English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio, translated from the author's LaTeX source. The Portuguese original is the authoritative version.

Contents

1 Preliminaries **2**
1.1 Lecture notes and problem sets 2
1.2 Auxiliary results 13

2 Connectedness exercises **16**

3 Compactness exercises **21**

4 Solutions to problem set 1 . . . . . **27**

5 Solutions to problem set 2 . . . . . **32**

AULA 10

Metrizabilidade e Axiomas de Separação

DEFINIÇÃO 10.1. Seja (Ω, τ) um espaço topológico não vazio. Temos as seguintes definições:

a) Seja $x \in \Omega$, diremos que $U \subset \Omega$ é uma *vizinhança* de x quando $x \in U^\circ$;
b) Dado $x \in \Omega$ *base de vizinhanças* $\mathscr{B}_x$ de x a família de vizinhanças de x com a seguinte propriedade: se U é qualquer vizinhança de x, então existe $V \subset \mathscr{B}_x$ tal que $V \subset U$;
c) Uma *cobertura aberta* de Ω é um subconjunto $\mathscr{F} \subset \tau$ tal que $\bigcup_{U \in \mathscr{F}} U = \Omega$; Uma *subcobertura* é um subconjunto de $\mathscr{F}$ que ainda é uma cobertura.

DEFINIÇÃO 10.2. Seja (Ω, τ) um espaço topológico não vazio. Temos as seguintes definições:

a) Diremos que Ω satisfaz o *o primeiro axioma de enumerabilidade* quando todo $x \in \Omega$ tem uma base de vizinhanças enumerável;
b) Diremos que Ω satisfaz o *o segundo axioma de enumerabilidade* quando a topologia τ tem um base enumerável;
c) O espaço Ω será dito *Lindelöf* quando toda cobertura de Ω possuir uma subcobertura enumerável;
d) Finalmente, diremos que Ω é *separável* sempre que Ω admitir uma conjunto enumerável denso.

PROPOSIÇÃO 10.3. *Em espaços métricos: Separável $\Rightarrow$ base enumerável $\Rightarrow$ Lindelöf $\Rightarrow$ separável.*

DEMONSTRAÇÃO. Prova de: Separável $\Rightarrow$ base enumerável: Se $\{x_n\}_{n\in\mathbb{N}}$ é um conjunto denso em Ω então é fácil ver que se $\{q_m\}_{m\in\mathbb{N}}$ é uma enumeração dos racionais então $\{B_d(x_n, q_m)\}_{n,m\in\mathbb{N}}$ é uma base enumerável para τ.]

Prova de: base enumerável $\Rightarrow$ Lindelöf: Seja $\mathscr{F}$ uma cobertura de Ω e $\mathscr{B}$ uma base enumerável para Ω. Defina

$$\mathscr{B}_{\mathscr{F}} = \{B \in \mathscr{B} : B \subset A \text{ para algum } A \in \mathscr{F}\}.$$

Sendo $\mathscr{B}$ uma base temos que todo elemento de $\mathscr{F}$ pode ser escrito como uma união contável de elementos de $\mathscr{B}$, portanto $\mathscr{B}_{\mathscr{F}}$ é uma cobertura enumerável de Ω. Agora, para cada $B_j \in \mathscr{B}_{\mathscr{F}}$ escolha $A_j \in \mathscr{F}$ tal que $B_j \subset A_j$. Então temos que $\Omega = \bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} A_j$. Donde segue o resultado.

□

AULA 12

Espaços Conexos

DEFINIÇÃO 12.1. Um espaço topológico Ω é dito *desconexo* se existem conjuntos abertos, disjuntos e não vazios A, B tais que $\Omega = A \cup B$. Caso contrário diremos que Ω é *conexo*. Um subconjunto $S \subset \Omega$ é dito desconexo se com a topologia induzida S é desconexo. Caso contrário diremos que S é conexo.

EXEMPLO 12.2. Cada intervalo fechado e limitado de $\mathbb{R}$ é conexo. De fato, suponha que o intervalo $[a, b]$, $a < b$ seja desconexo. Então podemos escrever $[a, b] = A \cup B$ onde A e B são dois abertos não vazios de $[a, b]$. Suponha sem perda de generalidade que $b \in B$. Como B é aberto tem-se que $(b - \epsilon, b] \subset B$ para algum $\epsilon > 0$. Seja $c = \sup A$, pela observação precedente devemos ter que $c < b$ e que $(c, b] \subset B$. Caso $c \in A$, como A é aberto, deve existir $\epsilon > 0$ tal que $[c, c + \epsilon) \subset A$, contrariando o fato de $c = \sup A$. Assim $c \in B$. Se $a < c$, sendo B aberto deve existir $\epsilon > 0$ tal que $(c - \epsilon, c] \subset B$, contrariando o fato de $c = \sup A$. Concluímos que $c = a$, assim $[a, b] = [c, b] \subset B$, outro absurdo. Portanto $[a, b]$ é conexo.

