Universidade de Brasília

Questions for preparing for the qualifying exam in Analysis (Second Area)

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Chapter 1

Continuous linear operators

Question 1.
Is E' Banach? Justify.

Solution: Yes, since $E' = \mathcal{L}(E, \mathbb{R})$, and $\mathcal{L}(E, F)$ is Banach whenever F is. Indeed:

Proposition (P.1.1). $L(E, F)$ is Banach whenever F is Banach.

Proof: Let $(T_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{L}(E, F)$. Given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|T_n - T_m\| \leq \varepsilon$ whenever $n, m \geq n_0$. Hence

$$\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\| \leq \|T_n - T_m\| \cdot \|x\| \leq \varepsilon \|x\| \tag{2.1}$$

for all $x \in E$ and $n, m \geq n_0$. It follows that for each $x \in E$, the sequence $(T_n(x))_{n=1}^{\infty}$ is Cauchy in F, hence convergent since F is Banach. We can then define

$$T : E \longrightarrow F, \quad T(x) = \lim_{n \to \infty} T_n(x).$$

The linearity of T follows from the properties of limits. Letting $m \longrightarrow \infty$ in (2.1) we obtain

$$\|(T_n - T)(x)\| = \|T_n(x) - T(x)\| \leq \varepsilon \|x\| \tag{2.2}$$

for all $x \in E$ and $n \geq n_0$. In particular,

$$\|(T_n - T)(x)\| = \|T_{n_0}(x) - T(x)\| \leq \varepsilon \|x\|$$

for every $x \in E$, which guarantees that $(T - T_{n_0}) \in \mathcal{L}(E, F)$. Therefore $T = (T - T_{n_0}) + T_{n_0} \in \mathcal{L}(E, F)$. From (2.2) it also follows that $\|T_n - T\| \leq \varepsilon$ for every $n \geq n_0$, and thus $T_n \longrightarrow T$ in $\mathcal{L}(E, F)$.

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Question 2.
What is a Schauder basis? Give an example of a complete orthonormal system for an infinite-dimensional space.

Solution: Let E be a Banach space. A Schauder basis for E is a set of vectors

$$\mathcal{B} = \{e_n\}_{n\in\mathbb{N}} \subset E$$

such that every vector $x \in E$ can be written uniquely as an infinite series

$$x = \sum_{n=1}^{\infty} \alpha_n e_n,$$

for some unique scalars $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$.

Definition (D.1.1). Let H be an inner product space. We say that $S = \{e_\lambda\}_{\lambda\in\Lambda} \subset H$ is an **orthonormal system** if $\langle e_\alpha, e_\beta\rangle = 0$ for all $\alpha \neq \beta$ and $\|e_\lambda\| = 1$ for every λ. We say that an orthonormal system S is **complete** if there is no orthonormal system in H that properly contains S.

Definition (D.1.2). Let $1 \leq p \leq \infty$. We define the space $\ell^p(n)$ as the space $\mathbb{R}^n$ endowed with the norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \tag{1.1}$$

if $1 \leq p < \infty$, and

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i|. \tag{1.2}$$

In $\ell^2(n)$, the canonical basis of $\mathbb{R}^n$ is a complete orthonormal system. In ℓ^2, the canonical basis

$$(e_n)_i = \begin{cases} 1, & \text{if } i = n, \\ 0, & \text{if } i \neq n, \end{cases}$$

is a complete orthonormal system. If Ω is a bounded open set, in $L^2(\Omega)$, the eigenfunctions of the Laplacian operator, that is, the linearly independent solutions of the Dirichlet problem

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

form a complete orthonormal system.

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Question 3.
State the Banach-Steinhaus, Open Mapping and Closed Graph theorems.

Solution:

Theorem (T.1.1) (Banach-Steinhauss/Uniform boundedness principle). *Let E be a Banach space, F a normed space and $(T_i)_{i\in I}$ a family of operators in $\mathcal{L}(E;F)$ satisfying the condition that for each $x \in E$ there exists $C_x < \infty$ such that*

$$\sup_{i\in I} \|T_i(x)\| < C_x. \tag{1.3}$$

Then $\sup_{i\in I} \|T_i\| < \infty$.

Theorem (T.1.2) (Open Mapping Theorem). *Let E and F be Banach spaces and $T : E \longrightarrow F$ linear, continuous and surjective. Then T is an open map. In particular, every continuous bijective linear operator between Banach spaces is an isomorphism.*

Theorem (T.1.3) (Closed Graph Theorem). *Let E and F be Banach spaces and $T : E \longrightarrow F$ a linear operator. Then T is continuous if and only if $G(T)$ is closed in $E \times F$.*

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Question 4.
What is the main theorem used in the proof of the Banach-Steinhaus Theorem? Application of this theorem.

Solution: The main theorem used is

Theorem (T.1.4) (Baire's Theorem). *Let (M, d) be a complete metric space and $(F_n)_{n=1}^{\infty}$ a sequence of closed subsets of M such that*

$$M = \bigcup_{n=1}^{\infty} F_n.$$

Then there exists $n_0 \in \mathbb{N}$ such that F_{n_0} has nonempty interior.

From Real Analysis we know that the uniform limit of continuous functions defined on an interval of the real line and taking values in the real line is a continuous function, whereas the pointwise limit of continuous functions may fail to be a continuous function. We will see below that, in the presence of linearity, pointwise convergence is enough.

Corollary (C.1.1). *Let E be a Banach space, F a normed space and $(T_n)_{n=1}^{\infty}$ a sequence in $\mathcal{L}(E;F)$ such that $(T_n(x))_{n=1}^{\infty}$ is convergent in F for every x in E. If we define*

$$T : E \longrightarrow F, \quad T(x) = \lim_{n\to\infty} T_n(x),$$

then T is a continuous linear operator.

Proof: The linearity of T follows from the arithmetic properties of limits. By hypothesis, for each $x \in E$ the sequence $(T_n(x))_{n=1}^{\infty}$ is convergent, and therefore bounded. Thus $\sup_{n\in\mathbb{N}} \|T_n(x)\| < \infty$ for every

$x \in E$. By Theorem 2.3.2 there exists $c > 0$ such that $\sup_{n\in\mathbb{N}} \|T_n\| \leq c$. It then follows that

$$\|T_n(x)\| \leq \|T_n\| \cdot \|x\| \leq c\|x\|$$

for all $x \in E$ and $n \in \mathbb{N}$. Letting $n \to \infty$ we obtain $\|T(x)\| \leq c\|x\|$ for every $x \in E$, which completes the proof that T is continuous.

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Question 5.
What is the main theorem used in the proof of the Open Mapping Theorem? Application of this theorem.

Solution: The main theorem used is Baire's Theorem, stated in **(T.1.4)**. Some very useful applications are the following:

Corollary (C.1.2). *Let E, F be Banach spaces. If $T : E \longrightarrow F$ is a bijective bounded linear map, then the linear map $T^{-1} : F \longrightarrow E$ is continuous.*

Corollary (C.1.3). *Let E be a Banach space. If $\|\cdot\|_1, \|\cdot\|_2$ are two norms such that there exists a constant $C > 0$ such that*

$$\|x\|_1 \leq C\|x\|_2 \quad \textit{for every } x \in E,$$

then they are equivalent norms.

■

Question 6.
We cannot dispense with the surjectivity hypothesis in the Open Mapping Theorem, why? The spaces have to be Banach, why?

Solution: Surjectivity is necessary, since every open linear map is surjective (given an open ball centered at the origin of the domain, its image under T contains a ball centered at the origin of the codomain, from which surjectivity follows easily by linearity). The example below shows that the hypothesis that the spaces are complete is essential:

Example (E.1.1). By c_0 we denote the set of all sequences of scalars that converge to zero, that is, fixing $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$$c_0 = \left\{(a_k)_{k=1}^{\infty} : a_k \in \mathbb{K} \text{ for every } k \in \mathbb{N} \text{ and } a_k \longrightarrow 0\right\}.$$

Clearly c_0 is a vector space with the usual operations on sequences (coordinatewise operations). It is easy to check that the expression

$$\left\|(a_k)_{k=1}^{\infty}\right\|_{\infty} = \sup\left\{|a_k| : k \in \mathbb{N}\right\}$$

makes c_0 a normed space. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in c_0. Say $x_n = (a_n^k)_{k=1}^{\infty}$ for each $n \in \mathbb{N}$. For each $j \in \mathbb{N}$, the inequality

$$\left|a_n^j - a_m^j\right| \leq \sup\left\{\left|a_n^k - a_m^k\right| : k \in \mathbb{N}\right\} = \left\|x_n - x_m\right\|_{\infty}$$

makes it clear that the sequence of scalars $(a_n^j)_{n=1}^{\infty}$ is Cauchy in $\mathbb{K}$, hence convergent. Say $a_n^j \xrightarrow{n\to\infty} a_j$ for each $j \in \mathbb{N}$. Setting $x = (a_j)_{j=1}^{\infty}$ it is not hard to check that $x \in c_0$ and that $x_n \longrightarrow x$ in c_0. We conclude then that c_0 is a Banach space.

Now let c_{00} denote the subspace of c_0 formed by the eventually null sequences, that is,

$$c_{00} = \left\{(a_k)_{k=1}^{\infty} \in c_0 : \text{there exists } k_0 \in \mathbb{N} \text{ such that } a_k = 0 \text{ for every } k \geq k_0\right\}.$$

Consider the following vectors of c_{00}:

$$x_1 = (1, 0, 0, 0, \dots), \quad x_2 = \left(1, \frac{1}{2}, 0, 0, \dots\right), \dots, \quad x_n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\right), \dots.$$

Clearly $(x_n)_{n=1}^{\infty}$ is a sequence in c_{00}. Taking $x = \left(\frac{1}{k}\right)_{k=1}^{\infty} \in c_0$, from

$$\|x_n - x\|_{\infty} = \frac{1}{n+1} \longrightarrow 0,$$

we conclude that $x_n \longrightarrow x$ in c_0. Since $x \notin c_{00}$ it follows that c_{00} is a non-closed subspace of c_0. From the following proposition, it follows that c_{00} is an incomplete normed space.

Proposition (P.1.2). *Let E be a Banach space and F a vector subspace of E. Then F is a Banach space, with the norm induced from E, if and only if F is closed in E.*

Proof: Suppose F is Banach and take $(x_n)_{n=1}^{\infty}$ a sequence in F such that $x_n \longrightarrow x \in E$. Then $(x_n)_{n=1}^{\infty}$ is Cauchy in F, and therefore convergent since F is complete by hypothesis. So there exists $y \in F$ such that $x_n \longrightarrow y$. By the uniqueness of the limit we have $x = y \in F$, proving that F is closed in E.

Conversely, suppose F is closed in E and let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in F. Then $(x_n)_{n=1}^{\infty}$ is Cauchy in E, and therefore there exists $x \in E$ such that $x_n \longrightarrow x$. Since F is closed it follows that $x \in F$, which proves that F is complete.

■

Now let $T : c_{00} \longrightarrow c_{00}$ be the linear operator given by

$$T\left((a_n)_{n=1}^{\infty}\right) = \left(a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots\right).$$

T is evidently linear, bijective and continuous, but the inverse operator T^{-1} is not continuous, since the image of the bounded set $\{e_n\}_{n\in\mathbb{N}}$ under T^{-1} is unbounded.

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Question 7.
What is the main theorem used in the proof of the Closed Graph Theorem? Application of this theorem.

Solution: The main theorem used is the Open Mapping Theorem.

The great usefulness of the Closed Graph Theorem lies in the following observation: to show the continuity of an operator $T : E \longrightarrow F$ (linear or not) via the definition, we must prove that, for every convergent sequence $x_n \longrightarrow x$ in E, it is true that:

(a) $(T(x_n))_{n=1}^{\infty}$ is convergent in F, that is, there exists $y \in F$ such that $T(x_n) \longrightarrow y$;

(b) $y = T(x)$.

In general, proving (a), that is, proving that $(T(x_n))_{n=1}^{\infty}$ converges, is a hard task, especially in infinite dimension. The Closed Graph Theorem gives us, in the case where T is linear and acts between Banach spaces, item (a) for free, that is, we may assume the convergence of $(T(x_n))_{n=1}^{\infty}$, and it only remains to verify (b). The following example illustrates what we have just said:

Example (E.1.2). Let E be a Banach space and $T : E \longrightarrow E'$ a *symmetric* linear operator, that is, T is linear and

$$T(x)(y) = T(y)(x) \text{ for all } x, y \in E.$$

Let us show that T is continuous. In view of the Closed Graph Theorem, it suffices to verify that $G(T)$ is closed. For this, let $(x_n)_{n=1}^{\infty}$ be a sequence converging to x in E and suppose that $T(x_n) \longrightarrow \varphi$ in E'. Fixing $y \in E$, taking the limit as $n \longrightarrow \infty$,

$$\varphi(y) \longleftarrow T(x_n)(y) = T(y)(x_n) \longrightarrow T(y)(x) = T(x)(y),$$

which allows us to conclude that $T(x)(y) = \varphi(y)$. Since $y \in E$ was chosen arbitrarily, it follows that $\varphi = T(x)$, and therefore $G(T)$ is closed.

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Question 8.
Are the Open Mapping and Closed Graph theorems equivalent? Justify.

Solution: Yes, they are equivalent. See this post on MSE and this blog.

■

Chapter 2

Hahn-Banach Theorems

Theorem (T.2.1) (Hahn–Banach Theorem – real case). *Let E be a vector space over the field of real numbers and $p : E \longrightarrow \mathbb{R}$ a function satisfying*

$$p(ax) = ap(x) \text{ for every } a > 0 \text{ and every } x \in E, \tag{2.1}$$

and

$$p(x + y) \leq p(x) + p(y) \text{ for any } x, y \in E.$$

Let also G be a vector subspace of E and $\varphi : G \longrightarrow \mathbb{R}$ a linear functional such that $\varphi(x) \leq p(x)$ for every $x \in G$. Then there exists a linear functional $\tilde{\varphi} : E \longrightarrow \mathbb{R}$ that extends φ, that is, $\tilde{\varphi}(x) = \varphi(x)$ for every $x \in G$, and that satisfies $\tilde{\varphi}(x) \leq p(x)$ for every $x \in E$.

Corollary (C.2.1) (Hahn–Banach Theorem). *Let G be a subspace of a normed space E over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $\varphi : G \to \mathbb{K}$ be a continuous linear functional. Then there exists a continuous linear functional $\tilde{\varphi} : E \to \mathbb{K}$ whose restriction to G coincides with φ and $\|\tilde{\varphi}\| = \|\varphi\|$.*

Corollary (C.2.2). *If $0 \neq x_0 \in E$, where E is a real v.s., $\exists f_0 \in E'$ such that $\langle f_0, x_0 \rangle = \|x_0\|^2$ and $\|f_0\| = \|x_0\|$.*

Proof: Put $G = \langle x_0 \rangle$ and define g by $G \ni tx_0 \mapsto g(tx_0) = t\|x_0\|^2$. It is easy to verify that g is bounded and has norm $= \|x_0\|$.

