A historical journey

English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio, translated from the author's LaTeX source. The Portuguese original is the authoritative version.

In the words of Carl Sagan, the Cosmos is all that is, or was, or ever will be. Einstein's theory of general relativity models the Cosmos as a pseudo-Riemannian manifold of dimension 4 (with 3 spatial dimensions and 1 temporal one), whose gravitational effects can be explained by the curvature of this manifold, which is commonly called *spacetime*. Explicitly, in this model gravity cannot be regarded as a force in the conventional sense: in fact, it is an intrinsic property of spacetime and is "born" as a result of its local curvature, which in turn reacts to the distribution of mass-energy, momentum and stress of a spatial region via the following equation:

$$\mathrm{Ric} - \frac{1}{2} \cdot \mathrm{Scal} \cdot g + \Lambda g = \frac{8\pi G}{c^4} T \tag{1}$$

where T denotes the energy-momentum tensor of spacetime and the constants Λ, G and c denote, respectively, the *cosmological constant* of the model, Newton's gravitational constant, and the speed of light in vacuum. In fact, the convention of denoting the Riemannian metric by g originated with Einstein and Grossman around 1910-13, who realized that the components $\{g_{\mu\nu}\}_{1\leq\mu,\nu\leq 4}$ determining the geometry of spacetime seemed to depend on the amount of gravitating matter in the region under study. And as seen in [2], there is a precise sense in which the quantities $\{g_{\mu\nu}\}_{1\leq\mu,\nu\leq 4}$ can be regarded as gravitational potentials. Equation (1) is usually presented in local coordinates, in the form

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} \cdot \mathrm{Scal} \cdot g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

which altogether constitute a system of 10 partial differential equations (where the 10 comes from the fact that the tensors involved have 10 independent components, as seen in [1]).

That being so, we may very well ask the following:

Question (P.1). *Given a fibration (the question of which fibration exactly we should consider is more subtle and for simplicity can be neglected, but it is worth mentioning that the existence of the cosmic microwave background radiation and the Copernican principle in a certain sense imply the existence of a canonical fibration) of the Cosmos by three-dimensional "time slices" (all isometric to each other, one such fiber being called the Universe), what are all the possible topologies and geometries of such slices?*

In other words, if $\mathcal{M}^3$ is a three-dimensional topological manifold, do there exist a finite set $\mathcal{F}$ of equivalence classes of manifolds (where the equivalence relation is determined by declaring two manifolds equivalent when they are homeomorphic), a Riemannian metric $g \in \Gamma(\mathrm{Sym}_2^+ \mathcal{M}^3)$ and a finite set of equivalence classes of Riemannian metrics $\mathcal{G}$ (where in this case the equivalence relation is determined by declaring two metrics equivalent when they are isometric) such that $\left[\mathcal{M}^3\right] \in \mathcal{F}$, $[g] \in \mathcal{G}$? We will see that the answer is negative, but we will manage to get reasonably close. If we do not assume that $\mathcal{M}^3$ is compact, the problem is far too difficult: for example, McMillan proved in [3] that there exists an uncountable set $\mathfrak{F}$ of open contractible subsets of $\mathbb{R}^3$ such that $[X] \neq [Y]$ whatever $X \neq Y \in \mathfrak{F}$ may be. For simplicity, we will therefore assume that $\mathcal{M}^3$ is compact and without boundary.

A naive first impression is that speculative questions of this kind, which arise purely from mathematical abstraction, cannot possibly have real, observable physical consequences in our own universe. However, the

very study of the classical differential geometry of surfaces strongly indicates that this is *not* the case. For example, we unconsciously use Gauss's Theorema Egregium every time we bend a book in order to be able to read it while holding it with only one hand, or when we fold a slice of pizza so that we can eat it without it falling. *Curvature creates strength* is one of the physical principles imposed by Gauss's Theorema Egregium that both engineers around the world and nature itself use in their constructions. See, for example, the shape of nuclear power plants or of blades of grass.

Of course, the relativistic model of the Cosmos is not a completely faithful portrait of reality. The Cosmos may be something much more complex than a pseudo-Riemannian manifold, and in the distant future it may very well happen that some mathematician or physicist discovers more general spaces that have manifolds as we know them today as particular cases and model it more precisely than the relativistic model. However, this does not constitute any great loss; after all, the very model of the Earth as the round sphere $\mathbb{S}^2$ is not completely accurate either: the Earth is flattened at the poles, and the existence of thousands of natural arches also shows that its genus is at least of the order of 10^3.

Note further that according to the theory of relativity, the concept of "time slices" is not a priori well defined: after all, the very concept of simultaneity is not absolute and depends on the frame of reference of each observer. For example, we could ask: with respect to which observer is the age of the universe (known to be $\approx 14 \times 10^9$ years) commonly measured? Fortunately, the use of *comoving coordinates* takes care of this problem and makes such concepts well posed. In particular, the age of the universe is measured with respect to an observer with zero comoving velocity (any such observer perceives the whole universe as isotropic; in fact, only such observers perceive it this way), that is, $\approx 14 \times 10^9$ years is the time elapsed since the Big Bang according to the clock of a comoving observer.

Before delving into the classification problem in dimension 3, we can analyze what happens in the lower-dimensional cases, that is: what are all the possible topologies and geometries of worlds (compact one/two-dimensional manifolds without boundary) of dimension ≤ 2? This question is fundamental and was, in a certain sense, contemplated by the Greeks more than two thousand two hundred years ago; that is where the origin of the word "geometry" comes from, which means "to measure" the Earth. Of course, today it is possible to board a spaceship and directly observe the round shape of the Earth. The ancient Greeks, however, did not have this option (and obviously, we will also never have such an option for observing the topology of spacetime, so that intrinsic methods become absolutely necessary). Thales of Miletus believed that the Earth was a flat disk floating on a giant ocean, while Anaximander, a disciple of Thales, thought that the Earth had the shape of a cylinder, with the continents located on one of its circular faces. However, it was Aristarchus of Samos, a Greek astronomer and mathematician who lived between 310 BC and 230 BC, in his fantastic work *On the Sizes and Distances of the Sun and Moon*, who paved the way for Eratosthenes, another great Greek scientist, to attack this problem with great success by, among other discoveries, observing that the Sun's rays were parallel.

