S}^3$ is also exact), it follows that there exists $\eta \in \Lambda^1(\mathbb{S}^2)$ such that $\mathrm{d}\eta = f^*(\theta)$.

- First, let ξ be another 1-form on $\mathbb{S}^2$ such that $\mathrm{d}\xi = f^*(\theta)$. We then have that

$$\begin{aligned}\mathrm{d}(\xi \wedge \eta) &= \mathrm{d}\xi \wedge \eta + (-1)^1 \cdot \xi \wedge \mathrm{d}\eta \\ &= (-1)^{2\cdot 1} \cdot \eta \wedge \mathrm{d}\xi - \xi \wedge \mathrm{d}\eta \\ &= \eta \wedge f^*(\theta) - \xi \wedge f^*(\theta)\end{aligned}$$

  Now, by Stokes' theorem

$$\int_{\mathbb{S}^3} \eta \wedge f^*(\theta) - \int_{\mathbb{S}^3} \xi \wedge f^*(\theta) = \int_{\mathbb{S}^3} \mathrm{d}(\xi \wedge \eta) = \int_{\partial \mathbb{S}^3 = \emptyset} \xi \wedge \eta = 0$$

  And therefore $H(f)$ does not depend on the choice of η. We will next show that $H(f)$ also does not depend on the choice of θ. So let ζ be another volume form on $\mathbb{S}^2$ that satisfies

$$\int_{\mathbb{S}^2} \zeta = 1$$

  Again, for dimensional reasons both θ and ζ are closed. Since $H^2_{\mathrm{dR}}(\mathbb{S}^2) = \mathbb{R}$, both θ and ζ generate $H^2_{\mathrm{dR}}(\mathbb{S}^2)$. And as we saw in the proof of Poincaré duality, the map

$$H^2_{\mathrm{dR}}(\mathbb{S}^2) \ni [\omega] \mapsto \int_{\mathbb{S}^2} \omega \in \mathbb{R}$$

  is an isomorphism. It then follows from the fact that

$$0 = \int_{\mathbb{S}^2} \theta - \int_{\mathbb{S}^2} \zeta = \int_{\mathbb{S}^2} (\theta - \zeta)$$

  that $[\theta] = [\zeta]$. By definition, we then have that θ and ζ differ by an exact form, that is, there exists $\beta \in \Lambda^1(\mathbb{S}^2)$ such that $\theta - \zeta = \mathrm{d}\beta$. Hence

$$\begin{aligned}\int_{\mathbb{S}^3} \eta \wedge f^*(\theta) - \int_{\mathbb{S}^3} \eta \wedge f^*(\zeta) &= \int_{\mathbb{S}^3} \eta \wedge f^*(\mathrm{d}\beta) \\ &= \int_{\mathbb{S}^3} \{\mathrm{d}\eta \wedge f^*(\beta) - \mathrm{d}(\eta \wedge f^*(\beta))\} \\ &= \int_{\mathbb{S}^3} \mathrm{d}\eta \wedge f^*(\beta) - \underbrace{\int_{\partial \mathbb{S}^3 = \emptyset} \eta \wedge f^*(\beta)}_{=0} \\ &= \int_{\mathbb{S}^3} f^*(\theta) \wedge f^*(\beta) \\ &= \int_{\mathbb{S}^3} f^*(\theta \wedge \beta) \\ &= 0\end{aligned}$$

  where in the last line we used the fact that $\theta \wedge \beta \equiv 0$ is a 3-form on $\mathbb{S}^2$ (and therefore it and its pullback vanish). We then conclude that $H(f)$ does not depend on the choice of θ and of η, as desired.

- We first need to compute $h^*(\theta)$ and find $\eta \in \Lambda^1(\mathbb{S}^3)$ such that $\mathrm{d}\eta = h^*(\theta)$. Since $h^*(\theta)$ is a 2-form on $\mathbb{S}^3$, there exist six smooth real functions $\{a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}\} \subset \mathcal{C}^\infty(\mathbb{S}^3)$ such that

$$h^*(\theta) = a_{12} \cdot \mathrm{d}x \wedge \mathrm{d}y + a_{13} \cdot \mathrm{d}x \wedge \mathrm{d}z + a_{14} \cdot \mathrm{d}x \wedge \mathrm{d}t + a_{23} \cdot \mathrm{d}y \wedge \mathrm{d}z + a_{24} \cdot \mathrm{d}y \wedge \mathrm{d}t + a_{34} \cdot \mathrm{d}z \wedge \mathrm{d}t$$