PROPOSIÇÃO 12.3. *Um espaço topológico (Ω, τ) é conexo se e semente se Ω e $\varnothing$ são os únicos subconjuntos de Ω simultaneamente abertos e fechados.*

PROPOSIÇÃO 12.4. *A imagem de um espaço conexo por aplicação contínua é um espaço conexo.*

COROLARIO 12.5. *Teorema do valor intermediário.*

EXERCÍCIO 12.1. Prove que a imagem contínua de um espaço conexo é um espaço conexo.

EXERCÍCIO 12.2. a) Mostre que se S é conexo então $\overline{S}$ também o é. b) Se S é conexo e $S \subset D \subset \overline{S}$ então D é conexo.

PROPOSIÇÃO 12.6. *Seja (Ω, τ) um espaço topológico. Suponha que $\Omega = \bigcup_{\alpha \in I} S_\alpha$, onde cada S_α é conexo e $\bigcap_{\alpha \in I} S_\alpha \neq \varnothing$. Então Ω é conexo.*

DEMONSTRAÇÃO. Suponha $\Omega = A \cup B$ onde A e B são abertos e disjuntos de Ω. Note que para cada $\alpha \in I$, temos que $S_\alpha \subset A$ ou $S_\alpha \subset B$, caso contrário S_α não seria conexo. Agora seja $x \in \bigcap_{\alpha \in I} S_\alpha$, se $x \in A$, então devemos ter pela observação acima que $S_\alpha \in A$ para todo $\alpha \in I$ donde $B = \varnothing$. Um raciocínio análogo caso $x \in B$ mostra que $A = \varnothing$. Logo Ω é conexo. □

EXERCÍCIO 12.3. a)Mostre que $\mathbb{R}$ é conexo. b) Mostre que $\mathbb{R}^n$ é conexo.

EXERCÍCIO 12.4. Seja (Ω, τ) um espaço topológico. Suponha que cada par de pontos $x, y \in \Omega$ pertença a um conjunto conexo $S_{xy} \subset \Omega$. Então Ω é conexo.

PROPOSIÇÃO 12.7. *Sejam (Ω_1, τ_1) e (Ω_2, τ_2) espaços topológicos não vazios. Então o produto Então o produto cartesiano $\Omega_1 \times \Omega_2$ é conexo se e somente se Ω_1 e Ω_2 são conexos conexo.*

DEMONSTRAÇÃO. Sendo o produto $\Omega_1 \times \Omega_2$ conexo e as projeções π_1, π_2 contínuas então pelo exercício ?? temos que $\pi_1(\Omega_1 \times \Omega_2) = \Omega_1$ e $\pi_2(\Omega_1 \times \Omega_2) = \Omega_2$ são conexos.

Reciprocamente suponha que Ω_1 e Ω_2 sejam conexos. Fixe $y \in \Omega_2$ e para cada $x \in \Omega_1$ defina $U_x = (\{x\} \times \Omega_2) \cup (\Omega_1 \times \{y\})$. Note que para cada $x \in \Omega_1$ o cojunto U_x é a união dos dois conjuntos conexos $\{x\} \times \Omega_2$ e $\Omega_1 \times \{y\}$ e que $(\{x\} \times \Omega_2) \cap (\Omega_1 \times \{y\}) = \{(x, y)\}$, assim sendo seegue da proposição ?? que U_x é conexo.

Agora note que $\Omega_1 \times \Omega_2 = \bigcup_{x \in \Omega_1} U_x$ e que $\bigcap_{x \in \Omega_1} = \Omega_1 \times \{y\} \neq \varnothing$. Portanto pela proposição?? segue o resultado. □

EXERCÍCIO 12.5. Complete a demonstração acima mostrando que $\{x\} \times \Omega_2$ é homemorfo a Ω_2 e que $\Omega_1 \times \{y\}$ é homeomorfo a Ω_1.

EXERCÍCIO 12.6. Mostre que $\mathbb{R}^{\mathbb{N}}$ com a topologia das caixas não é conexo. Sugestão: Decomponha $\mathbb{R}^{\mathbb{N}}$ no conjunto das sequências limitadas e das sequências não limitadas.