■

Corollary (C.2.3). *Let $x \in E$, where E is a normed vector space over the reals. Then*

$$\|x\| = \sup_{0 \neq f \in E'} \frac{|\langle f, x \rangle|}{\|f\|}$$

Question 9.
Does the Hahn-Banach Theorem hold for Hilbert spaces? Does it hold for finite-dimensional spaces? Do you know how to prove it?

Solution: Yes, it holds, since by definition every Hilbert space (or finite-dimensional space) is a vector space. However, in these cases the proof is much simpler. Indeed, if H is a Hilbert space, it suffices to use the following

Theorem (T.2.2) (Riesz representation theorem). *Let H be a Hilbert space and $\varphi : H \longrightarrow \mathbb{K}$ a continuous linear functional. Then there exists a unique $y_0 \in H$ such that*

$$\varphi(x) = \langle x, y_0 \rangle \text{ for every } x \in H.$$

Moreover, $\|\varphi\| = \|y_0\|$.

For finite-dimensional spaces see this post on MSE.

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Question 10.
Let H be a Hilbert space, F a closed subspace of H and $\varphi : F \to \mathbb{R}$ a bounded linear map (that is, $\varphi \in F'$). Prove that there exists a unique bounded linear extension $\psi : H \to \mathbb{R}$ of φ such that $\|\psi\| = \|\varphi\|$.

Solution: to be filled in

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Question 11.
Consider $F = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ with the sum norm, and $\varphi : F \to \mathbb{R}$ defined by

$$\varphi(x, 0) = x, \ \forall x \in \mathbb{R}^2.$$

Note that we can extend φ (even preserving norms, when endowing $\mathbb{R}^2$ with the sum norm) via

$$g(x, y) = x + y \quad \text{and} \quad h(x, y) = x - y.$$

Why does this not contradict the previous question?

Solution: to be filled in

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Question 12.
Is the extension in the Hahn-Banach Theorem unique? When is it unique? Examples of when the extension is unique and when it is not.

Solution: Consider $g : A \subset \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x_1, 0) = x_1$, where A is the x-axis of $\mathbb{R}^2$, with $p : \mathbb{R}^2 \to \mathbb{R}$ given by $p(a, b) = |a| + 3|b|$. Clearly $g(x) \leq p(x)$ for all $x \in A$, and evidently p is sublinear. Note also that the functions $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $f_1(a, b) = a+b$ and $f_2(a, b) = a+2b$ are both sublinear, extend g and satisfy $f_1(x) \leq p(x)$, $f_2(x) \leq p(x)$ for any $x \in \mathbb{R}^2$.

When the subspace is dense, we have uniqueness of the extension, as the following asserts

Proposition (P.2.1). Let $G \subset E$ be a dense subspace of a normed vector space E and $\varphi : G \to F$ a bounded linear operator, where F is a Banach space. Then there exists a unique bounded linear extension of φ. Moreover, the norm of such an extension coincides with that of φ.

Proof: Let $x \in E \setminus G$. Since $\overline{G} = E$, there exists a sequence $\{x_m\}_{m\in\mathbb{N}}$ in G such that $x_m \to x$. We define $\Psi : E \to \mathbb{R}$ by:

$$\Psi(a) = \begin{cases} \varphi(a), & \text{if } a \in G, \\ \lim\limits_{m\to\infty} \varphi(a_m), & \text{if } a \notin G, \text{ where } \{a_m\}_{m\in\mathbb{N}} \text{ is such that } a_m \to a. \end{cases}$$

We need to show that Ψ is well defined (it does not depend on the choice of the sequence and the limit exists), $\Psi \in G'$ and $\|\Psi\|_{E'} = \|\varphi\|_G$.

Indeed, if $x \in E \setminus G$ and $x_m \to x$, $y_m \to x$ (where $\{x_m\}_{m\in\mathbb{N}}$, $\{y_m\}_{m\in\mathbb{N}}$ are distinct sequences), we have

$$|\varphi(x_m) - \varphi(y_m)| \leq \|\varphi\|_6 \|x_m - y_m\| \quad \forall m, n \in \mathbb{N}. \tag{2.2}$$

Since $\varphi \in G'$, $\|\varphi\|_G = M$ for some $M \in \mathbb{R}$. And since $x_m \to x$, $\{x_m\}_{m\in\mathbb{N}}$ is Cauchy. By (2.2) we then have that $\{\varphi(x_m)\}_{m\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, which is Banach, so $\lim\limits_{m\to\infty} \varphi(x_m)$ exists.

Moreover, since:

$$\|\varphi(x_m) - \varphi(y_m)\|_{\mathbb{R}} \leq M\|x_m - y_m\|$$

it follows that $\varphi(x_m) \to \varphi(y_m)$ whenever $\lim\limits_{m\to\infty} x_m = \lim\limits_{m\to\infty} y_m$. Hence Ψ is well defined. By the properties of the limit, Ψ is linear. Since $\|\varphi(x_m)\| \leq \|\varphi\|\|x_m\|$ and $\|\cdot\|_{\mathbb{R}}$ is continuous, it follows that

$$\|\Psi(x)\| \leq \|\varphi\|\|x\| \quad \forall x \in E.$$

Hence Ψ is bounded and satisfies $\|\Psi\| \leq \|\varphi\|$. Since the norm cannot decrease when extending (because $\sup A \geq \sup B$ whenever $A \geq B$ and A and B are bounded above), we also have $\|\Psi\| \geq \|\varphi\|$. Hence $\|\Psi\|_E = \|\varphi\|_G$, which concludes the proof.

■

The following result, which we will not prove but which is classical, is also related to this context:

Proposition (P.2.2). Let (X, d_X) and (Y, d_Y) be metric spaces, with (Y, d_Y) complete. Let $A \subseteq X$ be a subset of X. Given $f : A \to Y$ uniformly continuous, f can be uniquely extended to $\bar{A}$ preserving uniform continuity.

■

Question 13.
State the first Geometric Form of the Hahn-Banach Theorem. Give an idea of the proof. Do you know an application?

Solution:

Definition (D.2.1). Let E be a vector space and $A, B \subset E$ arbitrary subsets.

We say that the hyperplane $H = f^{-1}(\alpha)$ **separates A and B in the broad sense** if $f(x) \leq \alpha$ for every $x \in A$ and $f(x) \geq \alpha$ for every $x \in B$.

We say that the hyperplane $H = f^{-1}(\alpha)$ **separates A and B in the strict sense** if there exists $\varepsilon > 0$ such that $f(x) \leq \alpha - \varepsilon$ for every $x \in A$ and $f(x) \geq \alpha + \varepsilon$ for every $x \in B$.

Theorem (T.2.3) (Hahn-Banach Theorem, first geometric form). *Let E be a normed vector space. Let $A, B \subset E$ be nonempty disjoint convex sets, with A open. Then there exists a closed hyperplane that separates A and B in the broad sense.*

Theorem (T.2.4) (First geometric form of the Hahn–Banach Theorem). *Let A and B be convex, nonempty and disjoint subsets of the normed space E. If A is open, then there exist a functional $\varphi \in E'$ and $a \in \mathbb{R}$ such that*

$$\varphi(x) < a \leq \varphi(y) \quad \text{for all } x \in A \text{ and } y \in B.$$

In this case one says that the closed hyperplane $[\varphi = a]$ separates A and B.

The idea of the proof is to work with a Minkowski functional defined on the set (to be shown open, convex and nonempty) $C = A - B = \{x - y \mid x \in A, y \in B\}$.

Since the first form is used to prove the second geometric form, one application is the second geometric form.

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Question 14.
State the second Geometric Form of the Hahn-Banach Theorem. Give an idea of the proof. Do you know an application?

Solution:

Theorem (T.2.5) (Hahn-Banach Theorem, second geometric form). *Let E be a normed vector space. Let $A, B \subset E$ be nonempty disjoint convex sets, with A closed and B compact. Then there exists a closed hyperplane that separates A and B in the strict sense.*

Theorem (T.2.6) (Second geometric form of the Hahn–Banach Theorem). *Let A and B be convex, nonempty and disjoint subsets of the normed space E. If A is closed and B is compact, then there exist a functional $\varphi \in E'$ and $a, b \in \mathbb{R}$ such that*

$$\varphi(x) \leq a < b \leq \varphi(y) \quad \text{for all } x \in A \text{ and } y \in B.$$

For $c \in (a, b)$ one says that the closed hyperplane $[\varphi = c]$ separates A and B strictly.

One application is the characterization of closed convex sets under the strong and weak topologies. Indeed, we have the following

Proposition (P.2.3). *Let E be a normed vector space and $C \subset E$ a convex set. Then C is closed in the weak topology if and only if C is closed in the strong topology.*

Proof: Suppose that C is strongly closed. Let $x_0 \notin C$. According to the Hahn-Banach theorem, second geometric form, there exist $f \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) < \alpha < f(x_0)$$

for every $y \in C$ (indeed, it suffices to take $A = C$ and $B = \{x_0\}$). Then $V = f^{-1}(\alpha, \infty)$ is a weak open neighborhood of x_0 that does not intersect C. Therefore $E \setminus C$ is weakly open, hence C is weakly closed.

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Question 15.
In the Second Geometric Form of the Hahn-Banach Theorem, if we remove the hypothesis that one of the sets is compact, but keep both closed and convex, does the result still hold? Justify.

Solution: No. Consider, for example, in $\mathbb{R}^2$ the sets

$$A = \{(x, y) \in \mathbb{R}^2 \mid y \geq e^x\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 \mid y \leq 0\}.$$

Both are closed, convex and disjoint (why?), but they cannot be strictly separated.

■

Chapter 3

Weak and Weak* Topologies

Given a set of functions $\{f_\lambda\}_{\lambda\in\Lambda}$ from a set X to a topological space Y, the *coarsest* (or *least fine*) topology $\mathcal{T}$ on X is the one with the minimal number of open sets that make all the functions f_λ continuous. It is evidently the topology having as a subbasis the collection $\{f_\lambda^{-1}(V) : V \text{ is open in } Y \text{ and } \lambda \in \Lambda\}$.

Definition (D.3.1). Let E be a normed vector space. **The weak topology** on E is the coarsest topology such that all linear functionals $f \in E^*$ are continuous.

Proposition (P.3.1). *Let $\tau \doteq \sigma(E, E')$ be the weak topology. Then:*

(a) For each $f \in E'$ the function $f\colon X \to \mathbb{R}$ is continuous with respect to τ.

(b) τ is the smallest topology on X such that (a) holds.

(c) τ is the intersection of all topologies on X with respect to which all $f \in E'$ are continuous.

(d) For each $x \in X$, the sets of the form $f_1^{-1}(A_1) \cap \cdots \cap f_n^{-1}(A_n)$ where $n \in \mathbb{N}$, $f_i \in E'$ for each $1 \le i \le n$, and A_j is a neighborhood of $f_j(x)$, $j = 1, \ldots, n$, form a neighborhood basis for x.

(e) Let (x_λ) be a sequence in X. Then $x_n \to x$ in (X, τ) if and only if $f(x_n) \to f(x)$ in $\mathbb{R}$ for any $f \in E'$.

(f) Let Z be a topological space and $h\colon Z \to (X, \tau)$. Then h is continuous if and only if $f \circ h\colon Z \to Y_i$ is continuous for any $f \in E'$.

(g) τ is Hausdorff.

Proposition (P.3.2). Let E be a normed space. Then:

(a) Continuous linear functionals are weakly continuous, that is, for every $\varphi \in E'$, $\varphi\colon (E, \sigma(E, E')) \to \mathbb{R}$ is continuous.

(b) For each $x_0 \in E$, the sets of the form

$$V_{J,\varepsilon} = \{x \in E \mid |\varphi_i(x) - \varphi_i(x_0)| < \varepsilon \text{ for every } i \in J\},$$

where J is a finite set, $\varphi_i \in E'$ for every $i \in J$ and $\varepsilon > 0$, form a basis of open neighborhoods of x_0 for the weak topology.

(c) Let $(x_n)_{n=1}^{\infty}$ be a sequence in E. Then $x_n \rightharpoonup x$ if and only if $\varphi(x_n) \to \varphi(x)$ for every $\varphi \in E'$.

Solution: (a) is evident. For (b), let U be a neighborhood of x_0 in the weak topology. By Proposition **(P.3.1)** there exist a finite set J, functionals $\varphi_j \in E'$ and open sets V_j in $\mathbb{R}$ containing $\varphi_j(x_0)$, $j \in J$, such that

$$\bigcap_{j \in J} \varphi_j^{-1}(V_j)$$

is an open set of the weak topology containing x_0 and contained in U. Since J is finite and $\varphi_j(x_0) \in V_j$ for every $j \in J$, there exists $\varepsilon > 0$ such that

$$B(\varphi_j(x_0), \varepsilon) \subseteq V_j$$

for every $j \in J$. From this it follows that

$$V_{J,\varepsilon} = \{x \in E \mid |\varphi_j(x) - \varphi_j(x_0)| < \varepsilon \text{ for every } j \in J\}$$

$$= \bigcap_{j \in J} \varphi_j^{-1}(B(\varphi_j(x_0), \varepsilon)) \subseteq \bigcap_{j \in J} \varphi_j^{-1}(V_j) \subseteq U.$$

For (c), suppose $x_n \rightharpoonup x$. Then $f(x_n) \to f(x)$ for every $f \in E^*$ because f is continuous in the weak topology. Conversely, suppose that $f(x_n) \to f(x)$ for every $f \in E^*$. To prove that $x_n \rightharpoonup x$, we will show that given any weak open neighborhood U of x, we have $x_n \in U$ for every sufficiently large n.

Indeed, let

$$\bigcap_{i=1}^{m} f_i^{-1}(V_i) \subset U$$

be a basis element of the weak topology containing x_0, with $V_1, \ldots, V_m$ open in $\mathbb{R}$. For each i, there exists $N_i \in \mathbb{N}$ such that $f_i(x_n) \in V_i$ for every $n > N_i$.

Taking $N = \max\{N_1, \ldots, N_m\}$, it follows that if $n > N$ then $f_i(x_n) \in V_i$ for every i, hence

$$x_n \in \bigcap_{i=1}^{m} f_i^{-1}(V_i) \subset U$$

for every $n > N$.

■

Question 16.
Give an application of the Hahn-Banach Theorem in the study of the Weak Topologies.

Solution: See proposition **(P.2.3)**. The fact that the weak topology is Hausdorff also follows from Hahn-Banach:

Proposition (P.3.3). *A normed vector space E under the weak topology is a Hausdorff space.*

Proof: If $x, y \in E$ with $x \neq y$, by the Hahn-Banach theorem (see corollary **??**) there exists $f \in E^*$ such that $f(x-y) = \|x-y\|^2$. In particular, $f(x) \neq f(y)$ and if $\alpha \in \mathbb{R}$ is such that $f(x) < \alpha < f(y)$, then the open sets $f^{-1}(-\infty, \alpha)$ and $f^{-1}(\alpha, \infty)$ separate x and y.