Evaluating both sides of the equality above at $(\mathbf{e}_i, \mathbf{e}_j)$ for each $1 \leq i, j \leq 3$ (with $i \neq j$, of course), we see that for each $q = (x, y, z, t) \in \mathbb{S}^3$, it holds that

$$a_{ij}(q) = (h^*(\theta))_q(\mathbf{e}_i, \mathbf{e}_j) = \theta_{h(q)}(\mathrm{d}h_q(\mathbf{e}_i), \mathrm{d}h_q(\mathbf{e}_j))$$

We will now then compute the matrix $\mathrm{d}h_q$. We have

$$\begin{aligned}\mathrm{d}h_q &= \begin{pmatrix}\mathrm{d}h_q(\mathbf{e}_1) & \mathrm{d}h_q(\mathbf{e}_2) & \mathrm{d}h_q(\mathbf{e}_3) & \mathrm{d}h_q(\mathbf{e}_4)\end{pmatrix} \\ &= \begin{pmatrix} 2z & 2t & 2x & 2y \\ -2t & 2z & 2y & -2x \\ 2x & 2y & -2z & -2t \end{pmatrix}\end{aligned}$$

Therefore,

$$\begin{aligned}4\pi \cdot a_{12}(q) &= 4\pi \cdot \theta_{h(q)}((2z, -2t, 2x), (2t, 2z, 2y)) \\ &= 2(xz + yt)\begin{vmatrix}-2t & 2x \\ 2z & 2y\end{vmatrix} - 2(yz - xt)\begin{vmatrix}2z & 2x \\ 2t & 2y\end{vmatrix} + (x^2 + y^2 - z^2 - t^2)\begin{vmatrix}2z & -2t \\ 2t & 2z\end{vmatrix} \\ &= 2(xz + yt)(-4yt - 4xz) - 2(yz - xt)(4yz - 4xt) + (x^2 + y^2 - z^2 - t^2)(4z^2 + 4t^2) \\ &= \cdots \text{ we will expand and group the terms next} \\ &= -4t^2(x^2 + y^2 + z^2 + t^2) - 4z^2(x^2 + y^2 + z^2 + t^2) \\ &= -4(t^2 + z^2)\end{aligned}$$

In an entirely analogous way, we obtain

$$\begin{aligned}4\pi \cdot a_{13}(q) &= -4(xt - yz) \\ 4\pi \cdot a_{14}(q) &= 4(xz + yt) \\ 4\pi \cdot a_{23}(q) &= -4(xz + yt) \\ 4\pi \cdot a_{24}(q) &= -4(xt - yz) \\ 4\pi \cdot a_{34}(q) &= -4(x^2 + y^2)\end{aligned}$$

Hence,

$$\begin{aligned}4\pi \cdot h^*(\theta) = &-4(z^2 + t^2) \cdot \mathrm{d}x \wedge \mathrm{d}y - 4(xt - yz) \cdot \mathrm{d}x \wedge \mathrm{d}z + 4(xz + yt) \cdot \mathrm{d}x \wedge \mathrm{d}t \\ &- 4(xz + yt) \cdot \mathrm{d}y \wedge \mathrm{d}z - 4(xt - yz) \cdot \mathrm{d}y \wedge \mathrm{d}t - 4(x^2 + y^2) \cdot \mathrm{d}z \wedge \mathrm{d}t\end{aligned} \tag{2}$$

Now, since

$$x^2 + y^2 + z^2 + t^2 = 1, \ \forall q = (x, y, z, t) \in \mathbb{S}^3$$

we have that

$$x \cdot \mathrm{d}x + y \cdot \mathrm{d}y + z \cdot \mathrm{d}z + t \cdot \mathrm{d}t = 0$$