Eratosthenes was director of the library of Alexandria, one of the greatest centers of knowledge of Antiquity. Through one of the library's manuscripts, he learned of a remarkable phenomenon that occurred at the Summer Solstice of June 21: in the morning, in Syene (present-day Aswan), the shadows of the columns of the temples (or of a rod fixed vertically in the ground, for example) grew shorter and shorter as noon approached. Precisely at the arrival of noon, these columns cast no shadow at all; analogously, a deep water well, which on the other days of the year was entirely covered by shadow, was completely lit by the Sun, which shone clearly on the water of the well (this instant, at which the sun is exactly at the "top" of the sky, is called the solar zenith).

This observation, which at first might seem insignificant, proved very fruitful: Eratosthenes had the shrewdness to ask himself whether the same phenomenon also happened simultaneously in Alexandria. He found that it did not: on the same day, in Alexandria, temple columns cast their usual long shadows. Eratosthenes asked himself how it could happen that, at the same instant, a rod in Syene cast no shadow at all while, 800 kilometers to the north, in Alexandria, a rod cast a very definite shadow. On a flat Earth

at a sufficiently large distance from the Sun (a hypothesis in fact attested by Aristarchus of Samos), the presence or absence of shadows would necessarily have to be seen simultaneously at any places. Consequently, the substantial presence of shadows in Alexandria together with their absence in Syene demonstrated the curvature of the Earth.

Moreover, the greater the curvature, the greater the difference in the length of the shadows: for, as had already been observed by Aristarchus of Samos, the Sun's rays are parallel, and therefore rods whose angles with the Sun are distinct will cast shadows of different lengths. It was therefore reasonable to suppose the sphericity of the Earth. Armed with this hypothesis and using trigonometry, Eratosthenes calculated that the observed difference in the length of the shadows implied that the distance between Alexandria and Syene corresponded to approximately seven degrees along the surface of the Earth (that is, imagining that the rods in Alexandria and Syene were extended to the center of the Earth, they would intersect at an angle of approximately seven degrees). Since seven degrees corresponds to approximately one fiftieth of a whole circle, Eratosthenes estimated the circumference of the Earth as fifty times the distance between Alexandria and Syene, which at the time was estimated at five thousand stadia, and therefore the circumference of the Earth was approximately two hundred and fifty thousand stadia. The precise measure of a stadion is unknown to this day (ranging from 157 to 209 meters), but in any case, the precision of Eratosthenes' calculation for his time is incredible, and attests to the power of human ingenuity.

Erroneously, Eratosthenes became known as the first person to prove the sphericity of the Earth: in fact, the success of his experiment relied heavily on the a priori assumption of this sphericity, and the only definitive conclusion possible besides the diameter of the Earth was being able to rule out a possible flat Earth. Other shapes (i.e. *topologies*) were still possible: it could happen that the world extended indefinitely, or even that it had an end (i.e. a topological *boundary*). These hypotheses were ruled out by the first circumnavigation of the globe, made by Ferdinand Magellan in the 16th century, but even so other non-spherical shapes still could not be ruled out, since not all regions of the world, including the poles, had been mapped. The maps of Battista Agnese were reasonably consistent with a toroidal Earth, for example (for details, see [4]), and we know today that a toroidal planet is not physically impossible, even though none has been discovered.

Having made this digression, let us now return to the problem from an entirely mathematical perspective: what are all the possible topologies of a closed Riemannian surface, and what is their relation to geometry? To approach these problems, it will be useful to explore the following concepts first:

- *the connected sum* of two closed manifolds: given two closed manifolds $\mathcal{M}$ and $\mathcal{N}$, there is a way of constructing a new manifold called the *connected sum* of $\mathcal{M}$ and $\mathcal{N}$, which we will denote by $\mathcal{M}\sharp\mathcal{N}$. We choose two balls V_1, V_2, in $\mathcal{M}$ and $\mathcal{N}$ respectively, small enough to be homeomorphic to balls. Next we remove the interior of these two balls, resulting in two boundaries ∂V_1 and ∂V_2 homeomorphic to $\mathbb{S}^n$. By gluing these sets and identifying these boundaries via an orientation-reversing diffeomorphism $\psi : V_1 \to V_2$, we obtain a new connected manifold. Formally,

$$\mathcal{M}\sharp\mathcal{N} = \frac{(\mathcal{M} \setminus \operatorname{int}(V_1)) \sqcup (\mathcal{N} \setminus \operatorname{int}(V_2))}{p \in \partial V_1 \sim \psi(p) \in \partial V_2}$$

  One can prove that this operation is well defined, is compatible with the orientation of each factor, is associative and commutative, in general has no inverse and has the sphere as its neutral element. Some useful visualizations are the following

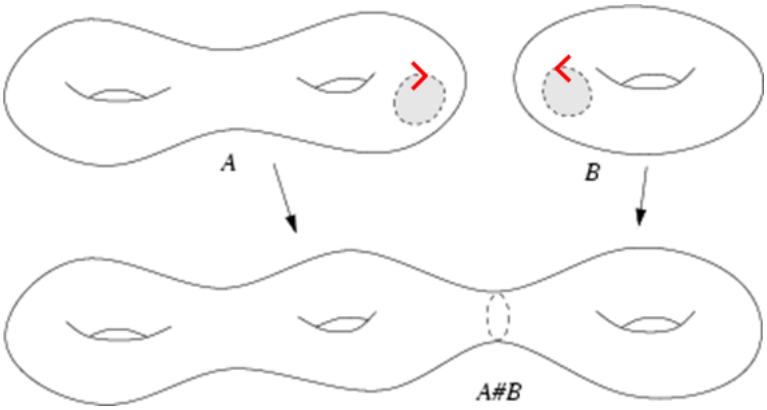


Figure 1: the connected sum of a torus with a double torus

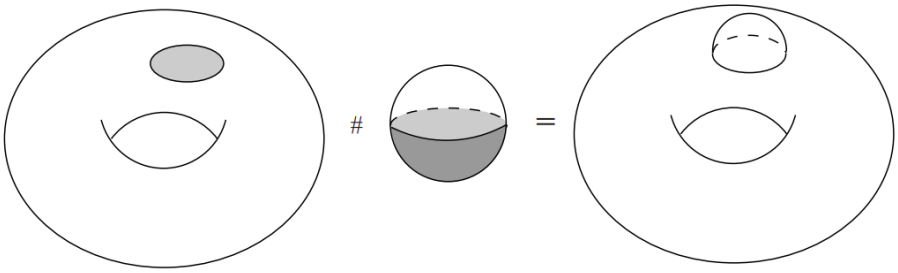


Figure 2: the connected sum of a torus with a sphere

- *the polygonal representation* of a compact surface: a polygonal representation of a surface is a polygon in which each side has a label and a direction (commonly represented as an arrow). When two sides have the same label, this indicates that they must be "glued" using a homeomorphism compatible with the direction of the corresponding arrows. For example,