■

One of the corollaries of Hahn-Banach also guarantees that weakly convergent sequences are bounded:

Proposition (P.3.4). *If $x_n \xrightarrow{w} x$ in E, then the sequence $(\|x_n\|)_{n=1}^{\infty}$ is bounded.*

Proof: Let $\{x_n\}$ be a weakly convergent sequence in X. Let us define $T_n \in X^{**}$ by $T_n(\ell) = \ell(x_n)$ for every $\ell \in X^*$. Let us fix an $\ell \in X^*$. For every $n \in \mathbb{N}$, the convergence of $\{\ell(x_n)\}$ guarantees that $\{T_n(\ell)\}$ is a bounded set. By the Uniform Boundedness Principle and by insert HB corollary Later, $\sup_{n\in\mathbb{N}} \|x_n\| = \sup_{n\in\mathbb{N}} \|T_n\| < \infty$, i.e. $\{x_n\}$ is bounded.

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Question 17.
Why is there a need to obtain compact sets in the study of the weak topologies?

Solution: A weakened topology naturally has more compact sets, and compactness is very important in existence and minimization techniques (usually in the form of sequential compactness). Read below the translation of this answer on MSE:

Many important problems in mathematics and mathematical physics can be formulated as problems of minimizing some integral functional, that is, we want to find

$$\overline{u} \in S \quad \text{such that} \quad \overline{u} = \min\{I(u) : u \in S\},$$

where

$$I(u) := \int_\Omega L(x, u(x), \nabla u(x))\, dx.$$

Here S is some function space, for example,

$$S = \{u \in W^{1,1} \mid u|_{\partial\Omega} = f\},$$

for some fixed $f \in W^{1,1}$, and Ω is a bounded domain in a Euclidean space. The integrand L represents the characteristic of the problem we want to solve. For example, L may represent the energy density for the Dirichlet energy functional or the internal energy of an elastic body.

One method to solve this is called the **Direct Method**. Suppose that the functional I is bounded below on S; we can then choose a minimizing sequence $(u_n)_n \subset S$ such that

$$I(u_n) \to \inf_S I.$$

The minimization problem is solved if we can establish the following:

(a). $(u_n)_n$ has a subsequence that converges (in an appropriate sense) to some $u \in S$.

(b). $I(u) \leq \liminf_{n\to\infty} I(u_n)$.

The appropriate mode of convergence is usually weak convergence, since weak compactness is easier to establish than strong compactness. Here we see that compactness is crucial to guarantee that $(u_n)_n$ has a convergent subsequence.

In this case, we want the functional I to be weakly lower semicontinuous so that the second condition is satisfied.

■

Question 18.
Construct the Weak Topology. Give examples of open sets of this topology. Does the Weak Topology have fewer continuous functions than the strong topology? Justify.

Solution: The construction was done above. A generic open set is a union of finite intersections described previously. For a function $f : (E, \sigma(E, E')) \to Y$ to be continuous, it is necessary that $f^{-1}(O)$ be open in $\sigma(E, E')$ for every open set O in Y, which is evidently harder to happen when we remove open sets of E, as we do when we weaken the topology.

We observe that an open set in the weak topology is never bounded, since it always contains a line, as we will see in the proof of the following result:

Proposition (P.3.5). *If E is an infinite-dimensional normed vector space, then the unit sphere is not closed in the weak topology.*

In fact, the weak closure of the unit sphere is the closed unit ball.

Proof: Let

$$S = \{x \in E : \|x\| = 1\}$$

be the unit sphere of E,

$$\overline{B_1(0)} = \{x \in E : \|x\| \leq 1\}$$

the closed unit ball, and denote by $\overline{S}^W$ the closure of S in the weak topology. We then want to prove that

$$\overline{S}^W = \overline{B_1(0)}.$$

Let $x_0 \in B_1(0)$ and U be any weak open set containing x_0. By Proposition 5.4, U contains a weak open neighborhood of x_0 of the form

$$V = \{x \in E : |f_i(x) - f_i(x_0)| < \varepsilon, \quad f_1, \ldots, f_n \in E^*\}.$$

Since E is infinite-dimensional, there exists $y_0 \in E$ such that

$$f_i(y_0) = 0 \quad \text{for every } i,$$

otherwise the linear map $T : E \to \mathbb{R}^n$ defined by

$$Tx = (f_1(x), \dots, f_n(x))$$

would be injective, which would imply $\dim E \leq n$. (another way to see this is to invoke proposition **(P.10.1)**: the kernel of a linear functional has codimension 1, so the intersection of a finite number of kernels of linear functionals in an infinite-dimensional vector space is also infinite-dimensional). In particular, V contains the line $t \mapsto x_0 + ty_0$, since

$$f_i(x_0 + t_0 y_0) - f_i(x_0) = f_i(x_0 + t_0 y_0 - x_0) = t_0 f_i(y_0) = 0$$

for every i. *(Thus, in an infinite-dimensional normed vector space, every weak open neighborhood of a point x_0 contains a line through x_0; in fact, infinitely many such lines.)*

Now consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \|x_0 + ty_0\|.$$

This function is continuous and satisfies $\varphi(0) < 1$ and $\lim_{t \to +\infty} \varphi(t) = +\infty$, which implies by the intermediate value theorem that there exists $t_0 > 0$ such that $g(t_0) = 1$, that is, $x_0 + t_0 y_0 \in S \cap V$. We conclude that every *weak* open neighborhood of x_0 intersects S, therefore $x_0 \in \overline{S}^W$.

This argument proves that $\overline{B_1(0)} = B_1(0) \cup S \subseteq \overline{S}^W$. Since $B_1(0)$ is closed in the weak topology by proposition **(P.2.3)**, there are no points of the weak closure of S outside $B_1(0)$. Therefore equality holds

$$\overline{S}^W = \overline{B_1(0)}.$$

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■

Question 19.
Does strong continuity imply weak continuity? Does the converse hold?

Solution: In the presence of linearity, strong continuity implies weak continuity. Indeed, suppose $T : E \to F$ strongly continuous. To prove weak continuity, by proposition **(P.3.1)**, it suffices to prove that $g \circ T$ is continuous for each $g \in F'$, which follows immediately from the fact that $g \circ T \in E'$.

In the absence of linearity, strong continuity does not imply weak continuity. Indeed, taking the sequence $(e_n)_{n \in \mathbb{N}} \subset \ell^2$, clearly (e_n) converges weakly to 0 in ℓ^2, but $|e_n|_2$ does not converge to $|0|_2 = 0$.

The norm topology contains all the open sets of the weak topology (and in infinite dimension, open sets that are not weak open sets), therefore every weakly continuous function $h : E \to F$ (that is, considering E with the weak topology) is strongly continuous.

■

Question 20.
What is the relation between the Weak and Weak* Topologies?

Solution:

Definition (D.3.2). Let E be a normed vector space and consider the canonical embedding $J : E \to E^{**}$. The **weak* topology** on the dual E^*, which we will denote by $\sigma(E^*, E)$, is the coarsest topology such that all linear functionals in the image $J(E)$ are continuous.

Evidently, if E is a reflexive space then the weak* topology coincides with the weak topology of E^*. Otherwise, in general the weak* topology of E^* is coarser than the weak topology of E^*, which in turn is coarser than the strong topology of E^*.

In fact, by the theorems of Banach–Alaoglu–Bourbaki and Kakutani, the weak and weak* topologies on the dual coincide if and only if the space is reflexive!

■

Question 21.
Examples of: (i) weak that is not strong; (ii) weak* that is not weak; (iii) weak* that is not strong.

Solution: The norm topology contains all the weak open sets, therefore there is no weak open set that is not strong. On the other hand, by the proof of proposition **(P.3.5)**, any bounded strong open set is not a weak open set. To exhibit a weak* open set that is not a weak open set it suffices to exhibit a weak* closed set that is not weakly closed. This is possible if and only if the space is not reflexive. In that case, taking $\xi \notin J(E)$, the set

$$F = \xi^{-1}(\{0\})$$

is evidently weakly closed but is not weak* closed due to the following result (Corollary 3.15. of Brezis)

Proposition (P.3.6). Let H be a hyperplane in E^* that is weak* closed. Then there exist $0 \neq x_0 \in E$ and $\alpha \in \mathbb{R}$ such that

$$H = \{f \in E^* \mid f(x_0) = \alpha\}$$

Indeed, if F were weak* closed, there would exist $0 \neq x_0 \in E$ and $\alpha \in \mathbb{R}$ such that

$$F = \ker(\xi) = \{f \in E^* \mid f(x_0) = \alpha\}$$

Since $\ker(\xi)$ is a subspace, we cannot have $\alpha \neq 0$. It follows that ξ and $J(x_0)$ have the same kernel, which implies that $\xi \in J(E)$ by proposition **(P.10.1)**.

There is no weak* open set that is not a strong open set, since the norm topology contains the weak topology, in particular it contains the weak* topology. However, the unit ball is a strong open set that is not a weak open set, therefore it is not a weak* open set, since every weak* open set is a weak open set.

■

Question 22.
What is the relation between separable and metrizable?

Solution: We have the following results:

Proposition (P.3.7). *If E is separable then $(B_E, \sigma(E', E))$ is metrizable.*

Proposition (P.3.8). *If E is separable then E' is separable in the weak-star topology.*

Proposition (P.3.9). *If E' is separable then $(B_E, \sigma(E, E'))$ is metrizable.*

Proposition (P.3.10). (a) *If E is an infinite-dimensional normed space, then the weak topology on E is not metrizable.*

(b) *If E is an infinite-dimensional Banach space, then the weak-star topology on E' is not metrizable.*

■

Question 23.
Why is $B_1(0)$ not open in the weak topology?

Solution: As we saw in the proof of proposition **(P.3.5)**, every weak open set is unbounded.

■

Question 24.
Give an idea of the proof of the following result: every strong closed convex set is weakly closed.

Solution: The idea is to use the second geometric form of Hahn-Banach to guarantee the existence of weak neighborhoods of each $x_0 \in E \setminus C$, as we saw in the proof of proposition **(P.2.3)**.

■

Question 25.
State the Banach-Alaoglu-Bourbaki Theorem. Give the central idea of the proof.

Solution:

Theorem (T.3.1) (Banach–Alaoglu–Bourbaki Theorem). *For every normed space E, the ball $B_{E'}$ is compact in the weak-star topology $\sigma(E', E)$ of E'.*

The central idea of the proof consists in identifying the ball with a product of compact disks and applying Tychonoff's theorem,

■

Question 26.
State Kakutani's Theorem. Give the central idea of the proof.

Solution:

Theorem (T.3.2) (Kakutani's Theorem). *A Banach space E is reflexive if and only if the closed unit ball B_E is compact in the weak topology $\sigma(E, E')$.*

The idea for the forward direction (assuming E reflexive) is to use the Banach-Alaoglu-Bourbaki Theorem and the fact that in reflexive spaces the canonical injection is a homeomorphism.

For the converse, the idea is to work with the image of B_E under the canonical injection, considering the weak-star topology on the bidual. Since the weak-star topology is Hausdorff, such an image is closed, and by Goldstine's theorem we will then have $J_E(B_E) = B_{E''}$, which by linearity guarantees that E is reflexive.

Theorem (T.3.3) (Goldstine's Theorem). *Let E be a Banach space and $J_E : E \to E''$ the canonical embedding. Then $J_E(B_E)$ is dense in $B_{E''}$ in the weak-star topology.*

■

Question 27.
Define uniformly convex spaces. Give examples of spaces that are and are not uniformly convex. Give an application.

Solution: Our geometric intuition clearly tells us that the closed unit ball of the Euclidean space $(\mathbb{R}^2, \|\cdot\|_2)$ is round, whereas those of the spaces $(\mathbb{R}^2, \|\cdot\|_1)$ and $(\mathbb{R}^2, \|\cdot\|_\infty)$ are not round at all; in fact they are square. These last two examples are extreme, since their unit spheres contain line segments.

To prevent the closed unit sphere $S_E = \{x \in E : \|x\| = 1\}$ of a space E from containing line segments, it suffices to require that if $x, y \in B_E$, then the midpoint of the segment joining x to y lies in the interior of the ball, that is,

$$\left\| \frac{x+y}{2} \right\| < 1 \quad \text{whenever} \quad \|x\| \leq 1 \quad \text{and} \quad \|y\| \leq 1.$$

But the unit sphere of $(\mathbb{R}^2, \|\cdot\|_2)$, that is, the circle, satisfies a much stronger property than simply not containing line segments. Fix a distance $\varepsilon > 0$ and take any two points x and y on the circle that are (at least) ε apart. Appealing again to geometric intuition, we see that, regardless of where the points x and y are located on the circle, the midpoint of the segment joining them remains inside the circle and, more importantly, at a safe distance from the circle. That is, we can vary the points x and y on the circle, just keeping them at a distance ε, and there is no risk of the midpoint of the interval joining them getting dangerously close to the circle. The normed spaces that share with Euclidean spaces this uniform behavior of the points of the unit ball are called uniformly convex spaces.

Definition (D.3.3). We say that a normed space E is *uniformly convex* or, more precisely, that its norm is *uniformly convex* if, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta \quad \text{whenever} \quad x, y \in B_E \quad \text{and} \quad \|x - y\| \geq \varepsilon.$$

Since two points of B_E are at most 2 apart, it suffices to show the condition above for $0 < \varepsilon \leq 2$.

Example (E.3.1). Every Hilbert space H is uniformly convex. Indeed, given $x, y \in B_H$ and $0 < \varepsilon \leq 2$, from the Parallelogram Law, it follows that if $\|x - y\| \geq \varepsilon$ then

$$\left\| \frac{x+y}{2} \right\|^2 = \frac{\|x\|^2}{2} + \frac{\|y\|^2}{2} - \frac{\|x-y\|^2}{4} \leq 1 - \frac{\varepsilon^2}{4}.$$

It then suffices to take $\delta = 1 - (1 - \varepsilon^2/4)^{\frac{1}{2}} > 0$.

Example (E.3.2). The spaces $(\mathbb{R}^2, \|\cdot\|_\infty)$ and $(\mathbb{R}^2, \|\cdot\|_1)$ are not uniformly convex. Indeed, given $0 < \varepsilon < 1$, it suffices to take two vectors that lie on the same line segment of the unit sphere and are ε apart from each other. Completing with zeros in the remaining coordinates, we turn these ordered pairs into sequences, which proves that the spaces c_0, ℓ_1 and ℓ_∞ are not uniformly convex.

Example (E.3.3). $L^p(\Omega)$ is uniformly convex for any open Ω in $\mathbb{R}^n$ and $1 < p < \infty$.

A widely used application is the following:

Proposition (P.3.11). *Let E be a uniformly convex Banach space and $(x_n)_{n=1}^\infty$ a sequence in E. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.*

■

Question 28.
When does the weak topology on the dual coincide with the weak-star topology?

Solution: They coincide if and only if the space is reflexive.

■

Question 29.
Give examples of a sequence that converges weakly but not strongly and of a sequence that converges in the weak-star topology but not in the weak topology.

Solution: to be filled in

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Question 30.
Relate separability and reflexivity.

Solution:

Theorem (T.3.4). *Let E be a Banach space. Then E is a reflexive space if and only if E^* is.*

Theorem (T.3.5). *Let E be a normed vector space. If E^* is separable, then so is E.*

Theorem (T.3.6). *If E is a separable reflexive space, then so is E^*.*

Therefore, a separable space is reflexive if and only if its dual is separable (and reflexive).

■

Chapter 4

Hilbert Spaces

WE WILL CONSIDER HILBERT SPACES ONLY OVER THE FIELD OF REAL NUMBERS.