PROPOSIÇÃO 12.8. *Seja $\{\Omega_\alpha\}_{\alpha \in I}$ uma família não vazia de espaços topológicos não vazios. Então o produto $\prod_{\alpha \in I} \Omega_\alpha$ é conexo se e somente se cada Ω_α é conexo.*

DEMONSTRAÇÃO. □

1. Lista de exercícios

1. Um espaço topológico Ω é dito ser *totalmente desconexo* quando os únicos subconjuntos conexos de Ω são os conjuntos unitários. Mostre que se Ω está equipado com a topologia discreta então Ω é totalmente desconexo. A recíproca vale?
2. Mostre que se $\{A_n\}_{n \in \mathbb{N}}$ é uma sequência de subespaços conexos de um espaço topológico Ω, tais que $A_n \cap A_{n+1} \neq \varnothing$ para todo n. Mostre que $\bigcup_{n \in \mathbb{N}} A_n$ é conexo.
3. O espaço $\mathbb{R}_\ell$ é conexo?
4. Mostre que um espaço topológico Ω é contínua se e somente se as únicas funções $f : \Omega \to \{0, 1\}$ contínuas são as constantes.
5. Seja Ω um espaço topológico e suponha que para cada par de pontos x, y existe um subconjunto conexo S_{xy} de Ω contendo x e y, então Ω é conexo.
6. Mostre que para $n \geq 2$, $\mathbb{R}^n \setminus \{x_1, \ldots, x_n\}$ é conexo.
7. Mostre que $\mathbb{R}^2 \setminus \mathbb{Q}^2$, onde $\mathbb{Q}^2$ representa o conjunto de todos os pontos de coordenadas racionais, também é conexo.
8. Prove que $\mathbb{R}^n$ e $\mathbb{R}$ não são homeomorfos para todo $n \geq 2$
9. Seja $E \subset \mathbb{R}^n$ um subespaço de codimensão ≥ 2. Mostre que $\mathbb{R}^n \setminus E$ é conexo.

AULA 13

Compacidade

DEFINIÇÃO 13.1. Um espaço topológico Ω é dito *compacto* quando toda cobertura aberta de Ω admite uma subcobertura finita. Um subconjunto $K \subset \Omega$ é dito um *subconjunto compacto* quando K com a topologia induzida for compacto.

EXERCÍCIO 13.1. Seja $K \subset \Omega$, então K é compacto se e somente se toda cobertura de K por abertos de Ω admite uma subcobertura finita.

EXEMPLO 13.2. A reta $\mathbb{R}$ não é compacta. De fato considere a cobertura $\mathscr{F} = \{(n, n+2), n \in \mathbb{Z}\}$, é fácil ver que $\mathscr{F}$ não tem subcobertura finita.

EXEMPLO 13.3. O seguinte subconjunto de $\mathbb{R}$ é compacto: $\Omega = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{Z}\}$.

DEFINIÇÃO 13.4. Diremos que um espaço topológico tem a *propriedade da intersecção finta* (p.i.f) se e somente se para qualquer família de conjuntos fechados $\{F_\alpha\}_{\alpha \in I}$ como a propriedade de que: $F_{\alpha_1} \cap \cdots \cap F_{\alpha_k} \neq \varnothing$ para todos $\alpha_1, \ldots, \alpha_k \in I$, tem-se que $\bigcap_{\alpha \in I} F_\alpha \neq \varnothing$.

PROPOSIÇÃO 13.5. *Um espaço topológico Ω é compacto se e somente se tem a propriedade da intersecção finita.*

DEMONSTRAÇÃO. Seja $\{U_\alpha\}_{\alpha \in I}$ uma família de abertos. Associamos a essa família a seguinte família de fechados: $\{F_\alpha\}_{\alpha \in I}$, onde $F_\alpha = \Omega \setminus U_\alpha$. Então temos que $F_{\alpha_1} \cap \cdots \cap F_{\alpha_i} \neq \varnothing$ se e somente se $\{U_{\alpha_i}\}_{i=1}^k$ não é uma cobertura de Ω, enquanto pela p.i.f $\bigcap_{\alpha \in I} F_\alpha \neq \varnothing \Leftrightarrow \bigcup_{\alpha \in I} U_\alpha \neq \Omega$. Portanto, a propriedade da intersecção finita diz que se nenhuma subfamília finita de $\{U_\alpha\}_{\alpha \in I}$ é uma cobertura então $\{U_\alpha\}_{\alpha \in I}$ também não é uma cobertura, o que é a contrapositiva da definição de compacidade. □