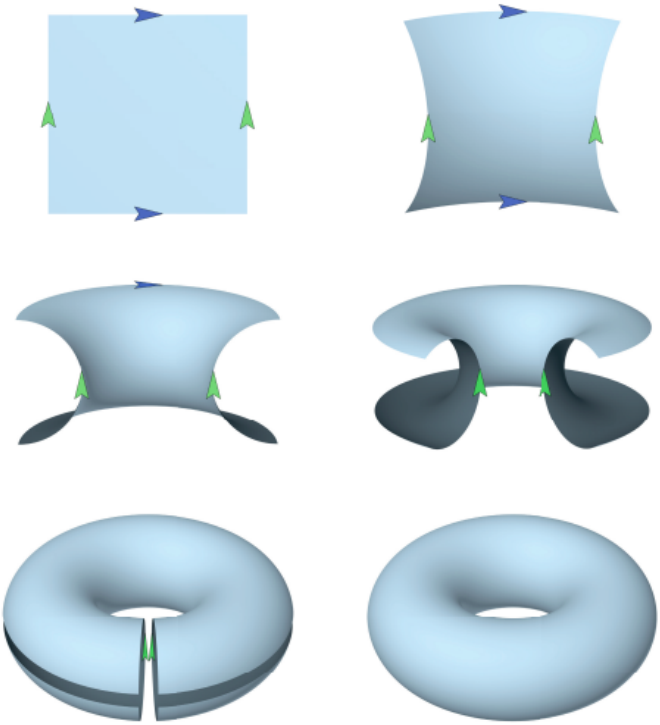


Figure 3: the polygonal representation of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$

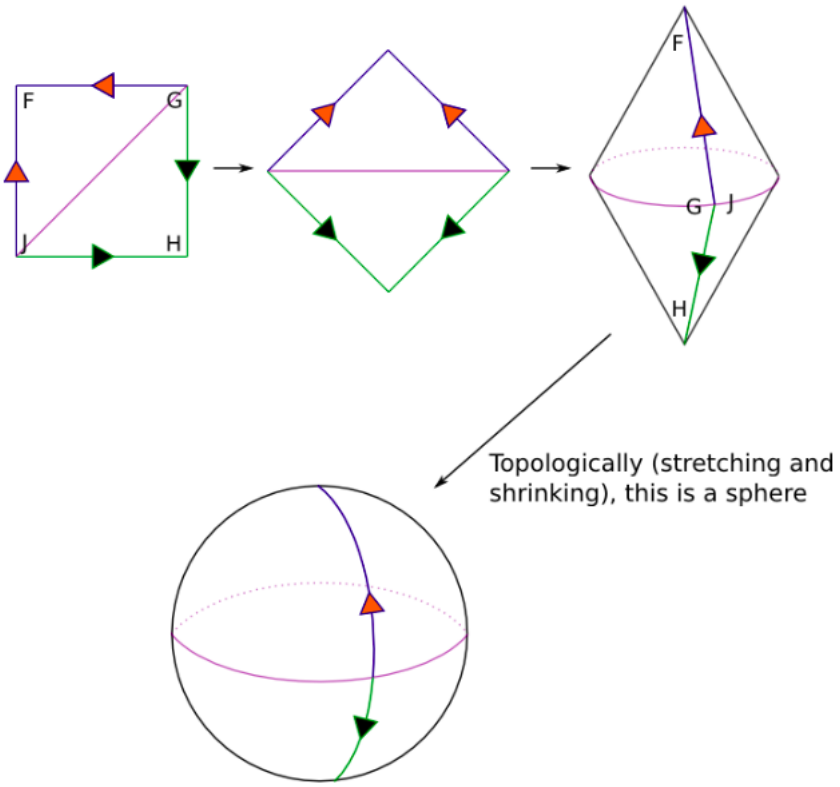


Figure 4: the polygonal representation of the sphere $\mathbb{S}^2$

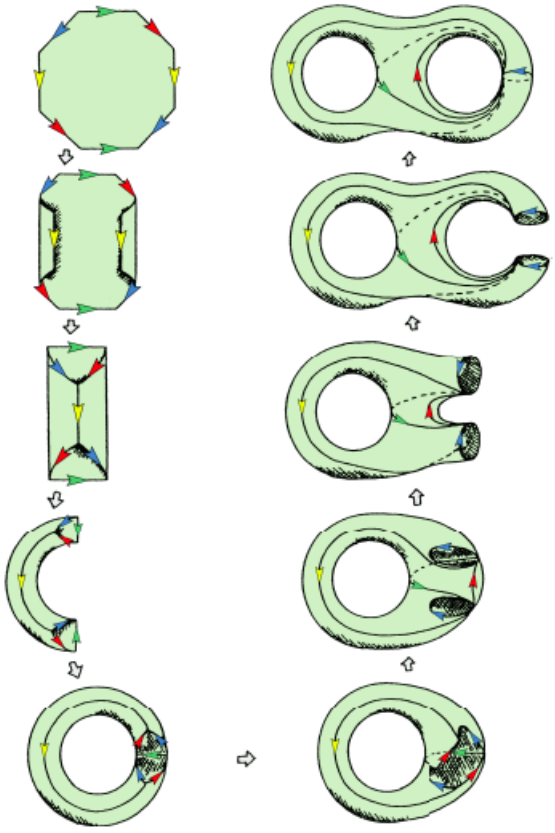


Figure 5: the polygonal representation of the double torus $\mathbb{T}^2 \sharp \mathbb{T}^2$