Question 31.
In Hilbert spaces, what is the relation between Schauder bases and Hilbert bases? Under which conditions is a Schauder basis a Hilbert basis? Is a Hilbert basis a Schauder basis, why?

Solution: A Hilbert basis is a complete orthonormal system, as defined in (D.1.1).

Theorem (T.4.1). *Let $S = \{x_i : i \in I\}$ be an orthonormal set in the Hilbert space H. The following statements are equivalent:*

(a) For each $x \in H$, $x = \sum_{i \in I} \langle x, x_i \rangle x_i$.

(b) S is a complete orthonormal system.

(c) $\overline{[S]} = H$.

(d) For each $x \in H$, $\|x\|^2 = \sum_{i \in I} |\langle x, x_i \rangle|^2$. (Parseval's identity)

(e) For all $x, y \in H$, $\langle x, y \rangle = \sum_{i \in I} \langle x, x_i \rangle \langle y, x_i \rangle$.

Proposition (P.4.1). In the separable case, the representation of Theorem **(T.4.1)**(a) is unique, that is, if $x = \sum_{n=1}^{\infty} b_n x_n$, then $b_n = \langle x, x_n \rangle$ for every $n \in \mathbb{N}$.

Theorem (T.4.2). *An infinite-dimensional Hilbert space H is separable if and only if there exists in H a countable complete orthonormal system.*

Combining the results above, we conclude that complete orthonormal systems in separable Hilbert spaces are Schauder bases. Multiplying a Hilbert basis by 2, for example, we get a Schauder basis that is not a Hilbert basis. In separable Hilbert spaces we can always turn a Schauder basis into a Hilbert basis by the Gram–Schmidt process.

■

Question 32.
State and give an idea of the proof of the Riesz Representation Theorem.

Solution: The proof of the theorem uses the following property of Hilbert spaces:

Theorem (T.4.3) (Orthogonal decomposition). *Let E be an inner product space and M a complete subspace of E. Then $E = M \oplus M^{\perp}$, that is, each $x \in E$ admits a unique representation of the form*

$$x = p + q \quad \text{with } p \in M \text{ and } q \in M^{\perp}.$$

Moreover,

$$\|x - p\| = dist(x, M).$$

The vector p is called the orthogonal projection of x onto M.

Theorem (T.4.4) (Riesz Representation Theorem). *Let H be a Hilbert space and $\varphi : H \to \mathbb{R}$ a continuous linear functional. Then there exists a unique $y_0 \in H$ such that*

$$\varphi(x) = \langle x, y_0 \rangle \quad \text{for every } x \in H.$$

Moreover, $\|\varphi\| = \|y_0\|$.

Proof: If φ is the identically zero functional, it suffices to take $y_0 = 0$. Let us now consider the case where φ is not identically zero. In this case the kernel of φ,

$$M := \{x \in H : \varphi(x) = 0\},$$

is a proper closed subspace of H. From the previous theorem we know that $M^{\perp} \neq \{0\}$, and so we can choose x_0 in $M^{\perp}$ of norm 1 (recall that $M^{\perp}$ is a vector subspace). Let us prove that $y_0 := \varphi(x_0)x_0$ is the desired vector. Given $x \in H$,

$$x = \left(x - \frac{\varphi(x)}{\varphi(x_0)}x_0\right) + \frac{\varphi(x)}{\varphi(x_0)}x_0,$$

where $\left(x - \dfrac{\varphi(x)}{\varphi(x_0)}x_0\right) \in M$ and $\dfrac{\varphi(x)}{\varphi(x_0)}x_0 \in M^{\perp}$. Since $y_0 \in M^{\perp}$ and $\|x_0\| = 1$,

$$\langle x, y_0 \rangle = \left\langle x - \frac{\varphi(x)}{\varphi(x_0)}x_0, y_0 \right\rangle + \left\langle \frac{\varphi(x)}{\varphi(x_0)}x_0, y_0 \right\rangle.$$

Since $\left\langle x - \dfrac{\varphi(x)}{\varphi(x_0)} x_0, y_0 \right\rangle = 0$, we have

$$\langle x, y_0 \rangle = \frac{\varphi(x)}{\varphi(x_0)} \langle x_0, \varphi(x_0) x_0 \rangle = \varphi(x).$$

From the Cauchy–Schwarz inequality we have $\|\varphi\| \leq \|y_0\|$. The equality $|\varphi(x_0)| = \|y_0\|$ guarantees the reverse inequality and therefore $\|\varphi\| = \|y_0\|$. Uniqueness is evident.

■

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Question 33.
Is every Hilbert space reflexive? Justify.

Solution: Yes, for example by the Riesz Representation Theorem.

■

Question 34.
When are the spaces L^p and ℓ^p separable spaces? And reflexive? And Hilbert?

Solution: ℓ^p is separable and reflexive for $1 \leq p < \infty$. L^p is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. Both are Hilbert spaces only for $p = 2$.

ℓ^∞ is not separable. ℓ^1 and ℓ^∞ are not reflexive. L^1 and L^∞ are not reflexive.

■

Question 35.
Is there any non-separable Hilbert space? Justify.

Solution: The space $l^2(\mathbb{R})$ is an example of a non-separable Hilbert space: it consists of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ only for a countable number of x, and

$$\sum_{x \in \mathbb{R}} f(x)^2 < \infty.$$

It is easy to see that this is a Hilbert space, being the countable union of countable sets.

The functions f_y defined by

$$f_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

form an uncountable set of elements at distance $\sqrt{2}$, hence $l^2(\mathbb{R})$ is not separable.

■

Question 36.
Give an application of the Riesz Representation Theorem in PDE.

Solution: See chapter 6.

■

Question 37.
State and give an idea of the proof of the Lax-Milgram Theorem.

Solution:

Theorem (T.4.5) (Lax-Milgram). *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space and $\mathcal{B} : H \times H \to \mathbb{R}$ a bilinear form satisfying*

(i) there exists $\alpha > 0$ such that, for every $u, v \in H$,

$$|\mathcal{B}[u, v]| \leq \alpha \|u\|_H \|v\|_H, \quad \forall u, v \in H.$$

(ii) there exists $\beta > 0$ such that,

$$\mathcal{B}[u, u] \geq \beta \|u\|^2, \quad \forall u \in H.$$

Then, given a continuous linear functional $T : H \to \mathbb{R}$, there exists a unique $u_T \in H$ such that

$$T(u) = \mathcal{B}[u_T, u], \quad \forall u \in H.$$

Proof: The idea of the proof is to use the Riesz representation theorem to represent B by a continuous endomorphism $A : H \to H$, and then reduce the original problem to showing that A is bijective. Letting $\varphi : H \to H^*$ be the isometric isomorphism whose existence is guaranteed by the Riesz representation theorem, we define the map A by

$$u \mapsto A(u) = \varphi\Big(\mathcal{B}[u, \cdot]\Big).$$

Note that A is linear and continuous, since $\mathcal{B}$ is bilinear and continuous. Moreover,

(a) A is injective (a direct consequence of the coercivity of $\mathcal{B}$),

(b) $\mathrm{Im}(A)$ is *closed*, since $\beta|v| \leq |Av| \quad \forall v \in H$ (also a consequence of the coercivity of $\mathcal{B}$). Hence, if $y_n = A(v_n) \to y$, then v_n is Cauchy in H and, by the completeness of H, there exists $v \in H$ such that $v_n \to v$. It follows from the continuity of A that $y = A(v) \in \mathrm{Im}(A)$. This guarantees the surjectivity of A (note that the coercivity of $\mathcal{B}$ guarantees that $(\mathrm{Im}(A))^\perp = \{0\}$; since we are in Hilbert spaces, this is equivalent to $\mathrm{Im}(A) = H$).

Now given $T \in H^*$, it suffices to apply the Riesz representation theorem to T and the surjectivity of A to conclude that there exists $u_T \in H$ such that $T(u) = \mathcal{B}[u_T, u]$, as desired. The uniqueness of u_T follows from the coercivity of A.

■

■

Question 38.
Give an example in which we can apply the Lax-Milgram Theorem but it is not possible to apply the Riesz representation Theorem.

Solution: Consider the problem

$$-\Delta u + \partial_x u = f, u \in H_0^1(\Omega),$$

for $\Omega \subset \mathbb{R}^n$ a bounded open set and $f \in L^2(\Omega)$. The bilinear form associated with the weak formulation of the problem is not symmetric, therefore we cannot apply the Riesz Representation Theorem. However, we can apply the Lax-Milgram Theorem.

■

Question 39.
Let $\mathcal{B} : H \times H \to \mathbb{R}$ be a bilinear form that satisfies the hypotheses of Lax-Milgram (that is, it is continuous and coercive) but that is also symmetric. In this case can we simply apply the Riesz Representation Theorem? Justify.

Solution: Yes, we can. $(H, \langle \cdot, \cdot \rangle_H)$ is a Hilbert space with a norm, say $\| \cdot \|_1$, which is induced by $\langle \cdot, \cdot \rangle_H$, that is,

$$\| \cdot \|_1(v) \doteq \sqrt{\langle v, v \rangle}, \ \forall v \in H.$$

Trivially $\mathcal{B}$ also defines an inner product on H. Now, given $T : H \to \mathbb{R}$ linear and continuous with respect to $\| \cdot \|_1$, that is, satisfying

$$|T(u)|_{\mathbb{R}} \leq C \|u\|_1, \text{ for any } u \in H \text{ and some constant } C > 0;$$

a priori, it could happen that T is not continuous with respect to the norm $\| \cdot \|_{\mathcal{B}}$ defined by:

$$H \ni v \mapsto \| \cdot \|_{\mathcal{B}}(v) \doteq \sqrt{\mathcal{B}(v, v)}$$

But, in fact, the continuity and coercivity of $\mathcal{B}$ guarantee that $\| \cdot \|_{\mathcal{B}}$ and $\| \cdot \|_1$ are equivalent, so that

$$T \text{ is continuous with respect to } \| \cdot \|_{\mathcal{B}} \iff T \text{ is continuous with respect to } \| \cdot \|_1.$$

Therefore, the Riesz Representation Theorem guarantees the existence of a unique $u_T \in H$ such that

$$T(u) = \mathcal{B}[u_T, u], \quad \forall u \in H.$$

■

Chapter 5

Spectral Theory of Compact and Self-Adjoint Operators

Question 40.
State the Fredholm Alternative.

Solution:

Theorem (T.5.1) (Fredholm Alternative). *Let H be a real Hilbert space and $K : H \to H$ a compact linear operator. Then*

(i) $\dim \ker(\mathrm{Id} - K) < \infty$*;*

(ii) $\mathrm{Im}(\mathrm{Id} - K)$ *is a closed subspace and* $\mathrm{Im}(\mathrm{Id} - K) = (\ker(\mathrm{Id} - K^*))^\perp$*;*

(iii) $(\mathrm{Id} - K)$ *is injective if and only if it is surjective;*

(iv) $\dim \ker(\mathrm{Id} - K) = \dim \ker(\mathrm{Id} - K^*)$.

Theorem (T.5.2) (Fredholm Alternative). *Let E be a Banach space and $T : E \longrightarrow E$ a compact operator. Then one and only one of the possibilities below occurs:*

(a) T has a nonzero fixed point.

(b) The equation $T(x) - x = y$ has a solution for every $y \in E$.

The Fredholm alternative deals with the solvability of the equation $u - Tu = f$. It states that

- either for every $f \in E$ the equation $u - Tu = f$ has a unique solution,

- or the homogeneous equation $u - Tu = 0$ admits $n \geq 1$ linearly independent solutions, and, in this case, the nonhomogeneous equation $u - Tu = f$ is solvable if and only if f satisfies n *orthogonality conditions*, that is,

$$f \in N(I - T^*)^{\perp}.$$

The equivalence of injectivity and surjectivity is familiar in finite-dimensional spaces. If $\dim E < \infty$, a linear operator from E into itself is *injective* (= one-to-one) if and only if it is *surjective* (= onto). However, *in infinite-dimensional spaces, a bounded operator may be injective without being surjective and vice versa*, as, for example, the right shift (resp. left shift) in ℓ^2. Therefore, statement (iii) is a remarkable property of the operators of the form $I - T$ with $T \in \mathcal{K}(E)$.

■

Question 41.
What is the point spectrum of the Volterra operator? Can you justify it?

Solution: We recall the following

Theorem (T.5.3) (Spectral theory of compact operators). *Let H be an infinite-dimensional real Hilbert space and $K : H \to H$ a compact operator. Then,*

(i) $0 \in \sigma(K)$;

(ii) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$;

(iii) if $\sigma(K) \setminus \{0\}$ is an infinite set then $\sigma(K) \setminus \{0\} = (\lambda_m)_{m \in \mathbb{N}}$ with $\lambda_m \to 0$ as $m \to \infty$.

The Volterra operator $T : L^2[0,1] \to L^2[0,1]$ is defined by

$$Tf(x) = \int_0^x f(t)\, dt$$

If we show that

$$\sigma(T) = \{0\}$$

then, by the theorem above, we will have $\sigma_p(T) = \{0\}$. Below is a translation (by ChatGPT, for lack of time at the moment of writing) of the following answer on MSE, which shows in particular that T has no nonzero eigenvalues (in particular its point spectrum is $\{0\}$).

Let $f \in L^2[0,1]$ with $\|f\|_2 \leq 1$. By the Cauchy-Schwarz inequality, we have

$$|(Tf)(x) - (Tf)(y)| \leq \int_y^x |1 \cdot f(t)|\, dt \leq \left(\int_y^x dt\right)^{1/2} \left(\int_y^x |f(t)|^2\, dt\right)^{1/2}$$

$$\leq \|f\|_2 |x - y|^{1/2} = |x - y|^{1/2}.$$

This shows that Tf is Hölder continuous with exponent $1/2$. By the Arzelà-Ascoli Theorem, it follows that the operator $T : L^2[0,1] \to C[0,1]$ is compact. Since the identity map from $C[0,1]$ into $L^2[0,1]$ is bounded, one concludes that $T : L^2[0,1] \to L^2[0,1]$ is also compact. By the Spectral Theorem for Compact Operators, all nonzero elements of the spectrum of T are eigenvalues.

Now, suppose that $\lambda \neq 0$ and that $f \neq 0$ is a corresponding eigenfunction of T, that is,

$$\int_0^x f(t)\,dt = \lambda f(x).$$

Since the left-hand side of the equation is continuous, it follows that the right-hand side is too, which implies that f is continuous. Since f is continuous, the integral on the left is differentiable, which implies that f is continuously differentiable. Applying the Fundamental Theorem of Calculus, we obtain

$$f(x) = \lambda f'(x),$$

whose general solution is of the form $f(x) = Ce^{x/\lambda}$. However, imposing the condition $f(0) = 0$, we obtain $C = 0$, which contradicts the hypothesis $f \neq 0$. Hence, T has no nonzero eigenvalues, and one concludes that $\sigma(T) = \{0\}$. Thus, it follows from the definition that the spectral radius of T is $r(T) = 0$.