From this, we obtain

$$\begin{aligned}-4(z^2 + t^2) \cdot \mathrm{d}x \wedge \mathrm{d}y &= -4(1 - x^2 - y^2) \cdot \mathrm{d}x \wedge \mathrm{d}y \\ &= -4 \cdot \mathrm{d}x \wedge \mathrm{d}y + 4x \cdot (x \cdot \mathrm{d}x) \wedge \mathrm{d}y + 4y \cdot \mathrm{d}x \wedge (y \cdot \mathrm{d}y) \\ &= -4 \cdot \mathrm{d}x \wedge \mathrm{d}y + 4x \cdot (-y \cdot \mathrm{d}y - z \cdot \mathrm{d}z - t \cdot \mathrm{d}t) \wedge \mathrm{d}y \\ &\quad + 4y \cdot \mathrm{d}x \wedge (-x \cdot \mathrm{d}x - z \cdot \mathrm{d}z - t \cdot \mathrm{d}t) \\ &= -4 \cdot \mathrm{d}x \wedge \mathrm{d}y + 4xz \cdot \mathrm{d}y \wedge \mathrm{d}z + 4xt \cdot \mathrm{d}y \wedge \mathrm{d}t - 4yz \cdot \mathrm{d}x \wedge \mathrm{d}z - 4yt \cdot \mathrm{d}x \wedge \mathrm{d}t\end{aligned} \tag{3}$$

Analogously, we have

$$\begin{aligned} -4(x^2+y^2)\cdot \mathrm{d}z\wedge \mathrm{d}t &= -4(1-z^2-t^2)\cdot \mathrm{d}z\wedge \mathrm{d}t \\ &= -4\cdot \mathrm{d}z\wedge \mathrm{d}t - 4xz\cdot \mathrm{d}x\wedge \mathrm{d}t - 4yz\cdot \mathrm{d}y\wedge \mathrm{d}t + 4xt\cdot \mathrm{d}x\wedge \mathrm{d}z + 4yt\cdot \mathrm{d}y\wedge \mathrm{d}z \end{aligned} \tag{4}$$

Substituting the expressions (3) and (4) into equation (2), we obtain

$$\begin{aligned} h^*(\theta) &= -\frac{1}{\pi}\cdot(\mathrm{d}x\wedge \mathrm{d}y + \mathrm{d}z\wedge \mathrm{d}t) \\ &= \mathrm{d}\left(-\frac{1}{\pi}\cdot(x\cdot \mathrm{d}y + z\cdot \mathrm{d}t)\right) \end{aligned}$$

We are then justified in taking η to be

$$\eta = -\frac{1}{\pi}\cdot(x\cdot \mathrm{d}y + z\cdot \mathrm{d}t)$$

Now, a direct computation shows that

$$\eta\wedge \mathrm{d}\eta = \frac{1}{\pi^2}\cdot(x\cdot \mathrm{d}y\wedge \mathrm{d}z\wedge \mathrm{d}t + z\cdot \mathrm{d}x\wedge \mathrm{d}y\wedge \mathrm{d}t)$$

Thus,

$$\begin{aligned} H(h) &= \int_{\mathbb{S}^3}\eta\wedge \mathrm{d}\eta \\ &= \frac{1}{\pi^2}\cdot\int_{\mathbb{S}^3}(x\cdot \mathrm{d}y\wedge \mathrm{d}z\wedge \mathrm{d}t + z\cdot \mathrm{d}x\wedge \mathrm{d}y\wedge \mathrm{d}t) \\ &= \frac{2}{\pi^2}\cdot\int_{\mathbb{S}^3} x\cdot \mathrm{d}y\wedge \mathrm{d}z\wedge \mathrm{d}t, \text{ by symmetry} \end{aligned}$$

Note that in the last equality, to formalize the symmetry argument, it suffices to apply the change of variables theorem to the change of coordinates $(x,y,z,t)\mapsto(z,y,x,t)$, whose Jacobian matrix has determinant -1. Using spherical coordinates in the form

$$\begin{aligned} x &= \sin(a)\sin(b)\cos(c) \\ y &= \sin(a)\sin(b)\sin(c) \\ z &= \sin(a)\cos(b) \\ t &= \cos(a) \end{aligned}$$

with $a,b\in[0,\pi]$ and $c\in[0,2\pi]$, we then obtain

$$\begin{aligned} H(h) &= \frac{2}{\pi^2}\cdot\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\sin^4(a)\sin^3(b)\cos^2(c)\ \mathrm{d}c\ \mathrm{d}b\ \mathrm{d}a \\ &= \frac{2}{\pi^2}\cdot\frac{\pi^2}{2} \\ &= 1 \end{aligned}$$

We then conclude that $H(h)=1$.