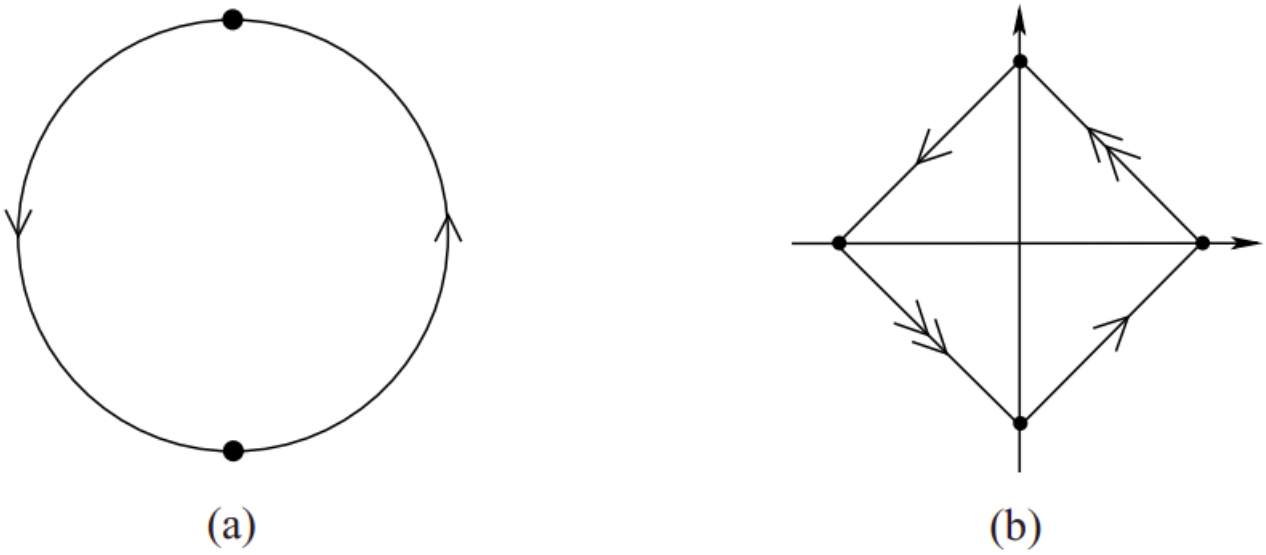


Figure 6: polygonal representations of the projective plane $\mathbb{RP}^2$

It is possible to show that every compact surface admits a polygonal representation. In general, we have the following:

Theorem (T.1). *Every connected and closed two-dimensional manifold $\mathcal{M}^2$ is homeomorphic to exactly one of the following spaces:*

- *the sphere $\mathbb{S}^2$*
- *the finite connected sum of one or more copies of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$*
- *the finite connected sum of one or more copies of the projective plane $\mathbb{RP}^2$*

It is hard to overestimate the importance of Theorem 1. In the words of V.I. Arnold (see [5]), one of the greatest mathematicians of the last century,

> The theorem of classification of surfaces is a top-class mathematical achievement, comparable with the discovery of America or X-rays. This is a genuine discovery of mathematical natural science and it is even difficult to say whether the fact itself is more attributable to physics or to mathematics. In its significance for both the applications and the development of correct Weltanschauung it by far surpasses such "achievements" of mathematics as the proof of Fermat's last theorem or the proof of the fact that any sufficiently large whole number can be represented as a sum of three prime numbers.

In dimension 2, geometry and topology determine each other through the following direct consequences of the uniformization theorem and of the Gauss-Bonnet theorem, stated below:

Theorem (T.2). *Every closed surface $\mathcal{M}^2$ admits a Riemannian metric of constant sectional curvature.*

Theorem (T.3). *The total curvature of a two-dimensional manifold $\mathcal{M}^2$ satisfies:*

$$\int_{\mathcal{M}^2} K \, \mathrm{d}A = 2\pi \cdot \chi(\mathcal{M}^2)$$

Consequently, the Euler characteristic and orientability are topological invariants that completely determine the topology of a closed surface. Those of genus 0 admit spherical geometry, and those of genus 1 the flat one (the torus, for example, admits a metric of zero curvature obtained by "descending" the metric of the covering $\mathbb{R}^2$ to the torus $\mathbb{T}^2$). All the others, of genus $g > 1$, are modeled by hyperbolic geometry, which is in this sense abundant, and unfortunately impossible to visualize faithfully, due to Hilbert's theorem. The construction of such hyperbolic metrics arises from a polygon with $4g$ sides with appropriate identifications, where all the vertices are identified with a single point. It is therefore necessary that the sum of the interior angles of such a polygon be equal to 2π. Note that the angles of Euclidean geometry are too "fat" to generate any hyperbolic metric: indeed,

$$(4g-2)\pi > 2\pi, \ \forall g \geq 2$$

On the other hand, in the Poincaré disk, the sum of the interior angles of a hyperbolic polygon is given by

$$(4g-2)\pi - A$$

where A denotes the area of the polygon. Therefore there exist hyperbolic polygons with interior angles $\frac{2\pi}{4g}$, so that every closed surface of genus $g > 1$ admits a hyperbolic metric.

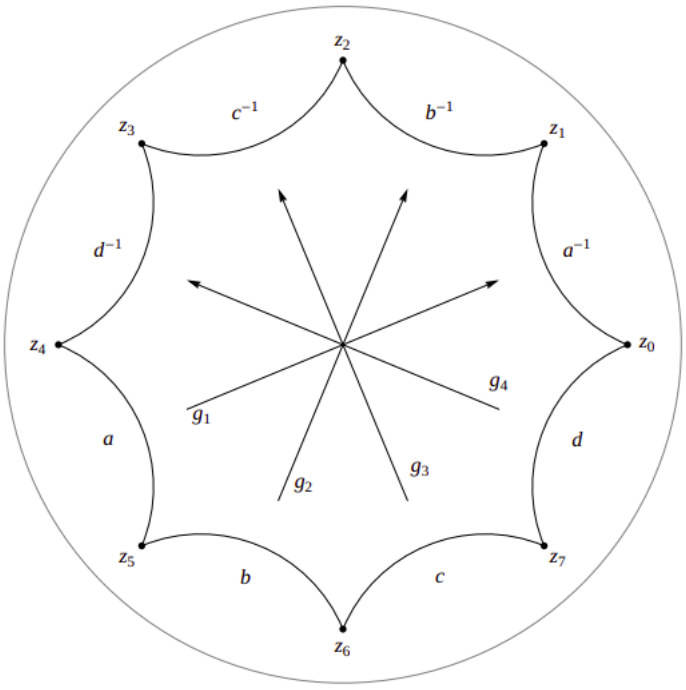


Figure 7: hyperbolic octagon with labeled identifications generating the double torus with hyperbolic geometry

In dimension 1, the situation is considerably simpler: any (connected) manifold of dimension 1 without boundary is diffeomorphic either to the circle $\mathbb{S}^1$ or to the real line $\mathbb{R}$. The presence of boundary introduces the additional cases $[0,1]$ and $[0,1)$. Compactness and the presence of boundary are therefore topological invariants that together completely determine manifolds of dimension 1. Moreover, no one-dimensional Riemannian manifold has intrinsic geometry: by virtue of the definition of the curvature tensor, in dimension 1 the curvature is always identically zero, so that any one-dimensional Riemannian manifold is locally isometric to $\mathbb{R}$ (but of course: an antisymmetric tensor in dimension 1 is necessarily identically zero!).