Now, consider the function $k : [0,1]^2 \to \mathbb{R}$ defined by

$$k(x,y) = \begin{cases} 1, & \text{if } x \geq y, \\ 0, & \text{otherwise.} \end{cases}$$

With this, we can express T as

$$(Tf)(x) = \int_0^x f(y)\,dy = \int_0^1 k(x,y) f(y)\,dy.$$

By Fubini's Theorem, it follows that

$$\begin{aligned} \langle g, Tf \rangle &= \int_0^1 \overline{g(x)} \int_0^1 k(x,y) f(y)\,dy\,dx \\ &= \int_0^1 f(y) \int_0^1 \overline{g(x)} k(x,y)\,dx\,dy \\ &= \int_0^1 \overline{\int_0^1 g(x)\overline{k(y,x)}\,dx}\, f(y)\,dy \\ &= \langle \int_0^1 g(x)\overline{k(y,x)}\,dx, f \rangle. \end{aligned}$$

In this way, we obtain the expression for the adjoint operator:

$$(T^* f)(y) = \int_y^1 f(t)\,dt.$$

We can now compute T^*T:

$$(T^*Tf)(x) = \int_x^1 \int_0^y f(t)\,dt\,dy.$$

To determine the norm of T, we use the B^* identity: $\|T\| = \sqrt{\|T^*T\|}$. Since $S = T^*T$ is compact and self-adjoint, its set of eigenvalues contains one of largest absolute value, which corresponds to the norm of S by the Spectral Theorem for Compact Self-Adjoint Operators.

Using an argument similar to the previous one, if $\lambda \neq 0$ and $f \neq 0$ are an eigenvalue and an eigenfunction of S, then f is twice continuously differentiable and satisfies

$$\lambda f''(x) = -f(x).$$

The solutions of this equation are of the form

$$f(x) = \alpha e^{i\omega x} + \beta e^{-i\omega x}, \quad \text{with } \omega^2 = 1/\lambda.$$

A direct computation shows that the eigenfunctions must be of the form

$$f(x) = 2\alpha \cos(\omega x),$$

with eigenvalues

$$\lambda_n = \frac{4}{(2n+1)^2\pi^2}.$$

Maximizing λ_n, we obtain that the largest eigenvalue occurs at $n = 0$, resulting in $\|S\| = 4/\pi^2$ and, therefore, the norm of T is

$$\|T\| = \sqrt{4/\pi^2} = \frac{2}{\pi}.$$

For further references on this subject, consult the literature on the *Volterra Operator* and *Hilbert-Schmidt Kernels*.

■

Question 42.
When does the point spectrum of an operator (distinguish the space on which this operator is defined) coincide with the spectrum?

Solution: Given a real Hilbert space H, we recall that the point spectrum is defined by:

$$\sigma_p(T) = \{\lambda \in \sigma(T) : \ker(T - \lambda \mathrm{Id}) \neq \{0\}\}$$

When $\lambda \in \sigma_p(T)$, we say that λ is an eigenvalue of T. The spectrum is given by:

$$\sigma(T) = \{\lambda \in \mathbb{R} : T - \lambda \mathrm{Id} \text{ is not bijective}\}.$$

By Theorem **(T.5.3)**, in infinite dimension we always have $0 \in \sigma(K)$, and the spectrum of a compact operator consists entirely of eigenvalues, with the only possible exception being 0. It follows that the point spectrum coincides with the spectrum whenever 0 is an eigenvalue. In finite dimension we do not

always have $0 \in \sigma(K)$ (recall also that in finite dimension every linear operator is compact), but in this case the spectrum and the point spectrum always coincide.

■

Question 43.
What can be said about the spectrum of a self-adjoint operator on a Hilbert space?

Solution:

Theorem (T.5.4). *Let H be a Hilbert space and $T \in \mathcal{L}(H)$ a self-adjoint operator. Let*

$$m = \inf_{x \in B_H} \langle Tx, x \rangle = \inf_{x \in H \setminus \{0\}} \frac{\langle Tx, x \rangle}{\|x\|^2},$$

$$M = \sup_{x \in B_H} \langle Tx, x \rangle = \sup_{x \in H \setminus \{0\}} \frac{\langle Tx, x \rangle}{\|x\|^2}.$$

Then $\sigma(T) \subset [m, M]$. Moreover, $m, M \in \sigma(T)$ and

$$\|T\| = \max(|m|, |M|).$$

■

Question 44.
What can be said about a self-adjoint operator on a Hilbert space whose spectrum is $\{0\}$?

Solution: By the answer to the previous question, such an operator is identically zero.

■

Question 45.
What can be said about the eigenvectors of compact self-adjoint operators on a Hilbert space?

Solution:

Theorem (T.5.5) (Spectral Decomposition of Compact Self-Adjoint Operators). *Let H be a separable Hilbert space and $T \in \mathcal{L}(H)$ a compact self-adjoint operator. Then the linearly independent eigenvectors of T form a Hilbert basis for H.*

■

Chapter 6

Sobolev Spaces

Question 46.
Define weak derivative.

Solution:

Definition (D.6.1) (Space of locally integrable functions). *The space $L^1_{loc}(\Omega)$ consists of the Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that, for every compact set $K \subset \Omega$, we have*

$$\int_K |u(x)| \, \mathrm{d}x < \infty.$$

Definition (D.6.2). The space $C_0^\infty(\Omega)$ is the set of functions $\varphi : \Omega \to \mathbb{R}$ that are infinitely differentiable and have compact support in Ω, i.e., there exists a compact set $K \subset \Omega$ such that

$$\operatorname{supp}(\varphi) = \overline{\{x \in \Omega \mid \varphi(x) \neq 0\}} \subset K;$$

in particular $\varphi(x) = 0$ for every $x \in \Omega \setminus K$.

Definition (D.6.3). *Given an open set $\Omega \subset \mathbb{R}^n$, a function $u \in L^1_{loc}(\Omega)$ and a multi-index α, we say that $v \in L^1_{loc}(\Omega)$ is an α-th weak derivative of u if*

$$\int_\Omega u \cdot D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \cdot \varphi \, \mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\Omega).$$

■

Question 47.
Why do we require the weak derivative to be in $L^1_{loc}(\Omega)$?

Solution: To guarantee that the weak derivative is well defined (that is, it is unique, i.e., two functions that are weak derivatives of the same function coincide a.e. in Ω).

■

Question 48.
Exhibit an example of a function that has no classical derivative but has a weak derivative.

Solution: g is the weak derivative of $f(x) = |x|$ if and only if, $\forall \phi \in C_c^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f\,\phi'\,\mathrm{d}x = -\int_{\mathbb{R}} g\,\phi\,\mathrm{d}x \tag{6.1}$$

Let us define

$$g(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \alpha & x = 0 \end{cases}$$

where $\alpha \in \mathbb{R}$ is an arbitrary constant. Let us verify that (6.1) is satisfied. Indeed,

$$\int_{\mathbb{R}} |x|\phi'(x)\,\mathrm{d}x = \int_{-\infty}^{0} (-x)\phi'(x)\,\mathrm{d}x + \int_{0}^{+\infty} x\phi'(x)\,\mathrm{d}x.$$

We integrate both integrals by parts:

$$\int_{-\infty}^{0} (-x)\phi'(x)\,\mathrm{d}x = [-x\phi(x)]_{-\infty}^{0} + \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x.$$

Since ϕ has compact support, the term at $-\infty$ is zero, leaving:

$$-0 \cdot \phi(0) + \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x = \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x.$$

Analogously,

$$\int_{0}^{+\infty} x\phi'(x)\,\mathrm{d}x = [x\phi(x)]_{0}^{+\infty} - \int_{0}^{+\infty} \phi(x)\,\mathrm{d}x.$$

The term at $+\infty$ also vanishes, resulting in

$$0 \cdot \phi(0) - \int_{0}^{+\infty} \phi(x)\,\mathrm{d}x = -\int_{0}^{+\infty} \phi(x)\,\mathrm{d}x.$$

Adding the contributions,

$$\int_{\mathbb{R}} |x|\phi'(x)\,\mathrm{d}x = \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x - \int_{0}^{+\infty} \phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}} g(x)\phi(x)\,\mathrm{d}x.$$

Hence, (6.1) is satisfied with arbitrary α, which shows that g is the weak derivative of f.

■

Question 49.
Exhibit an example of a function that has no weak derivative.

Solution: We consider on $\mathbb{R}$ the Heaviside function

$$H : x \mapsto \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

Then u has no weak derivative on $\mathbb{R}$. For the proof, we consider $v \in L^1_{\text{loc}}(\mathbb{R})$. Let $\phi \in C_c^\infty(\mathbb{R}, [0,1])$ with compact support in $(-1,1)$ and such that $\phi(0) = 1$. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$, we define $\phi_n(x) = \phi(nx)$. Thus, $\phi_n \in C_c^\infty(\mathbb{R})$ for every $n \in \mathbb{N}^*$. On the one hand, by the Dominated Convergence Theorem, we have

$$\left| \int_{\mathbb{R}} v(x)\phi_n(x)\,dx \right| \leq \int_{x=-\frac{1}{n}}^{\frac{1}{n}} |v(x)|\,dx \xrightarrow[n\to+\infty]{} 0.$$

On the other hand, we have

$$-\int_{\mathbb{R}} H(x)\phi_n'(x)\,dx = -\int_0^{+\infty} \phi_n'(x)\,dx = \phi_n(0) = 1.$$

Therefore, no $v \in L^1_{\text{loc}}(\mathbb{R})$ is a weak derivative for H.

■

Question 50.
Define Sobolev spaces.

Solution:

Definition (D.6.4). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leq p \leq \infty$ and $k \in \mathbb{N} \cup \{0\}$. We define the Sobolev space $W^{k,p}(\Omega)$ as*

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \text{ for every multi-index } \alpha \text{ such that } |\alpha| \leq k\},$$

with the derivatives $D^\alpha u$ above being taken in the weak sense.

Definition (D.6.5). *The norm on $W^{k,p}(\Omega)$ is defined by*

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int |D^\alpha u|^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}, & \text{if } p = \infty. \end{cases}$$

■

Question 51.
Define the spaces $W_0^{m,p}$. What characterization of these spaces do you know?

Solution:

Definition (D.6.6). Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leq p \leq \infty$ and $k \in \mathbb{N} \cup \{0\}$. The space $W_0^{k,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{k,p}$, i.e.,

$$W_0^{k,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}.$$

Theorem (T.6.1) (Trace Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1 and $1 \leq p < \infty$. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

such that,

(i) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$;

(ii) there exists $C = C(p, \Omega) > 0$ such that

$$\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

Theorem (T.6.2) (Characterization of $W_0^{1,p}$ in terms of the trace). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1 and $1 \leq p < \infty$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $Tu = 0$ on $\partial\Omega$.*

■

Question 52.
State the continuous embeddings of the Sobolev spaces.

Solution:

Theorem (T.6.3). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then,*

(i) if $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega),$$

for every $1 \leq q \leq \dfrac{np}{n - kp}$;

(ii) if $kp = n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega),$$

for every $q \geq 1$;

(iii) if $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C^{k-\lfloor \frac{n}{p} \rfloor - 1, \gamma}(\Omega),$$

where

$$\gamma = \begin{cases} \lfloor \dfrac{n}{p} \rfloor + 1 - \dfrac{n}{p}, & \text{if } \dfrac{n}{p} \notin \mathbb{Z}, \\ \text{any number in } (0,1), & \text{if } \dfrac{n}{p} \in \mathbb{Z}, \end{cases}$$

and $\lfloor n/p \rfloor$ is the largest integer less than or equal to n/p.

The theorem remains valid if $\Omega \subset \mathbb{R}^n$ is an arbitrary (possibly unbounded) open set of class C^1 with the restriction $q \geq p$ in the first two items.

■

Question 53.
State the compact embeddings of the Sobolev spaces.

Solution:

Theorem (T.6.4) (Rellich-Kondrachov). *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set of class C^1.*

(i) If $1 \leq p < n$ and $1 \leq q < p^*$, then

$$W^{1,p}(\Omega) \overset{cpt.}{\hookrightarrow} L^q(\Omega).$$

(ii) If $p = n$ and $q \geq 1$, then

$$W^{1,p}(\Omega) \overset{cpt.}{\hookrightarrow} L^q(\Omega).$$

(iii) If $n < p$ and $0 < \gamma < 1 - (n/p)$, then

$$W^{1,p}(\Omega) \overset{cpt.}{\hookrightarrow} C^{0,\gamma}(\Omega).$$

Moreover, the embeddings of $W_0^{1,p}(\Omega)$ into the spaces above are always compact, regardless of the regularity of Ω.

■

Question 54.
Define the Hilbert spaces H^k. Into which other spaces of higher regularity are these spaces embedded?

Solution:

Definition (D.6.7). When $p = 2$, we will denote $W^{k,p}$ simply by $H^k(\Omega)$, that is, $H^k(\Omega) = W^{k,2}(\Omega)$.

Theorem (T.6.5). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1, $k \in \mathbb{N}$. Then,*

(i) if $2k < n$

$$H^k(\Omega) \hookrightarrow L^q(\Omega),$$

for every $1 \leq q \leq \dfrac{2n}{n-2k}$;

(ii) if $2k = n$

$$H^k(\Omega) \hookrightarrow L^q(\Omega),$$

for every $q \geq 1$*;*

(iii) if $2k > n$

$$H^k(\Omega) \hookrightarrow C^{k-\lfloor \frac{n}{2} \rfloor - 1, \gamma}(\Omega),$$

where

$$\gamma = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1 - \frac{n}{2}, & \text{if } \frac{n}{2} \notin \mathbb{Z}, \\ \text{any number in } (0,1), & \text{if } \frac{n}{2} \in \mathbb{Z}, \end{cases}$$

and $\lfloor n/2 \rfloor$ *is the largest integer less than or equal to* $n/2$*.*

The theorem remains valid if $\Omega \subset \mathbb{R}^n$ *is an arbitrary (possibly unbounded) open set of class* C^1 *with the restriction* $q \geq 2$ *in the first two items.*

■

Question 55.
When is $W^{m,p}$ reflexive $1 < p < \infty$? When is it separable?

Solution:

Theorem (T.6.6). *The space* $W^{k,p}(\Omega)$ *is reflexive if* $1 < p < \infty$ *and separable if* $1 \leq p < \infty$*.*

Proof: Let us consider the general case $k \geq 1$. Observe that $W^{k,p}(\Omega)$ can be isometrically embedded into $(L^p(\Omega))^N$ for $N = \binom{n+k}{k}$ through the map

$$I : W^{k,p}(\Omega) \longrightarrow (L^p(\Omega))^N$$

defined by

$$I(u) := (D^\alpha u)_{|\alpha| \leq k},$$

where $D^\alpha u$ represents the weak derivatives of u up to order k, and the space $X := (L^p(\Omega))^N$ is endowed with the norm

$$\|(v_\alpha)_{|\alpha| \leq k}\|_{(L^p(\Omega))^N} = \left(\sum_{|\alpha| \leq k} \|v_\alpha\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

for every $(v_\alpha)_{|\alpha| \leq k} \in (L^p(\Omega))^N$.

This means that we can identify $W^{k,p}(\Omega)$ with the corresponding subspace $Y := I(W^{k,p}(\Omega))$ of X. Since $W^{k,p}(\Omega)$ is complete, it follows that Y is a closed subspace of X. But X is reflexive when $1 < p < \infty$ and separable when $1 \leq p < \infty$, which shows that the closed subspace Y (and therefore $W^{k,p}(\Omega)$) has these same properties.