■

Question 11. *Let $\pi : E \to \mathcal{M}$ be a vector bundle over $\mathcal{M}$ and let $f : \mathcal{N} \to \mathcal{M}$ be a differentiable map between differentiable manifolds. Consider:*

$$f^*(E) = \{(p, u) \in \mathcal{N} \times E \mid f(p) = \pi(u)\}$$

Show that the projection onto the first factor $\pi_1 : f^*(E) \to \mathcal{N}$ defines a vector bundle with total space $f^*(E)$ and base space $\mathcal{N}$ (called the pullback bundle by the map f).

Remark (O.6). For completeness, we will now include the definition of a vector bundle. A vector bundle of rank n over a differentiable manifold is a differentiable manifold E together with a smooth surjective map $\pi : E \to \mathcal{M}$ that satisfies the following conditions

1. For each $p \in \mathcal{M}$, the fiber
$$E_p \doteq \pi^{-1}(\{p\})$$
is a real vector space of dimension n.

2. For each $p \in \mathcal{M}$ there exist a neighborhood $U_p \ni p$ of $p \in \mathcal{M}$ and a diffeomorphism
$$\psi_p : \pi^{-1}(U_p) \to U \times \mathbb{R}^n$$
such that, if $\mathrm{pr}_1 : U_p \times \mathbb{R}^n \to U_p$ denotes the projection onto the first coordinate, then
$$\pi = \mathrm{pr}_1 \circ \psi_p$$

3. For each $q \in U$, the map
$$\psi_p|_{E_q} : E_q \to \{q\} \times \mathbb{R}^n \cong \mathbb{R}^n$$
is an isomorphism of vector spaces.

Remark (O.7). We will denote the map $\psi_p|_{E_q}$ of item (iii) above by $\xi^E_{p,q}$. Moreover, to eliminate possible confusion, we will denote the map $\pi : E \to \mathcal{M}$ by π_E, the bundle $f^*(E)$ by $\widetilde{E}$, the map $\pi_1 : \widetilde{E} \to \mathcal{N}$ by $\pi_{\widetilde{E}}$, and the projection onto the second coordinate by $\mathrm{pr}_2 : U_p \times \mathbb{R}^n \to \mathbb{R}^n$.

Remark (O.8). To prove that a set E admits a bundle structure, there is no need to first prove that E admits a differentiable manifold structure and only afterwards prove that it satisfies conditions (i) to (iii). This is because these conditions already imply the existence of such a structure: indeed, if $\{U_\alpha\}_{\alpha \in I}$ is an atlas of $\mathcal{M}$, the collection $\{\psi_\alpha\}_{\alpha \in I}$,
$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$$
naturally induces an atlas on E (and therefore also induces a topological structure and a differentiable structure on E).

Solution: Suppose that E has rank n over $\mathcal{M}$. First, note that for each $p \in \mathcal{N}$, we have

$$\begin{aligned}
\widetilde{E}_p = \pi^{-1}_{\widetilde{E}}(\{p\}) &= \{p\} \times \{u \in E \mid f(p) = \pi_E(u)\} \\
&= \{p\} \times \pi_E^{-1}(\{f(p)\}) \\
&= \{p\} \times E_{f(p)} \\
&\cong E_{f(p)}
\end{aligned}$$

Hence, it follows from the fact that E is (by hypothesis) a vector bundle of rank n over $\mathcal{M}$ that $\widetilde{E}_p$ is a real vector space of dimension n, which shows that condition (ii) is satisfied.

Now let $p \in \mathcal{N}$ be arbitrary. Consider the neighborhood $\widetilde{U}_p$ of p defined by $\widetilde{U}_p \doteq f^{-1}(U_{f(p)})$, where $U_{f(p)}$ is a neighborhood of $f(p) \in \mathcal{M}$ (note that since f is smooth, and *a fortiori* continuous, $\widetilde{U}_p$ is indeed a neighborhood of p) satisfying hypotheses (ii) and (iii) (whose existence is guaranteed by the hypothesis that E is a vector bundle over $\mathcal{M}$). We then define the map

$$\begin{aligned}\varphi_p : \pi_{\widetilde{E}}^{-1}(\widetilde{U}_p) \subset \widetilde{E} &\to \widetilde{U}_p \times \mathbb{R}^n \\ (q,u) &\mapsto (q, \xi^E_{f(p),f(q)}(u))\end{aligned}$$

By construction, it is evident that $\pi_{\widetilde{E}} = \mathrm{pr}_1 \circ \varphi_p$ (where this equality makes sense, of course). It is also clear (again, by construction) that the inverse of φ_p is given by

$$\begin{aligned}\varphi_p^{-1} : \widetilde{U}_p \times \mathbb{R}^n &\to \pi_{\widetilde{E}}^{-1}(\widetilde{U}_p) \subset \widetilde{E} \\ (q,v) &\mapsto (q, (\xi^E_{f(p),f(q)})^{-1}(v))\end{aligned}$$