The situation in dimension 3 becomes extremely interesting: before Einstein's incredible theory of relativity became established, it was common to think that three-dimensional Euclidean geometry modeled the Universe with complete fidelity. In the words of the fantastic physicist and mathematician Roger Penrose in his book [6] (another incredible work by the same author is [7]),

> The fact that Euclidean geometry seems so accurately to reflect the structure of the "space" of our world has fooled us (or our ancestors!) into thinking that this geometry is a logical necessity, or into thinking that we have an innate *a priori* intuitive grasp that Euclidean geometry must apply to the world in which we live. (Even the great philosopher Immanuel Kant claimed this.) This real break with Euclidean geometry only came with Einstein's general relativity, which was put forward many years later. Far from Euclidean geometry being a logical necessity, it is an *empirical observational fact* that this geometry applies so accurately – though not quite exactly – to the structure of our physical space! Euclidean geometry was indeed, all along, a (SUPERB) *physical theory*. This was in addition to its being an elegant and logical piece of pure mathematics.

One of the countless apocryphal stories about Gauss also portrays him measuring the angles of a large triangle formed by the mountain peaks of Hohenhagen, Inselberg and Brocken, in an effort to obtain evidence that the geometry of space is non-Euclidean. However, although Gauss was in fact involved in the mapping of the Kingdom of Hanover between 1818 and 1832 (having worked with some "test triangles" formed by mountain peaks), a deviation from Euclidean geometry large enough to be noticeable on the scale of the Earth would imply enormous distortions on an astronomical scale, which would have been noticed long before Gauss. On the other hand, this idea was in fact very promising: in 1829, Lobachevsky, a contemporary of Gauss, proposed an analogous test, carried out by considering the parallax of the stars Eridan 29, Rigel and Sirius. Using a parallax value of 1.24 arcseconds for Sirius, he concluded that the sum of the interior angles of a triangle formed by the sun, the Earth and Sirius deviated from the Euclidean value of π radians by at most 0.00000372 arcseconds. Although this strongly suggested that space could be Euclidean, Lobachevsky refrained from any definitive conclusion, realizing that, although in principle there was the possibility of proving the Universe to be non-Euclidean, one could never definitively prove it to be Euclidean (one could always argue that the area of space used in such an experiment was too small).

In [8], Bernhard Riemann also proposed the possibility of the Universe being modeled by the three-dimensional sphere $\mathbb{S}^3$,

> In the extension of space-construction to the infinitely great, we must distinguish between *unboundedness* and *infinite extent*, the former belongs to the extent relations, the latter to the measure-relations. That space is an unbounded three-fold manifoldness, is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought object are constructed, and which by these applications is for ever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means follows from this; on the other hand if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value. If we prolong all the geodesics starting in a given-surface element, we should obtain an unbounded surface of constant curvature, *i.e.,* a surface which in a *flat* manifoldness of three dimensions would take the form a sphere, and consequently be finite.

In [9], Poincaré himself imagined the perspective of inhabitants of a hyperbolic Universe $\mathbb{H}^3$:

> Suppose, for example, a world enclosed in a large sphere and subject to the following laws: The temperature is not uniform; it is greatest at the centre, and gradually decreases as we move towards the circumference of the sphere, where it is absolute zero. The law of this temperature is as follows: if R be the radius of the sphere, and r the distance of the point considered from the centre, the absolute temperature will be proportional to $R^2 - r^2$. Further, I shall suppose that in this world all bodies have the same coefficient of dilation so that the linear dilatation of any body is proportional to its absolute temperature. Finally, I shall suppose that a body transported from one point to another of different temperature is instantaneously in thermal equilibrium with its

> new environment. There is nothing in these hypotheses either contradictory or unimaginable. A moving object will become smaller and smaller as it approaches the circumference of the sphere. Let us observe, in the first place that although from the point of view of our ordinary geometry this world is finite, to its inhabitants it will appear infinite. As they approach the surface of the sphere they become colder, and at the same time smaller and smaller. The steps they take are therefore also smaller and smaller, so that they can never reach the boundary of the sphere. If to us geometry is only the study of laws according to which invariable solids move, to these imaginary beings it will be the study of laws of motion *deformed by the differences of temperature alluded to....*
>
> Let us make another hypothesis: suppose that light passes through media of different refractive indices, such that the index of refraction is inversely proportional to $R^2 - r^2$. Under these conditions it is clear that the rays of light will no longer be rectilinear, but circular.... If they [the beings in such a world] construct a geometry, it will not be like ours, which is the study of movements of our invariable solids; it will be the study of the changes of position which they will have thus distinguished, and will be "non-Euclidean displacements," and *this will be non-Euclidean geometry.* So that beings like ourselves, educated in such a world, will not have the same geometry as ours.

The works of Gauss and Riemann bore extraordinary fruit: in 1919, more than two thousand years after Eratosthenes' primordial experiment, British astronomers led by Arthur Stanley Eddington tested Einstein's theory of relativity and empirically observed the non-Euclidean nature of the Cosmos. They measured the angle subtended by two stars and the Earth itself, once when the Sun was not blocking the path of the light from these stars to the Earth and once when the path of the light from one of the stars passed very close to the Sun (now having to wait for a total solar eclipse, which was observed both from the island of Príncipe and on Brazilian soil, in Sobral). The difference between the two measurements was exactly what Einstein's theory predicted, which does not happen because light is somehow distorted (in fact, it follows intrinsically straight paths: geodesics!), but because the enormous mass of the Sun distorts the geometry of the Cosmos in its surroundings, precisely as Einstein predicted.

These considerations show a little of the grandeur of a possible result analogous to 1 for dimension 3, which would potentially answer the following

Question (P.2). *What are all the closed* 3*-manifolds?*

In the search for the answer, an immediate initial obstacle arises: the Euler characteristic is a useless topological invariant in this case; indeed, a direct consequence of Poincaré duality is that the Euler characteristic of any closed manifold of odd dimension is always zero. In the search for topological invariants in higher dimensions, Poincaré practically founded topology as we know it today, in a series of six papers published in renowned journals. Initially, he showed that knowledge of the Betti numbers did not determine the topology of a manifold, exhibiting an infinite family of closed three-dimensional manifolds that are not homeomorphic to each other, but with identical Betti numbers, equal to the Betti numbers of $\mathbb{S}^3$. Some of his first questions were the following:

> It would be interesting to investigate the following questions:
>
> - Given a group G defined by its generators and relations, can it be the fundamental group of an n-dimensional manifold?
> - How can such a manifold be formed?
> - Are two manifolds with the same dimension and the same fundamental group always homeomorphic?