■

■

Question 56.
State the Extension Theorem and give an example (application).

Solution:

Theorem (T.6.7) (Extension Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1, $1 \leq p \leq \infty$ and $\widetilde{\Omega} \subset \mathbb{R}^n$ an open set such that $\Omega \Subset \widetilde{\Omega}$. There exists a bounded linear operator*

$$E : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

such that, for every $u \in W^{1,p}(\mathbb{R}^n)$, we have

(i) $(Eu)(x) = u(x)$ *a.e. in* Ω;

(ii) $\operatorname{supp} Eu \subset \widetilde{\Omega}$;

(iii) there exists $C = C(p, \Omega, \widetilde{\Omega}) > 0$ *such that*

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

■

Question 57.
State the Trace Theorem and give an example (application).

Solution:

Theorem (T.6.8) (Trace Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1 and $1 \leq p < \infty$. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

such that,

(i) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$;

(ii) there exists $C = C(p, \Omega) > 0$ *such that*

$$\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

Theorem (T.6.9) (Characterization of $W_0^{1,p}$ in terms of the trace). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1 and $1 \leq p < \infty$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $Tu = 0$ on $\partial\Omega$.*

■

Question 58.
Is it possible to define the trace of functions in L^p?

Solution: No. Since $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $u_n \to 1$. Therefore, we have $1 = T(1) = \lim_{n\to\infty} T(u_n) = 0$, which is absurd.

■

Question 59.
What do we gain by working with the space H_0^1? Why not simply work with the space $\mathscr{C}_0^\infty$?

Solution: C_0^∞ is not a Hilbert space (in particular, we cannot apply the Riesz or Lax-Milgram theorems), and by working only with C_0^∞ we are asking for classical solutions, which are much harder to find than weak solutions.

Moreover, the space H_0^1 naturally incorporates the Dirichlet boundary conditions.

■

Question 60.
State the Poincaré inequality. What is the importance of this inequality? How is it proved and how is it used? Does it hold for unbounded open sets?

Solution: to be filled in

■

Question 61.
What is the best constant for the Poincaré inequality? Justify.

Solution: to be filled in

■

Question 62.
Given a function $f : (0, 1) \to \mathbb{R}$, how do we extend it to the closure?

Solution: If it is uniformly continuous.

■

Question 63.
Where is $W^{1,2}$ embedded? And compactly embedded where?

Solution:

(i) If $n = 1$, then

$$W^{1,2}(\Omega) \overset{\text{cpt.}}{\hookrightarrow} L^q(\Omega), \quad \forall 1 \leq q < 2^* = \frac{2n}{n-2}.$$

(ii) If $n = 2$, then

$$W^{1,2}(\Omega) \overset{\text{cpt.}}{\hookrightarrow} L^q(\Omega), \quad \forall q \geq 1.$$

(iii) If $n > 2$ and $0 < \gamma < 1 - \frac{2}{n}$, then

$$W^{1,2}(\Omega) \overset{\text{cpt.}}{\hookrightarrow} C^{0,\gamma}(\Omega).$$

■

Chapter 7

Second-order Linear Elliptic Equations

Question 64.
Is it necessary to impose boundary conditions in order to obtain uniqueness of solutions?

Solution: Yes. As a concrete example, note that in $(0, \infty) \times \mathbb{R}$, the functions $(x_1, x_2) \mapsto x_1$ and $(x_1, x_2) \mapsto x_1 + x_2$ are both obviously harmonic, but their values on the boundary $\{0\} \times \mathbb{R}$ are different.

■

Question 65.
Give an example of some elliptic PDE and solve it.

Solution: to be filled in

■

Question 66.
Give an example of an elliptic PDE that has no weak solution.

Solution: to be filled in

■

Question 67.
Give an example of an elliptic PDE that has no classical solution.

Solution: to be filled in

■

Question 68.
Is a weak solution that has enough regularity, say $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, a classical solution?

Solution: Yes, it suffices to integrate by parts backwards and use the Fundamental Lemma of the Calculus of Variations.

■

Question 69.
Give an example of an elliptic PDE that has a weak solution but not a classical one.

Solution: Let $B \subset \mathbb{R}^3$ be the open unit ball and consider the function

$$w(x) = |x|^{-\lambda} - 1, \quad \lambda < \frac{1}{2}.$$

This function is clearly not continuous, but $w \in H_0^1(B)$.

Define $f = -\Delta w$ and consider the problem

$$\begin{cases} -\Delta u = f, & \text{in } B, \\ u \in H_0^1(B). \end{cases}$$

This equation has a unique weak solution $u = w$ (where we refer to the definition of weak solution given in the question). This function is not continuous at 0, therefore, it is not a classical solution.

■

Question 70.
Consider the following PDE, where $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P_{2\pi}) \quad \begin{cases} -\Delta u = 2\pi, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Does $(P_{2\pi})$ have a weak solution? If it does, is the weak solution unique?

Solution: Yes, since $(\Omega \ni x \mapsto 2\pi) \in L^2(\Omega)$ (because Ω is bounded).

■

Question 71.
Consider the following PDE, where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P_1) \quad \begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Impose conditions on Ω and f in order to carry out the weak formulation. Define weak solution and use the Riesz Representation Theorem to guarantee the existence of a solution.

Solution: to be filled in

■

Question 72.
Consider the following PDE, where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P) \quad \begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

What should be imposed on g to make the problem well posed? Does (P) have a weak solution? If it does, is the weak solution unique?

Solution: to be filled in

■

Question 73.
Consider the following PDE, where $f \in L^2(\Omega)$, $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Impose conditions on Ω and λ in order to carry out the weak formulation. When is it possible to obtain some weak solution? Is such a solution unique?

Solution: to be filled in

■

Question 74.
Consider the following PDE, where $f \in L^2(\Omega)$, $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P_1) \quad \begin{cases} -\Delta u + \lambda u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Impose conditions on Ω, f and λ in order to carry out the weak formulation. Define weak solution and use the Lax-Milgram Theorem to guarantee the existence of a solution. Could the Riesz Representation Theorem be used in this case? Justify.

Solution: See the Appendix. The continuity and coercivity of the bilinear form associated with the weak formulation guarantee that the norm induced by B is equivalent to the norm of $H_0^1(\Omega)$, which makes it possible to apply the Riesz Representation Theorem.

■

Question 75.
State the Weak Maximum Principle ($c \geq 0$) and give an idea of the proof. Give an application of this principle.

Solution:

Let $u : \Omega \to \mathbb{R}$ be any function. We define the *positive part* u^+ and the *negative part* u^- of the function u as

$$u^+(x) := \max\{u(x), 0\}, \quad u^-(x) := \max\{-u(x), 0\}, \quad \text{for } x \in \Omega.$$

Observe that the functions above are nonnegative and satisfy the following equalities:

$$|u| = u^+ + u^-, \quad u = u^+ - u^-.$$

We will consider uniformly elliptic operators of the form

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x)u$$

Theorem (T.7.1) (Weak Maximum Principle). *Let L be a uniformly elliptic operator in Ω with $c \leq 0$ in Ω. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then the following items hold:*

(i) If $Lu \geq 0$ in Ω, then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+.$$

(ii) If $Lu \leq 0$ in Ω, then

$$\min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-.$$

(iii) If $Lu = 0$ in Ω, then

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

If we consider, as Evans does, uniformly elliptic operators of the form

$$\tilde{L}u = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$$

then we have the corresponding version for $c \geq 0$ (it suffices to work with $L \doteq -\tilde{L}$), that is,

Theorem (T.7.2). *Let L be a uniformly elliptic operator in Ω with $c \leq 0$ in Ω. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then the following items hold:*

(i) If $Lu \leq 0$ in Ω, then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+.$$

(ii) If $Lu \geq 0$ in Ω, then

$$\min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-.$$

(iii) If $Lu = 0$ in Ω, then

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

A common application is to guarantee uniqueness of classical solutions. The idea of the proof is to use the maximum principle for $c \equiv 0$ for the operator $Ku = Lu - c(x)u$, working on the set $\Omega^+ \doteq u^{-1}((0, \infty))$.

■

Question 76.
State Hopf's Lemma. What is the most relevant information it provides?

Solution:

Theorem (T.7.3). *Suppose that L is uniformly elliptic and $Lu \geq 0$ in Ω, with $u \in C^2(\Omega)$. Let $x_0 \in \partial\Omega$ be such that*

(i) u is continuous at x_0;

(ii) $u(x_0) > u(x)$, for every $x \in \Omega$;

(iii) there exists an open ball $B \subseteq \Omega$ such that $x_0 \in \partial B$.

Then the outer normal derivative at x_0, if it exists, satisfies

$$\frac{\partial u}{\partial \eta}(x_0) > 0,$$

provided that one of the hypotheses below holds:

(a). $c \equiv 0$ *in* B*;*

(b). $c \leq 0$ *in* B *and* $u(x_0) \geq 0$.

The most relevant information is that the outer normal derivative of a function that attains a nonnegative maximum at a boundary point is *strictly* positive.

■

Question 77.
State the Strong Maximum Principle and give an idea of the proof. Give an application of this principle.

Solution:

Theorem (T.7.4) (Strong Maximum Principle). *Let* $\Omega \subset \mathbb{R}^n$ *be a connected open set,* L *a uniformly elliptic operator in* Ω *with* $c \equiv 0$ *and* $u \in C^2(\Omega) \cap C(\overline{\Omega})$*. Then*

(i) if $Lu \geq 0$ *in* Ω *and* u *attains a maximum in* Ω*, then* u *is constant in* Ω*;*

(ii) if $Lu \leq 0$ *in* Ω *and* u *attains a minimum in* Ω*, then* u *is constant in* Ω*.*

In the case $c \leq 0$ *the following holds:*

(iii) if $Lu \geq 0$ *in* Ω *and* u *attains a nonnegative maximum in* Ω*, then* u *is constant in* Ω*;*

(iv) if $Lu \leq 0$ *in* Ω *and* u *attains a nonpositive minimum in* Ω*, then* u *is constant in* Ω*.*

The proof is based on an argument by contradiction and on the application of *Hopf's Lemma*. The main steps are:

(a). Suppose that u attains a maximum at an interior point of Ω and define the set

$$\Omega_M = \{x \in \Omega : u(x) = M\},$$

where $M = \max_{\overline{\Omega}} u$.

(b). Suppose, by contradiction, that $\Omega_M \neq \Omega$, that is, there exists a region $\Sigma = \Omega \setminus \Omega_M$ where $u(x) < M$.

(c). Choose a point $y \in \Sigma$ closer to Ω_M and construct a ball around y contained in Σ.

(d). Since $x_0 \in \partial B \cap \Omega_M$ is a maximum point in Ω, the gradient of u vanishes at x_0, that is,

$$\nabla u(x_0) = 0.$$

(e). By *Hopf's Lemma*, if u is not constant, the normal derivative at x_0 must be strictly positive, which contradicts $\nabla u(x_0) = 0$.

(f). We conclude that $\Omega_M = \Omega$, that is, u is constant in Ω.

The same argument holds for the case of the minimum, considering $v = -u$. If $c \leq 0$, the proof follows analogously, using the corresponding version of *Hopf's Lemma*.

■

Chapter 8

Regularity of Solutions of Elliptic PDEs

Suppose that $U \subset \mathbb{R}^n$ is a bounded open set. Suppose also that $u \in H_0^1(U)$ is a weak solution of the PDE $Lu = f$ (in U) , where L has the divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{8.1}$$

We continue to require the uniform ellipticity condition (i.e., for each $x \in U$, the matrix $A(x) = ((a^{ij}(x)))$ is positive definite, with its smallest eigenvalue being $\geq \theta > 0$) and, as needed, we will make several additional assumptions about the regularity of the coefficients a^{ij}, b^i, c.

Theorem (T.8.1) (Interior H^2 regularity - see 6.3.1 of Evans, page 327 of the book). *Suppose that*

$$a^{ij} \in C^1(U), \quad b^i, c \in L^\infty(U) \quad (i, j = 1, \ldots, n) \tag{8.2}$$

and that

$$f \in L^2(U). \tag{8.3}$$

Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \quad \text{in } U. \tag{8.4}$$

Then

$$u \in H^2_{loc}(U); \tag{8.5}$$

and, for each open subset $V \subset\subset U$, we have the estimate

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right), \tag{8.6}$$

where the constant C depends only on V, U and the coefficients of L.

Remark (O.1). (i) Observe carefully that we do not require $u \in H_0^1(U)$; that is, we are not necessarily assuming the boundary condition $u = 0$ on ∂U in the trace sense.

(ii) Observe additionally that, since $u \in H^2_{\text{loc}}(U)$, we have

$$Lu = f \quad \text{a.e. in } U.$$

Thus, u does in fact solve the PDE, at least for almost every point inside U. (To see this, observe that for each $v \in C_c^\infty(U)$, we have

$$B[u, v] = (f, v).$$

Since $u \in H^2_{\text{loc}}(U)$, we can integrate by parts and obtain:

$$B[u, v] = (Lu, v).$$

Thus, $(Lu - f, v) = 0$ for every $v \in C_c^\infty(U)$, and therefore $Lu = f$ a.e.)

Theorem (T.8.2) (Higher interior regularity - page 332 of Evans). *Let m be a nonnegative integer, and suppose that*

$$a^{ij}, b^i, c \in C^{m+1}(U) \quad (i, j = 1, \dots, n) \tag{8.7}$$

and that

$$f \in H^m(U). \tag{8.8}$$

Suppose that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \quad \text{in } U. \tag{8.9}$$

Then

$$u \in H^{m+2}_{loc}(U); \tag{8.10}$$

and, for each $V \subset\subset U$, we have the estimate

$$\|u\|_{H^{m+2}(V)} \leq C \left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)} \right), \tag{8.11}$$

where the constant C depends only on m, U, V and the coefficients of L.

Applying Theorem **(T.8.2)** recursively together with the Sobolev embedding, we obtain the following result:

Theorem (T.8.3) (Infinite differentiability in the interior - page 334 of Evans). *Suppose that*

$$a^{ij}, b^i, c \in C^\infty(U) \quad (i, j = 1, \dots, n) \tag{8.12}$$

and that

$$f \in C^\infty(U). \tag{8.13}$$

Suppose that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \quad \text{in } U. \tag{8.14}$$

Then

$$u \in C^\infty(U). \tag{8.15}$$

Remark (O.2). Note again that we are not making any assumption about the behavior of u on ∂U. Therefore, we are asserting that any possible singularities of u on the boundary do not propagate into the interior.

In particular, for *any open set, bounded or unbounded,* $\Omega \subset \mathbb{R}^n$, the weak solution u of the elliptic PDE $Lu = f$ is smooth (and therefore classical!) in $\operatorname{int}\Omega = \Omega$ whenever the coefficients of L are smooth, regardless of any singularities that may exist on $\partial\Omega$. Indeed, it suffices to recall that smoothness is a local property and to apply Theorem **(T.8.3)** to each ball $B_r(x) \subset \Omega$ contained in Ω. We can do this, since Theorem **(T.8.3)** makes no assumption about the behavior of u on the boundary of the ball (you could argue that it is not guaranteed that $u = 0$ on the boundary of the ball $\partial B_r(x_0)$, and indeed it is not, but Theorem **(T.8.3)** does not care about that).