Remark (O.9). We recall that a continuous function $F : \mathcal{M}_1 \to \mathcal{M}_2$ between two manifolds of dimensions n and m, respectively, is said to be differentiable at $p \in \mathcal{M}_1$ if there exist a chart (φ, U) around p and a chart (ξ, V) around $F(p)$ such that the composition $\xi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U)$ is differentiable at $\varphi(p)$. When this happens at every $p \in \mathcal{M}_1$, we say that F is differentiable.

Therefore, it is also obvious by construction that φ_p and its inverse are smooth: indeed, given $(q,u) \in \pi_{\widetilde{E}}^{-1}(\widetilde{U}_p)$, φ_p itself is a chart around the point (q,u), and taking a chart x_q around $q \in \mathcal{N}$ (which could without loss of generality be taken as an initial chart around p itself, since differentiability is a local property, and can therefore be verified on arbitrarily small open sets), which induces the chart

$$\widetilde{U}_q \times \mathbb{R}^n \ni (w,v) \mapsto (x_q \times \mathrm{Id})(w,v) = (x_q(w), v) \in \mathbb{R}^{\dim(\mathcal{N})} \times \mathbb{R}^n$$

around $\varphi_p(q)$, the compositions

$$\begin{aligned}&(x_q \times \mathrm{Id}) \circ \varphi_p \circ (\varphi_p)^{-1} \\ &\varphi_p \circ (\varphi_p)^{-1} \circ (x_q \times \mathrm{Id})^{-1}\end{aligned}$$

are trivially differentiable. Alternatively and more simply, we could use the fact that the components of φ_p and its inverse are trivially smooth. We then conclude that φ_p is a diffeomorphism.

Finally, since for each $q \in \widetilde{U}_p$, we have

$$\varphi_p|_{E_q} = \xi_{f(p),f(q)} : E_{f(q)} \to \mathbb{R}^n$$

it follows that $\varphi_p|_{E_q}$ is an isomorphism for each $q \in \widetilde{U}_p$ (again, by the hypothesis that E is a vector bundle over $\mathcal{M}$, which guarantees that $\xi_{f(p),f(q)}$ is an isomorphism). By what we have done so far, it is proved that the projection onto the first factor $\pi_1 : f^*(E) \to \mathcal{N}$ defines a vector bundle with total space $f^*(E)$ and base space $\mathcal{N}$, as desired.

■

Question 12. *Let $\mathcal{M}$ be a compact, connected, orientable differentiable manifold. Show that if the dimension of $\mathcal{M}$ is odd, then the Euler characteristic of $\mathcal{M}$ is zero.*

Solution: Since in this case the de Rham cohomology groups all have finite dimension, it follows from Poincaré duality that

$$\dim(H^k_{\mathrm{dR}}(\mathcal{M})) = \dim(H^{n-k}_{\mathrm{dR}}(\mathcal{M})),\ \forall k \in \{0, \cdots, n\}$$

where we denote $n = \dim(\mathcal{M})$. It follows from the hypothesis that $\dim(\mathcal{M})$ is odd that there exists $m \in \mathbb{Z}$ such that $n = 2m + 1$. Let us call $d_i \doteq \dim(H^i_{\mathrm{dR}}(\mathcal{M}))$ and $a_i = (-1)^i d_i$ for each $i \in \{0, \cdots, n\}$. We know that

$$\chi(\mathcal{M}) = \sum_{i=0}^{2m+1} (-1)^i \dim(H^i_{\mathrm{dR}}(\mathcal{M})) = \sum_{i=0}^{2m+1} a_i = \sum_{i=0}^{m} (a_i + a_{n-i})$$

Note that if $i \in \{0, \cdots, m\}$ is even, we have $a_i = d_i = -a_{n-i}$, whereas if $i \in \{0, \cdots, m\}$ is odd, we have $a_i = -d_i = -a_{n-i}$. Therefore $a_i + a_{n-i} = 0$ for every $i \in \{0, \cdots, m\}$. We then conclude that

$$\chi(\mathcal{M}) = \sum_{i=0}^{m} (a_i + a_{n-i}) = \sum_{i=0}^{m} 0 = 0$$

■