These questions involve difficult studies and long developments, of which I will not speak here.

Poincaré tried to answer a very weakened version of Question **(P.2)**: instead of trying to characterize all closed three-dimensional manifolds, he initially tried to characterize the simplest closed three-dimensional manifold of all, the sphere $\mathbb{S}^3$. In his first initial effort in this direction, he went too far by asserting that any closed three-dimensional manifold with "the homology of $\mathbb{S}^3$" (*id est* having the same homology groups as $\mathbb{S}^3$) was homeomorphic to $\mathbb{S}^3$, having even asserted that he had a proof of this claim.

Poincaré himself discovered the first counterexample to his claim, having in the process invented the concept of the fundamental group. He described the counterexample as a pair of solid double tori glued in an appropriate way (a *Heegaard diagram*). The resulting space had not only all the same homology groups as $\mathbb{S}^3$, but also a nontrivial fundamental group (and therefore could not be homeomorphic to $\mathbb{S}^3$). Today, this manifold is known as the Poincaré *dodecahedron*:

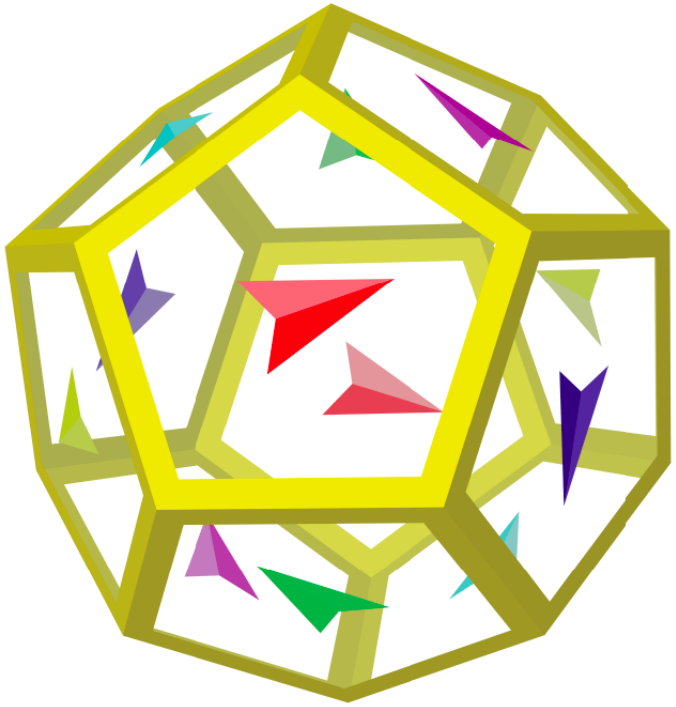


Figure 8: visualization of the Poincaré dodecahedron: glue opposite faces after a tenth of a turn counter-clockwise $\left(\text{rotation by } \frac{2\pi}{10} = \frac{\pi}{5} \text{ radians}\right)$

The description of this space as a dodecahedron with opposite faces identified was not introduced by Poincaré, but rather by C.Weber and H.Seifert in [10]. Recently, much consideration has been given to this manifold as a possible model for the shape/topology of the Universe itself (see, for example, [11]). With this identification of the faces, the twenty corners of the Euclidean dodecahedron come together in five groups of four, so that their interior angles are too "thin" to fit together properly. However, analogously to what we did to uniformize the geometry of a double torus, we can consider a "spherical" dodecahedron to fix this problem, as sketched in the figure below

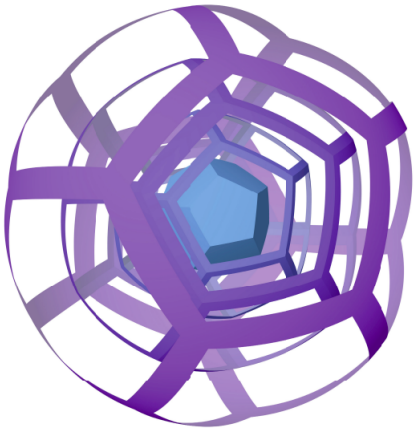


Figure 9: spherical geometry of the Poincaré dodecahedron: we expand the dodecahedron immersed in $\mathbb{S}^3$ until its corners have the right size to fit together in groups of four

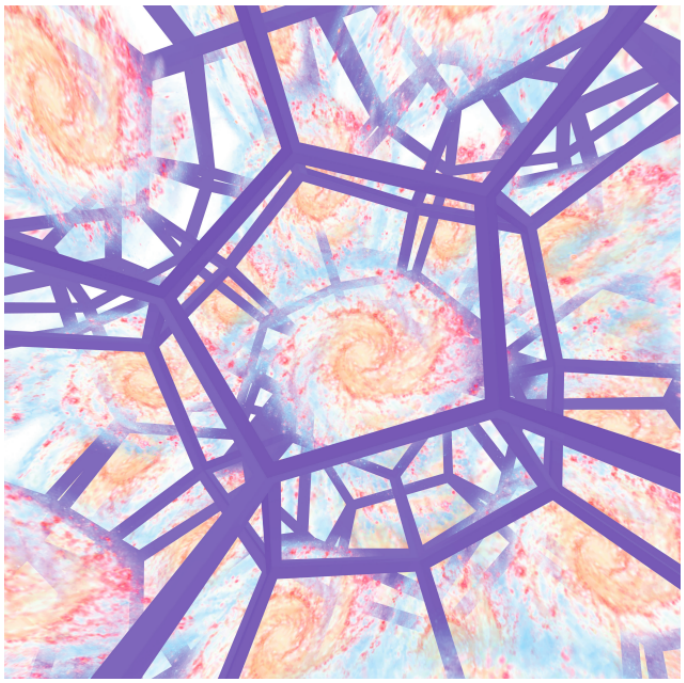


Figure 10: view inside a one-galaxy universe modeled by the Poincaré dodecahedron

Poincaré concluded the paper in which he presented his counterexample as follows:

> There are therefore two loops in the manifold that are not equivalent to a point; therefore the manifold cannot be homeomorphic to a sphere.
>
> In other words... **[He states the result again, explaining that the fundamental group is not trivial].**
>
> There still remains one question to deal with: is it possible that the fundamental group of a three-dimensional manifold could be trivial, but that the manifold not be homeomorphic to the three-dimensional sphere $\mathbb{S}^3$ **[This is the Poincaré conjecture!]**
>
> In other words... **[He carefully explains, in terms of the example constructed, what would need to happen to obtain such a manifold].**
>
> But this question would take us too far.