Theorem (T.8.4) (H^2 regularity up to the boundary - page 334 of Evans). *Suppose that*

$$a^{ij} \in C^1(\overline{U}), \quad b^i, c \in L^\infty(U) \quad (i,j = 1,\dots,n) \tag{8.16}$$

and that

$$f \in L^2(U). \tag{8.17}$$

Suppose that $u \in H_0^1(U)$ is a weak solution of the elliptic boundary value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{8.18}$$

Suppose, finally, that

$$\partial U \text{ is } C^2. \tag{8.19}$$

Then

$$u \in H^2(U), \tag{8.20}$$

and we have the estimate

$$\|u\|_{H^2(U)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right), \tag{8.21}$$

where the constant C depends only on U and the coefficients of L.

Remark (O.3). Observe that, in sharp contrast with Theorem **(T.8.1)**, we are assuming that $u \in H_0^1(U)$, that is, we are imposing the boundary condition $u = 0$ on ∂U in the trace sense.

Theorem (T.8.5) (Higher regularity up to the boundary). *Let m be a nonnegative integer, and suppose that*

$$a^{ij}, b^i, c \in C^{m+1}(\overline{U}) \quad (i, j = 1, \dots, n) \tag{8.22}$$

and that

$$f \in H^m(U). \tag{8.23}$$

Suppose that $u \in H_0^1(U)$ is a weak solution of the boundary value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{8.24}$$

Suppose, finally, that

$$\partial U \text{ is } C^{m+2}. \tag{8.25}$$

Then

$$u \in H^{m+2}(U), \tag{8.26}$$

and we have the estimate

$$\|u\|_{H^{m+2}(U)} \leq C \left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)} \right), \tag{8.27}$$

where the constant C depends only on m, U and the coefficients of L.

Iterating the conclusions of Theorem **(T.8.5)** infinitely many times, we obtain the following result:

Theorem (T.8.6) (Infinite differentiability up to the boundary). *Suppose that*

$$a^{ij}, b^i, c \in C^\infty(\overline{U}) \quad (i, j = 1, \dots, n) \tag{8.28}$$

and that

$$f \in C^\infty(\overline{U}). \tag{8.29}$$

Suppose that $u \in H_0^1(U)$ is a weak solution of the boundary value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{8.30}$$

Suppose also that ∂U is C^∞. Then

$$u \in C^\infty(\overline{U}). \tag{8.31}$$

Question 78.
Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ be a harmonic function on a possibly unbounded open set $\Omega \subset \mathbb{R}^n$. What can we say about the interior regularity of u?

Solution: *Regardless* of whether Ω is bounded or not, the fact that u is harmonic implies that it satisfies the mean value property, which guarantees the smoothness of u in $\operatorname{int}(\Omega) = \Omega$.

■

Question 79.
Suppose that $f \in C^0(\Omega)$, where Ω is a smooth domain. Does the problem $-\Delta u = f$ admit a classical solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ in this case? Note that we are not imposing boundary conditions.

Solution: No! See the examples below.

Example (E.8.1). The function $\rho(x) = \rho(x_1, x_2)$ defined on the disk $B_a = \{x : x_1^2 + x_2^2 < a\}$ of radius $a < 1$ by

$$\rho(x) = \begin{cases} |x|^{-2} \left[\log |x|\right]^{-1} x_1 x_2, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is a continuous function on B_a, whose corresponding Newtonian potential is not C^2 at $x = 0$.

Example (E.8.2). ($f \in L^\infty$, but $u \notin C^{1,1}$).

$$u(x_1, x_2) = |x_1||x_2| \log(|x_1| + |x_2|).$$

We compute (for $x_1, x_2 > 0$)

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left[x_2 \log(x_1 + x_2)\right] = \frac{2x_2}{x_1 + x_2} - \frac{x_1 x_2}{(x_1 + x_2)^2};$$

$$\frac{\partial^2 u}{\partial x_2^2} = \frac{2x_1}{x_1 + x_2} - \frac{x_1 x_2}{(x_1 + x_2)^2}.$$

Thus,

$$\Delta u = 2 - \frac{2x_1 x_2}{(x_1 + x_2)^2}$$

and the right-hand side is a bounded function. However, note also that

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \log(x_1 + x_2) + 1 - \frac{x_1 x_2}{(x_1 + x_2)^2} \notin L^\infty.$$

Example (E.8.3). (f continuous, but $u \notin C^{1,1}$).

$$\Delta u = f(x) = \frac{x_2^2 - x_1^2}{2|x|^2}\left[\frac{n+2}{(-\log|x|)^{1/2}} + \frac{1}{2(-\log|x|)^{3/2}}\right], \quad x \in B_R \subset \mathbb{R}^n.$$

$f(x)$ is continuous after defining $f(0) = 0$. However, the solution

$$u(x) = (x_2^2 - x_1^2)(-\log|x|)^{1/2}$$

satisfies

$$\frac{\partial^2 u}{\partial x_1^2} \to \infty \quad \text{as} \quad x \to 0.$$

Therefore, $u \notin C^{1,1}$.

■

Question 80.
Consider the following PDE, where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P_1) \quad \begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Impose conditions on Ω and f in order to carry out the weak formulation. When is it possible to obtain weak solutions? In that case, what is the best regularity we can get for such solutions in terms of the regularity of f and of $\partial\Omega$?

Solution: As we saw earlier, it is possible to obtain weak solutions (and uniqueness) for $f \in L^2(\Omega)$ and Ω bounded. The best regularity we can get for such solutions is $u \in C^{1,1-\varepsilon}(\Omega)$; see this post on MSE. Regularity up to the boundary is guaranteed provided the boundary satisfies regularity hypotheses.

■

The examples we have just seen show that the space of merely continuous functions on Ω is not adequate for the classical formulation of the problem. As we will see in the theorems below, the natural spaces for the classical formulation are the Hölder spaces defined on open sets with regular and Hölder boundary.

Theorem (T.8.7). *If $\Omega \subset \mathbb{R}^n$ is a bounded open set of class $C^{2,\gamma}$ and $g \in C^{2,\gamma}(\partial\Omega)$, then the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

has exactly one solution in $C^{2,\gamma}(\overline{\Omega})$.

Using the Newtonian potential, we can also prove the following result:

Theorem (T.8.8). *If $\Omega \subset \mathbb{R}^n$ is a bounded open set of class C^2, $f \in C^{0,\gamma}(\overline{\Omega})$ and $g \in C(\partial\Omega)$, then the problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \qquad (P)$$

has exactly one solution in $C^2(\Omega) \cap C(\overline{\Omega})$.

We can use the result above to discuss the existence of a solution for the problem

$$(P) \quad \begin{cases} Lu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set of class $C^{2,\gamma}$, $0 < \gamma \leq 1$, $f \in C^{0,\gamma}(\overline{\Omega})$, $g \in C^{2,\gamma}(\partial\Omega)$, L is a 2nd-order differential operator of the form

$$Lu := \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x)u,$$

In this context, the main result used to obtain existence of a solution is the following:

Theorem (T.8.9) (Continuation Principle). *Let X be a Banach space, Y a normed vector space and $\mathscr{L}_0, \mathscr{L}_1 : X \to Y$ bounded linear operators. For $t \in [0,1]$, consider the operator*

$$\mathscr{L}_t u := (1-t)\mathscr{L}_0 u + t\mathscr{L}_1 u, \quad u \in X.$$

Suppose that there exists $c > 0$ such that

$$\|u\|_X \leq c\|\mathscr{L}_t u\|_Y, \quad \forall u \in X,\ t \in [0,1]. \qquad \textbf{(8.32)}$$

Then $\mathscr{L}_0$ is surjective if and only if $\mathscr{L}_1$ is surjective.

We use the spaces

$$X \doteq \{u \in C^{2,\gamma}(\overline{\Omega}) \mid u \equiv 0 \text{ on } \partial\Omega\}, \quad Y \doteq C^{0,\gamma}(\overline{\Omega}),$$

and the family of operators

$$L_t := (1-t)L + t\Delta, \quad t \in [0,1].$$

By Theorem **(T.8.9)**, the existence of a solution for the operator $L_0 = L$ is equivalent to the existence of a solution for the problem $L_1 = -\Delta$, a case we have already treated in Theorem **(T.8.8)**.

Question 81.
Consider the following PDE, where $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P_{2\pi}) \quad \begin{cases} -\Delta u = 2\pi, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

When does $(P_{2\pi})$ have a classical solution? What can be said about the interior regularity of possible weak solutions? And what about regularity up to the boundary? What can be said about the sign of classical solutions?

Solution: As we saw earlier, the problem has a unique weak solution. By the interior regularity estimates stated at the beginning of this chapter, such a solution is in fact smooth in the interior of Ω. Smoothness up to the boundary is guaranteed provided the boundary is smooth. By the Maximum Principle, such a solution is positive in Ω.

■

Question 82.
Consider the following PDE, where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

What restriction must g satisfy for (P) to have a classical solution? If $g > 0$ on $\partial\Omega$, what can we say regarding the sign of the weak solution of this PDE? What can be said about the interior regularity of possible weak solutions in terms of the regularity of f? And what about regularity up to the boundary?

Solution:

Definition (D.8.1). Let $f \in L^2(\Omega)$ and $g \in W^{1,2}(\Omega)$. We say that $u \in W^{1,2}(\Omega)$ is a **weak solution** for the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{8.12}$$

if

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \text{for every } v \in W_0^{1,2}(\Omega)$$

and

$$u - g \in W_0^{1,2}(\Omega).$$

If the data of the Dirichlet problem (8.12) are sufficiently regular and the weak solution is also sufficiently regular, then it is a classical solution:

Theorem (T.8.10). *Let $f \in L^2(\Omega)$ and $g \in W^{1,2}(\Omega)$. Then there exists a unique weak solution $u \in W^{1,2}(\Omega)$ for the problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{8.33}$$

Moreover, $u \in W^{1,2}(\Omega)$ minimizes the functional $I : E \to \mathbb{R}$ given by

$$I(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x,$$

where

$$E = \left\{ v \in W^{1,2}(\Omega) : v - g \in W_0^{1,2}(\Omega) \right\}$$

is the space of admissible functions for (8.33). That is, $u \in E$ satisfies

$$I(u) = \min_{v \in E} \left(\frac{1}{2}\int_\Omega |\nabla v|^2 \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x \right).$$

For regularity, see the beginning of this chapter, where we stated the interior and up-to-the-boundary regularity theorems. The regularity will be determined by the regularity of f, g and of the boundary $\partial\Omega$. The sign of the solution depends on the sign of f. If $f > 0$, the solution will be positive.

■

Question 83.
Consider the following PDE, where $f \in L^2(\Omega)$, $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Does (P_λ) have a weak solution for every $\lambda \in \mathbb{R}$? If the answer is no, for which λ does the problem admit a weak solution? Is such a solution unique? What is the best interior regularity of possible weak solutions? And about regularity up to the boundary? Could Agmon-Douglis-Nirenberg be used? State all the theorems used to guarantee the existence, uniqueness and regularity of the solutions.

Solution: It does not admit a solution for every $\lambda \in \mathbb{R}$ (see the next question). Even if Ω is possibly unbounded and with irregular boundary, when the eigenfunctions exist they will be smooth in the interior. The regularity of the eigenfunction up to the boundary depends on the regularity of the boundary.

■

Question 84.
State everything you know about the eigenfunctions of the Laplacian. In fact, there exists a countable set $\Sigma = \{\lambda_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ such that (P_λ) admits a unique nontrivial solution if and only if $\lambda \in \Sigma$.

Solution:

(i) There exists a countable set $\Sigma = \{\lambda_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ such that (P_λ) admits a unique nontrivial solution if and only if $\lambda \in \Sigma$. The associated eigenfunctions satisfy (P_λ).

(ii) The eigenfunctions of the Laplacian form an orthonormal basis of $L^2(\Omega)$ (also of $H_0^1(\Omega)$), are always smooth in the interior, and smooth on the boundary provided the boundary is smooth.

(iii) The first eigenfunction satisfies $-\Delta\varphi_1 = \lambda_1\varphi_1$, where

$$\lambda_1 = \inf_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega v^2 \,\mathrm{d}x} > 0.$$

(iv) The first eigenfunction does not change sign in Ω and the associated eigenvalue is simple, that is, it has multiplicity 1.

(v) For every $j \geq 2$ the associated eigenfunctions φ_j change sign in Ω (a direct consequence of orthogonality).

■

Question 85.
Consider the following PDE, where $f \in L^2(\Omega)$, $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$(P_1) \quad \begin{cases} -\Delta u + \lambda u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

If we only have $f \in C(\Omega)$, is it possible to obtain weak or classical solutions? And if $f \in C\left(\overline{\Omega}\right)$? When it is possible to obtain weak solutions, what is the best regularity we can get for such solutions in terms of the regularity of f and of $\partial\Omega$?

Solution: to be filled in

■

Question 86.
Give an idea of *Bootstrapping* (Regularity).

Solution: to be filled in

■

Chapter 9

Questions from my exam

- I started by stating the Riesz Representation Theorem and then presented the proof of the Lax-Milgram Theorem. I defined an operator $A : H \to H$ determined by $\langle A(x), y\rangle = \mathcal{B}(x, y)$.
- Liliane asked whether I knew some version of the Riesz Representation Theorem for spaces that are not Hilbert spaces. I talked about the L^p spaces and stated the result for $1 < p < \infty$. Mayra asked whether the inequalities in p had to be strict, in particular whether it did not also hold for $p = 1$.
- Liliane asked me to show some application of Lax-Milgram. I talked about the problem

$$\begin{cases} -\Delta u + \partial_{x_m} u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

  and mentioned that this was a case that exhibits more concretely the strength of the Lax-Milgram Theorem, since the bilinear form associated with the weak formulation is not symmetric, making it impossible to apply the Riesz Representation Theorem. I explained that, given a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and a symmetric, continuous and coercive bilinear form $\mathcal{B} : H \times H \to \mathbb{R}$, the norm induced by $\mathcal{B}$ coincides with the original norm of H (precisely because of the continuity and coercivity), making it possible to apply the Riesz Representation Theorem directly.
- They asked me to define the weak formulation of the problem and to explain why the associated bilinear form is not symmetric.
- They asked what I used to obtain the weak formulation of the problem. I talked about integration by parts via the Divergence Theorem and about the motivation coming from the computations with classical derivatives.
- Liliane observed that, in the proof of the Lax-Milgram Theorem, I showed separately that the operator $A : H \to H$ determined by $\langle A(x), y\rangle = \mathcal{B}(x, y)$ was injective and surjective. She then asked whether there would be a need to show injectivity and surjectivity separately if the space were finite-dimensional. I answered no, since in that context they are equivalent.

- Right after, she asked whether in infinite dimension injectivity and surjectivity are equivalent. I said no, and she asked for examples of an operator that is injective but not surjective and vice versa. I talked about the shift operators on ℓ^2.

- Next, she asked whether I knew some result that guaranteed the equivalence of injectivity and surjectivity in infinite dimension. I stated one of the items of the Fredholm alternative, that if K is compact then $\mathrm{Id} - K$ is injective if and only if it is surjective.