PROPOSIÇÃO 13.6. *Seja Ω um espaço topológico compacto. Então temos o seguinte:*

a) *Todo subconjunto $F \subset \Omega$ fechado também é compacto;*
b) *Se Ω for Hausdorff e $K \subset \Omega$ for um subconjunto compacto então K é fechado em Ω;*
c) *Todo espaço compacto Hausdorff é normal;*

DEMONSTRAÇÃO. Prova de a): Seja $\{V_\alpha\}_{\alpha \in I}$ uma cobertura aberta de F. Então existem abertos U_α de Ω tais que $V_\alpha = F \cap U_\alpha$. Note que $\{\Omega \setminus F\} \cup \{U_\alpha\}$ é uma cobertura aberta de Ω, portanto existem indíces $\alpha_1, \ldots, \alpha_n$ tais que $\{\Omega \setminus F\} \cup \{U_{\alpha_i}\}_{i=1}^n$ é uma subcobertura finita de Ω. Portanto $\{V_{\alpha_i}\}_{i=1}^n$ é uma subcobertura finita de F.

Prova de b): Para provar o desejado vamos precisar usar o seguinte fato: em um espaço Hausdorff dado um compacto K e um ponto $x \notin K$, existem abertos disjuntos U_x e V_x tais que $x \in U_x$ e $K \subset V_x$. De fato, fixado $x \notin K$ e $y \in K$, obtemos, pois Ω é Hausdorff, abertos disjuntos $U_{xy} \ni x$ e $V_{xy} \ni y$. Assim temos que $\{V_{xy}\}_{y\in K}$ é uma cobertura aberta de K. Usando a compacidade de K obtemos $y_1, \ldots, y_n$ tais que

$$K \subset V_{xy_1} \cup \cdots \cup V_{xy_n} \equiv V.$$

Defina $U_x \equiv U_{xy_1} \cap \cdots U_{xy_n}$ note que U_x é aberto pois é a intersecção finita de abertos e que $U_x \cap V_x = \varnothing$. Provando a afirmação.

Com isso em mãos podemos proceder à prova de b). Para cada $x \notin K$ tome $U_x \ni x$ e $V_x \supset K$ abertos disjuntos. Então $\Omega \setminus K = \bigcup_{x\in\Omega\setminus K} U_x$ é aberto, donde K é fechado.

Prova de c): Sejam K e L fechados em Ω, pelo item a) são ambos compactos. Usando o fato que provamos na prova de b) para cada $x \in L$, temos abertos disjuntos $U_x \ni X$ $V_x \supset K$. Então, $\{U_x\}_{x\in L}$ é uma cobertura de L, por compacidade existem $x_1, \ldots x_n$ tais que $B \subset U \equiv U_{x_1} \cup \cdots U_{x_n}$. Tomando $V \equiv \bigcap_{i=1}^{n} V_{x_i}$ temos que U e V são abertos disjuntos $K \subset V$ e $L \subset U$

□

Definição 13.7. Um espaço topológico Ω é dito *fracamente sequencialmente compacto* (f.s.c) se toda sequência em Ω tem um ponto de acumulação. Se toda sequência tem uma subsequência convergente diremos que Ω é *sequêncialmente compacto.* Note que em espaços métricos essas duas noções coincidem.

> Exercício 13.2. Mostre que um espaço métrico é *fracamente sequencialmente compacto* se e somente se é sequêncialmente compacto. Se Ω for apenas um espaço topológico, qual dessas noções implica a outra? Dê um exemplo de um espaço não metrizável em que essas noções não coincidem.

Proposição 13.8. *Todo espaço topológico compacto é fracamente sequencialmente compacto.*

Demonstração. Seja $\{x_n\}_{n\in\mathbb{N}}$ uma sequência em um espaço compacto Ω. Suponha que $\{x_n\}_{n\in\mathbb{N}}$ não tenha ponto de acumulação (em particular $\{x_n\}_{n\in\mathbb{N}}$ é não eventualmente constante). Então para cada $x \in \Omega$ existe uma vizinhaça aberta U_x de x contendo apenas um número finito de termos de $\{x_n\}_{n\in\mathbb{N}}$. Veja que $\{U_x\}_{x\in\Omega}$ é uma cobertura aberta de Ω por compacidade deve existir $x_1, \ldots, x_n$ em Ω tais que $\Omega = U_{x_1} \cup \cdots \cup U_{x_n}$. Mas isso implica que $\{x_n\}_{n\in\mathbb{N}}$ tem apenas um número finito de termos, o que é absurdo. □

> Exercício 13.3. Seja $\Omega = \{0, 1\}$ e considere em Ω a topologia $\tau = \{\varnothing, \Omega\}$. Equipe $\mathbb{N}$ com a topologia discreta e considere o produto $\mathbb{N} \times \Omega$. Use esse exemplo para mostrar que a reciproca da proposição acima não é verdadeira.