The negative answer to the conjecture only came much later. In dimension 3, we still have the three model geometries of constant curvature, namely $\mathbb{S}^3$, $\mathbb{R}^3$ and $\mathbb{H}^3$. But there is no hope of a classification as good as in dimension 2: indeed, there are infinitely many three-dimensional manifolds that admit no metric of constant sectional curvature (consider, for example, $\mathbb{S}^2 \times \mathbb{S}^1$, whose universal cover is $\mathbb{R}$). Five other model geometries arise from Cartesian or "twisted" products of lower-dimensional geometries:

- the product $\mathbb{S}^2 \times \mathbb{R}$
- the product $\mathbb{H}^2 \times \mathbb{R}$
- the universal cover $\widetilde{SL}(2, \mathbb{R})$ (a twisted bundle over $\mathbb{H}^2$)
- the Heisenberg group (a twisted bundle over $\mathbb{R}^2$); and
- the Sol manifold (a twisted $\mathbb{T}^2$-bundle over $\mathbb{S}^1$)

Thurston's geometrization conjecture states that every closed 3-manifold can be appropriately decomposed in such a way that each piece of the decomposition admits one of the geometries mentioned above. More precisely,

Thurston's geometrization conjecture. *Let $\mathcal{M}^3$ be a closed, orientable and prime manifold (id est, it cannot be expressed as a nontrivial connected sum, that is, with none of the factors being a sphere, of two three-dimensional manifolds). Then there exists an embedding of a disjoint union of tori and Klein bottles $\cup_i T_i^2 \subset \mathcal{M}$ such that each component of the complement admits a locally homogeneous Riemannian metric of finite volume.*

We will say that a closed surface $S \subset \mathcal{M}^3$ of genus $g \geq 1$ is incompressible if there exists an injection of its fundamental group $\pi_1(S)$ into the fundamental group $\pi_1(\mathcal{M}^3)$ of $\mathcal{M}^3$. One can prove that tori and Klein bottles are incompressible. A reformulation of the conjecture is then that there exists a decomposition of $\mathcal{M}^3$ along incompressible tori and Klein bottles into pieces whose interiors admit locally homogeneous metrics of finite volume. Since closed 3-manifolds with finite fundamental group have no incompressible tori or Klein bottles, in such cases this decomposition is trivial. And since the only compact model geometry is $\mathbb{S}^3$, we conclude that if $\pi_1(\mathcal{M}^3)$ is finite then the universal cover of $\mathcal{M}^3$ is the sphere $\mathbb{S}^3$. *A fortiori,*

$$\text{Thurston's geometrization} \implies \text{Poincaré conjecture}$$

Thurston verified that a large class of manifolds, called *Haken manifolds*, satisfied his conjecture. This work was extremely important. In the words of John Morgan,

> From my perspective, before Thurston's work on hyperbolic 3-manifolds and his formalization of the Geometrization Conjecture, there was no consensus among the experts as to the validity of the Poincaré conjecture. After Thurston's work (notwithstanding the fact that it had no direct bearing on the Poincaré Conjecture), a consensus developed that both the Poincaré Conjecture and the Geometrization Conjecture were true.

Somewhat roughly, the classification/geometrization problem for closed manifolds can be summarized as follows:

- Dimension 1: any (connected) manifold of dimension 1 without boundary is diffeomorphic either to the circle $\mathbb{S}^1$ or to the real line $\mathbb{R}$. The presence of boundary introduces the additional cases $[0, 1]$ and $[0, 1)$. Compactness and the presence of boundary are therefore topological invariants that together completely determine the topology of manifolds of dimension 1. Moreover, the geometrization problem in dimension 1 is trivial: there is no intrinsic geometric invariant that distinguishes any two manifolds of dimension 1.

- Dimension 2: Every closed surface of genus g admits a metric of constant Gaussian curvature K. Moreover, $K > 0$ if and only if $g = 0$, $K = 0$ if and only if $g = 1$, and $K < 0$ if and only if $g \geq 2$.

- Dimension 3: solved by Perelman through his positive resolution of Thurston's geometrization conjecture.

- Dimension 4: it is not possible to obtain anything as comprehensive as in lower dimensions. Indeed, since every finitely presented group is the fundamental group of some four-dimensional manifold (for a proof of this fact, see [15]), classifying closed four-dimensional manifolds is in a certain sense as difficult as classifying such groups. This problem is known as the word problem for groups, and has been shown to be undecidable.

A closed three-dimensional manifold cannot be immersed in a Euclidean space (in fact, it is simple to prove that there is no immersion of any closed manifold into another noncompact manifold of the same dimension), and therefore faithful visualizations of them are impossible. Even so, the efforts in this direction presented in the fantastic book [12], some of which we present next, are quite satisfactory.

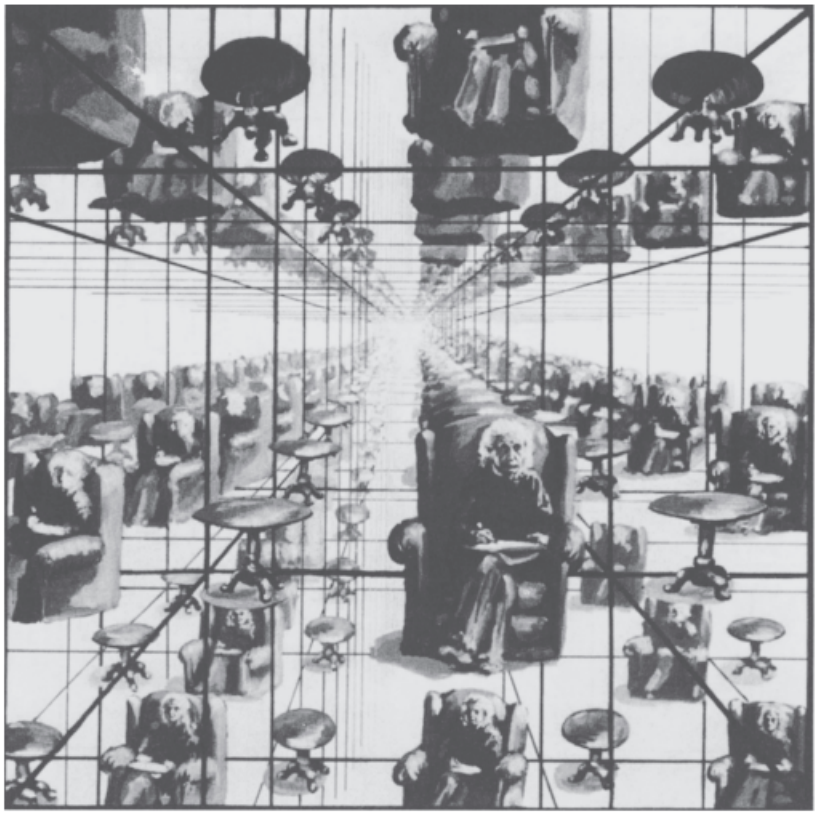


Figure 11: artistic drawing of the view in a universe modeled by the torus $\mathbb{T}^3$