- Liliane asked whether I knew an application of the Fredholm alternative to some problem. I talked about the problem

$$\begin{cases} -\Delta u + \lambda u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

  and showed that the problem has a solution provided $\lambda \notin \{\lambda_i\}_{i\in\mathbb{N}}$, where λ_i are the eigenvalues of the (minus) Laplacian $-\Delta$, and I also said that, in case $\lambda \in \{\lambda_i\}_{i\in\mathbb{N}}$, the existence of functions f for which the problem admits no solution is guaranteed, and moreover, when the problem does have a solution in this case, it will never be unique.

  Remark (O.4). Actually, as Gabriel observed after the exam, existence and uniqueness of solutions is guaranteed provided $\lambda \notin \{-\lambda_i\}_{i\in\mathbb{N}}$ (in particular if $\lambda \geq -\lambda_1$), not $\lambda \notin \{\lambda_i\}_{i\in\mathbb{N}}$.

- Liliane talked about the problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

  and asked for which $p \geq 1$ it was possible to establish existence and uniqueness of a solution. I remembered the optimal lower bound $p \geq \dfrac{2n}{n+2}$, they asked me to prove it, I got tangled up and they moved on to the next question.

- While doing the details of the application of the Fredholm alternative, I mentioned that, using the compact embeddings of the Sobolev spaces and the Riesz Representation Theorem, one can prove that the operator $L = (-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact. Liliane asked me to show that L^{-1}, the Laplacian itself, is not a compact operator.

- At this part, I started to get a bit tangled up. Romildo interrupted and asked me to define weak derivative. Liliane asked for an example of a function that has no weak derivative. I gave an example, and she and Mayra asked for an idea of how to prove that the example I gave has no weak derivative. I talked about the idea of taking a sequence of smooth functions with compact support and showing that, in the limit, the equalities in the definition of weak derivative break down.

- We went back to the problem of showing that the Laplacian is not compact. I managed to show it using the eigenfunctions of the Laplacian.

  Remark (O.5). Gabriel told me afterwards that, actually, the inverse of a compact operator (when such an inverse exists) is never compact. In fact, this can only happen in finite dimension.

Indeed, suppose that $T : X \to X$ is a compact linear operator on a normed vector space X. Then $T^{-1}(B_X) \subseteq KB_X$ for some $K > 0$. Applying T to both sides, we obtain

$$B_X = T(T^{-1}(B_X)) \subseteq T(KB_X) = KT(B_X).$$

Since $\overline{T(B_X)}$ is compact, it follows that B_X is a closed subset of a compact set and, therefore, is compact, from which one concludes that X is finite-dimensional. In fact, the inverse is only bounded if the space is finite-dimensional. Indeed, whenever the inverse is compact, the identity map (as a composition of a compact operator and a bounded one) is compact, which guarantees the compactness of the ball and, consequently, the finite dimension of X. This whole argument presupposes the existence of the inverse; however, by the spectral theory of compact operators in infinite-dimensional spaces, zero is in the spectrum of any compact operator, which guarantees that the inverse will always be, at most, partially defined.

- They asked for examples of compact operators. I talked about the finite-dimensional case and they asked for examples in infinite dimension. I talked about the Volterra operator, which was compact but not self-adjoint. They asked whether I knew how to show that it was compact. I said I would not manage it on the spot, they asked whether I would at least know some idea and I talked about using Arzelà-Ascoli.

- We went back to the problem
$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$
but now in the classical case. They asked which conditions on f and on the domain guarantee existence of classical solutions, and what the best regularity of such solutions is depending on the regularity of f and of the domain. Liliane asked how I would show uniqueness of solutions. I showed uniqueness using the maximum principle for harmonic functions/functions satisfying the mean value property.

- Mayra asked whether uniqueness could also be proved using the earlier parts of the presentation, i.e., using the Riesz Representation Theorem. I said yes, since the R.R.T. guarantees uniqueness of weak solutions, and since every classical solution is also weak, weak uniqueness guarantees classical uniqueness.

- Mayra then asked how to construct the solution explicitly when it exists, which method could be used. I talked about Green's functions and the Newtonian Potential.

- They asked how to guarantee that the solution of the problem is smooth if f is smooth and the domain is regular. I talked about the regularity results, about applying Agmon-Douglis-Nirenberg and Schauder recursively (bootstrap).

- Mayra asked me to state all the Sobolev embeddings I knew, when they are continuous, when they are compact, when compactness is lost.

- Mayra asked whether the regularity of the embedding into $L^q(\Omega)$ for every $q \geq 1$ could be improved to $L^\infty(\Omega)$. I said it could not, since $f \in L^q(\Omega)$ for every $q \geq 1$ does not guarantee that $f \in L^\infty(\Omega)$. They asked for an explicit counterexample of this.

- They asked what happens in the cases $n = 1$ and $n = 2$, what can be said about the critical Sobolev exponent and how in these cases the regularity and compactness of the embeddings are improved.

- We went back to Functional Analysis. Mayra asked me to explain the importance/usefulness of reflexive spaces, and also to define reflexivity. I talked about compactness (Liliane asked me at another moment what can be said about the compactness of the ball in infinite dimension) in the weak and weak-star topologies, i.e., the theorems of Banach-Alaoglu-Bourbaki and Kakutani, Eberlein-Smulain. Mayra asked me to give an idea of the proof of each of the theorems.

- Liliane asked for an example of a sequence that does not converge strongly but converges in the weak topology.

- Romildo asked me to define uniformly convex spaces.

- Mayra asked about the relation between reflexive and uniformly convex spaces. I said that every uniformly convex space is reflexive, and she asked for an idea of the proof.

- Mayra asked whether every reflexive space is uniformly convex. I answered no, since uniform convexity is a property of the norm of the space, and not of the space itself, and it is even possible to have equivalent norms on the same reflexive space such that one norm is uniformly convex and the other is not.

- Liliane asked under which conditions weak convergence guarantees strong convergence. I said that in uniformly convex spaces this happens provided the norm of the sequence converges strongly. Then she asked whether I knew how to prove this result. I said not for the general case, but I did it in the case of Hilbert spaces.

- Liliane asked me to talk about the Hahn-Banach Theorem, useful corollaries and also applications in the study of the weak and weak-star topologies. I talked about the theorem, about how to obtain the corollary that if $x \neq y$ there exists $f \in E'$ such that $f(x) \neq f(y)$, and mentioned how this guarantees that the weak topology is Hausdorff, in particular uniqueness of limits holds.

- Liliane asked for an example where the Hahn-Banach extension is not unique.

- Romildo asked about the Banach-Steinhauss and Closed Graph Theorems: to state them and give applications, what they are good for.

- The last problem of the exam was about the eigenvalue problem for the Laplacian on the ball. Liliane asked me how I would guarantee the existence of eigenfunctions and eigenvalues for the problem on the unit ball. I talked about the spectral theory of compact operators and the variational characterization of the first eigenvalue. Next, she asked how I could relate the first eigenvalue λ_1 of the unit ball to the first eigenvalue of the ball of radius R, say $\lambda_{1,R}$. I said my

intuition would say they were inversely proportional, and Liliane and Mayra asked how to do the computation explicitly. I talked about the change of variables theorem, they asked me to do the computation and we arrived at

$$\lambda_{1,R} = \frac{\lambda_1}{R^2}.$$

Liliane asked what conclusion can then be drawn about the eigenvalue problem for the Laplacian in $\mathbb{R}^n$. I said that the first eigenfunction does not exist in this case, since there cannot be an eigenfunction associated with a zero eigenvalue.

Appendix

Proposition (P.10.1). Let $0 \neq f \in E'$. Then $\ker(f)$ has codimension 1.

Proof: Since $0 \neq f$ by hypothesis, there exists $v_0 \in E$ such that $f(v_0) \neq 0$. We can then write

$$x = \left(x - \frac{f(x)}{f(v_0)}v_0\right) + \frac{f(x)}{f(v_0)}v_0$$

Evidently,

$$\left(x - \frac{f(x)}{f(v_0)}v_0\right) \in \ker(f).$$

■

Theorem (T.10.1). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f \in L^2(\Omega)$ and $\lambda \geq 0$. Then the problem*

$$\begin{cases} -\Delta u + \lambda u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution in $H_0^1(\Omega)$.

Solution: First, note that the weak formulation of the problem is

$$\int_\Omega \langle \nabla u, \nabla v \rangle + \lambda \int_\Omega uv = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega)$$

$$\Longleftrightarrow$$

$$\langle \nabla u, \nabla v \rangle_{L^2(\Omega)} + \lambda \langle u, v \rangle_{L^2(\Omega)} = \langle f, v \rangle_{L^2(\Omega)} \tag{10.1}$$

This motivates us to define the following bilinear form:

$$B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

$$(u, v) \mapsto \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} + \lambda \langle u, v \rangle_{L^2(\Omega)}$$

so that the weak formulation of the problem is equivalent to finding $u \in H_0^1(\Omega)$ such that

$$B(u,\cdot) = \left(H_0^1(\Omega) \ni v \mapsto \int_\Omega fv\right) \doteq T_f$$

Once we have shown that

- $$T_f \in (H_0^1(\Omega))^*;$$
- $$B \text{ is continuous;}$$
- $$B \text{ is coercive,}$$

the desired result will follow from the Lax-Milgram Theorem.

Indeed, T_f is evidently linear, and given $v \in H_0^1(\Omega)$ we have

$$|T_f(v)| = \left|\int_\Omega fv\right| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}\|v\|_{H_0^1(\Omega)} = \tilde{C}\|v\|_{H_0^1(\Omega)}$$

Moreover,

- given $u, v \in H_0^1(\Omega)$,

$$\begin{aligned} |B(u,v)| &= \left|\langle \nabla u, \nabla v\rangle_{L^2(\Omega)} + \lambda\langle u, v\rangle_{L^2(\Omega)}\right| \\ &\leq \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \lambda\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq (1+\lambda C^2)\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, \end{aligned} \tag{10.2}$$

  where we used the Holder and Poincaré inequalities. This shows the continuity of B.

- $\forall v \in H_0^1(\Omega)$, we have:

$$\begin{aligned} B(v,v) &= \langle \nabla v, \nabla v\rangle_{L^2(\Omega)} + \lambda\langle v, v\rangle_{L^2(\Omega)} \\ &= \|\nabla v\|^2_{L^2(\Omega)} + \lambda\|v\|^2_{L^2(\Omega)} \\ &\geq \|\nabla v\|^2_{L^2(\Omega)} \\ &\geq C\|v\|^2_{L^2(\Omega)}, \end{aligned} \tag{10.3}$$

  where we used the triangle inequality and the Poincaré inequality. This shows the coercivity of B.

■

Question 87.
Is it possible to improve the hypothesis $\lambda \geq 0$ to $\lambda > \alpha$ for some $\alpha < 0$?

Answer: Yes! Let

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$$

so that

$$\int_\Omega |\nabla u|^2 \geq \lambda_1 \int_\Omega |u|^2 \Rightarrow \|u\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda_1}\|\nabla u\|^2_{L^2(\Omega)}$$

and let us suppose $0 > \lambda > -\lambda_1$. Then **(10.2)** becomes

$$\begin{aligned} |B(u,v)| &\leq \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} - \lambda\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq \|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} - \frac{\lambda}{\lambda_1}\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} \\ &= \left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}. \end{aligned}$$

The hypothesis $0 > \lambda$ guarantees in particular that $\lambda < \lambda_1$, and therefore also that $1 - \frac{\lambda}{\lambda_1} > 0$, which shows that B is continuous.

The proof of coercivity given in **(10.3)** needs to be adapted. It suffices to note that for $v \in H_0^1(\Omega)$, we have

$$\begin{aligned} B(v,v) &= \langle \nabla v, \nabla v\rangle_{L^2(\Omega)} + \lambda\langle v, v\rangle_{L^2(\Omega)} \\ &\geq \|\nabla v\|^2_{L^2(\Omega)} + \frac{\lambda}{\lambda_1}\|v\|^2_{L^2(\Omega)} \\ &= \left(\frac{\lambda_1 + \lambda}{\lambda_1}\right)\|\nabla v\|^2_{L^2(\Omega)} \end{aligned}$$

■

Question 88.
Is it possible to replace the constant λ by some function $c(x)$? If so, which hypotheses must be placed on $c(x)$?

Solution: Yes! Yes, provided we assume that $c \in L^\infty(\Omega)$ and $\|c\|_{L^\infty(\Omega)} < \lambda_1$.

In this case, the associated bilinear form is:

$$B(u,v) = \int_\Omega \langle \nabla u, \nabla v\rangle + \int_\Omega c(x)uv$$

which satisfies

$$|B(u,v)| \leq \|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} + \|c\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$

$$\leq \left(1 + \frac{\|c\|_{L^\infty}}{\lambda_1}\right)\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)};$$

Besides

$$\begin{aligned} B(u,u) &= \|u\|^2_{H^1_0(\Omega)} + \int_\Omega c(x)u^2 \\ &\geq \|u\|^2_{H^1_0(\Omega)} - \|c\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} \\ &\geq \|u\|^2_{H^1_0(\Omega)} - \frac{\|c\|_{L^\infty(\Omega)}}{\lambda_1}\|u\|^2_{H^1_0(\Omega)} \\ &= \left(1 - \frac{\|c\|_{L^\infty(\Omega)}}{\lambda_1}\right)\|u\|^2_{H^1_0(\Omega)} \end{aligned}$$

where we used the Poincaré inequality and the fact that

$$|c(x)| \leq \|c\|_{L^\infty(\Omega)} \text{ a.e. in } \Omega, \text{ in particular } -\|c\|_{L^\infty(\Omega)} \leq c(x) \leq \|c\|_{L^\infty(\Omega)}$$

It follows from the hypothesis

$$\|c\|_{L^\infty(\Omega)} < \lambda_1$$

that $M \doteq 1 - \dfrac{\|c\|_{L^\infty(\Omega)}}{\lambda_1} > 0$, and therefore

$$|B(u,u)| > M\|u\|^2_{H^1_0(\Omega)} \ \forall u \in H^1_0(\Omega).$$

■

Question 89.
For the original problem, what can be said in the case $\lambda \leq -\lambda_1$?

Answer: By the results we have just proved, the solution operator

$$L = (-\Delta)^{-1} : L^2(\Omega) \to H^1_0(\Omega)$$

is well defined, so that our original problem is equivalent to finding

$$(\mathrm{Id} - K)(u) = f,$$

where $K = i \circ -\lambda L$ for some compact embedding $i : H^1_0(\Omega) \to L^2(\Omega)$. If $\lambda = -\lambda_j$, it is evident that $\mathrm{Id} - K$ will not be injective and, in particular (by the Fredholm alternative), will not be surjective. In this scenario, the existence of weak solutions is not guaranteed (in fact, it is *guaranteed* that there exist functions f for which the problem admits no weak solution). Uniqueness is also lost, since, when the problem admits some solution, say u, then $u + c\varphi_j$ (where φ_j is an eigenfunction of $-\Delta$) will also be a solution for any $c \in \mathbb{R}$.

Finally, if $\lambda < -\lambda_1$ and $\lambda \notin \{\lambda_i\}_{i\in\mathbb{N}}$, $\mathrm{Id} - K$ is injective and, therefore, surjective. In this case, the problem admits exactly one weak solution.

■

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