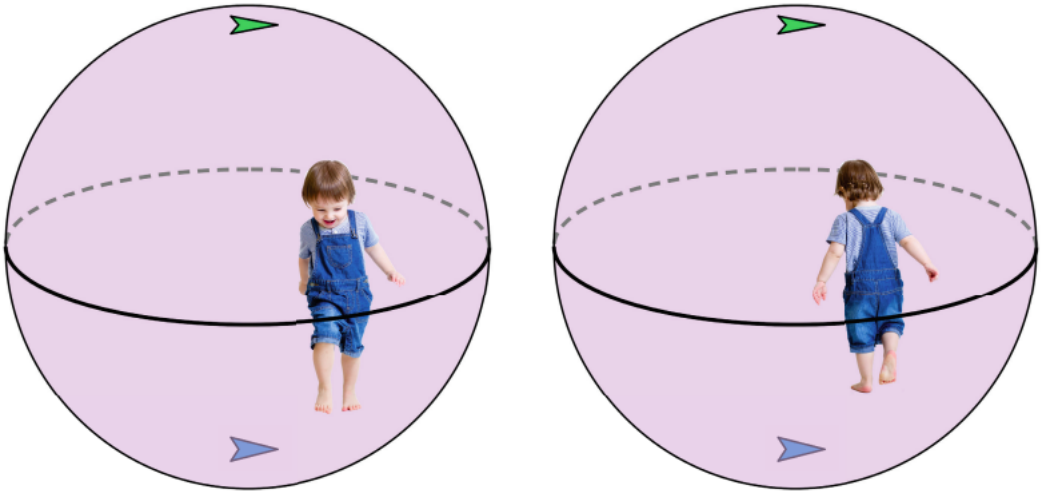


Figure 12: visualization of $\mathbb{S}^3$ as two balls glued along the boundary $\mathbb{S}^2$

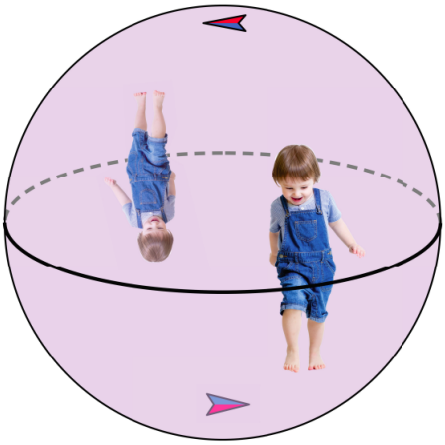


Figure 13: visualization of $\mathbb{RP}^3$ as the sphere $\mathbb{S}^2$ with antipodal points identified

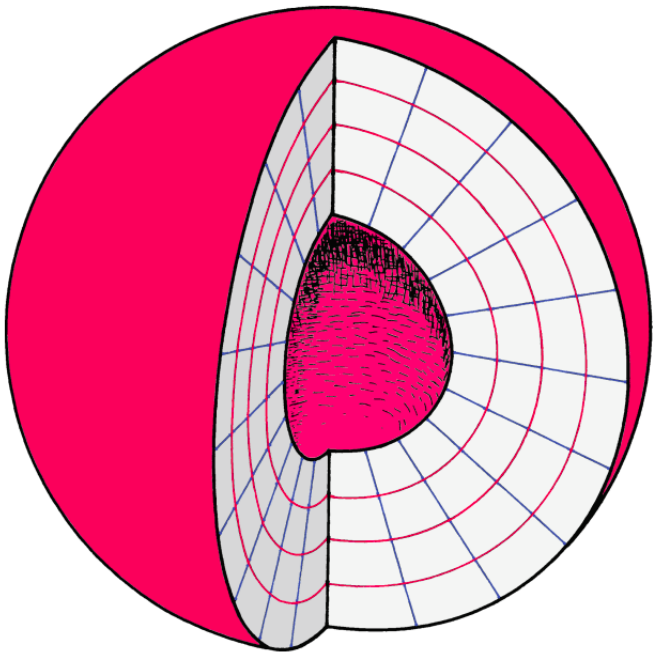


Figure 14: visualization of $\mathbb{S}^2 \times [0,1]$ as a ball with a hollow center. To imagine $\mathbb{S}^2 \times \mathbb{S}^1$, we glue the inner spherical boundary to the outer spherical boundary, pretending (in contrast with a four-dimensional being, who would not have to pretend) that the various spherical layers of the shell seen above all have the same size.

At first, a naive guess of how to recognize the topology of the Universe would suggest using telescopes to look for copies of our galaxy in the sky. If we saw such copies, we could determine the "gluings" of the fundamental domain of the Universe and therefore its topology (the copies seen in a Universe modeled by the torus $\mathbb{T}^3$ would be different from those seen in a universe modeled by the Poincaré dodecahedron, for example). And what if, for example, one of the many very distant galaxies discovered so far were in fact the Milky Way itself? Unfortunately, this technique is not at all promising: detecting copies of the same object (quasars, gamma-ray bursts, other galaxies, etc...) is unfeasible due to the finiteness of the speed of light, which means that, in practice, we "travel" to the past when looking through telescopes. Since galaxies evolve dramatically over time and the observable Universe is absurdly enormous, it is extremely unlikely that this method will eventually succeed.

Even so, the search for the answer to **(P.1)** constitutes a very active research subject in Physics, and other, more promising methods, which make heavy use of Riemannian Geometry, have been developed recently. See, for example, [13]. Mother Nature, in all her kindness and generosity, has provided us with the basis for some of these methods via the cosmic microwave background radiation, which suggests that the Universe can be modeled by a space form (justifying this name very well). According to recent experiments, there is evidence consistent with a Universe modeled by a flat three-dimensional manifold (*id est*, identically zero sectional curvature), of which in dimension 3 there are only eighteen (ten orientable and eight non-orientable, as proved in [14]). Due to equation **(1)**, such models are good global approximations but are not completely faithful (which, again, does not constitute a great loss of generality: the Earth itself is not a sphere either, but can be faithfully modeled by $\mathbb{S}^2$).

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