Definição 13.9. Um espaço merico (Ω, d) é dito *totalmente limitado* se e somente se para todo $\epsilon > 0$ existe um conjunto finito $\{x_1, \ldots, x_n\} \subset \Omega$ tal que $\Omega \subset \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$.

Proposição 13.10. *Seja (Ω, d) um espaço métrico. Então temos o seguinte:*

a) *Todo espaço métrico totalmente limitado é separável;*
b) *Todo espaço métrico totalmente limitado é sequencialmente compacto.*

DEMONSTRAÇÃO. Prova de a): Para cada n considere o conjunto $\{x_j^{(n)}\}_{j=1}^{N_n}$ tal que $\Omega = \bigcup_{j=1}^{N_n} B_d(x_j^{(n)}, 1/n)$. Afirmamos que $\{x_j^{(n)}\}_{j=1,\dots,N_n;n\in\mathbb{N}}$ é denso em Ω. Com efeito, seja $x \in \Omega$ e $U \ni x$ um aberto, então existe n tal que $B_d(x, 1/n) \subset U$. Agora veja que pela construção de $\{x_j^{(n)}\}_{j=1}^{N_n}$ existe j tal que $x \in B_d(x_j^{(n)}, 1/n)$, assim $x_j^{(n)} \in B_d(x, 1/n)$. Portanto $\overline{\{x_j^{(n)}\}} = \Omega$.

Prova de b): Seja $\{y_m\}_{n\in\mathbb{N}}$ uma sequência em Ω. Considere $\{x_j^{(n)}\}_{j=1}^{N_n}$ a sequência definida no item anterior. O princípio da casa dos pombos garante que pelo menos uma das bolas $B_d(x_j^{(1)}, 1)$, $j = 1, \dots, N_n$, contém infinitos termos de $\{y_m\}_{m\in\mathbb{N}}$, escolha uma dessas bolas e denote-a por B_1. Portanto existe uma subsequência $\{y_n^{(1)}\} \subset B_1$. Usando indução obtemos uma bola B_ℓ de raio $1/\ell$ e uma subsequência $\{y_m^{(\ell)}\}$ em B_ℓ. Então a subsequência $z_\ell \equiv y_\ell^{(\ell)}$, tem a propriedade de que $\{z_j\}_{j=\ell}^{\infty} \subset B_\ell$. Portanto temos que $d(z_m, z_j) < 1/\ell$ sempre que $m, j \geq \ell$, ou seja, $\{z_j\}_{j\in\mathbb{N}}$ é uma sequência de Cauchy, assim existe z tal que $z_j \to z$. Portanto z é limite de uma subsequência de $\{y_m\}_{n\in\mathbb{N}}$. □

TEOREMA 13.11. *Em um espaço métrico (Ω, d) as seguintes afirmações são equivalentes:*

a) Ω é compacto;
b) Ω é sequencialmente compacto;
c) Ω é completo e totalmente limitado.

DEMONSTRAÇÃO. a)⇒ b): Isso é o conteúdo da Proposição 13.8.

b)⇒ c): Dado $\epsilon > 0$ suponha que para quaisquer $\{x_1, \dots, x_N\}$ tenhamos que $\bigcup_{i=1}^{N} B_d(x_i, \epsilon) \neq \Omega$. Vamos construir uma sequência $\{x_n\}_{n\in\mathbb{N}}$ que não possui nenhuma subsequência convergente. Para, isso escolha, x_1 arbitrariamente e indutivamente escolha x_{n+1} de modo que $x_{n+1} \notin \bigcup_{i=1}^{n} B_d(x_n, \epsilon)$. Assim $d(x_n, x_m) \geq \epsilon$ para todos $n \neq m$. Concluímos que $\{x_n\}_{n\in\mathbb{N}}$ não possui subsequência convergente, o que uma contradição com a hipótese.

c)⇒ a): A proposição 13.10 garante que sendo Ω completo e totalmente limitado então Ω é separável. Logo, pela proposição 10.3 Ω é Lindelöf, ou seja, toda cobertura aberta de Ω adimite uma subcobertura enumerável. Assim, seja $\{U_n\}_{n\in\mathbb{N}}$ uma cobertura enumerável de Ω, tal que nehuma subcoleção finita $\{U_1, \dots, U_n\}$ seja uma subcobertura de Ω.

Seja $A_n = \Omega \setminus \bigcup_{j=1}^{n} U_n \neq \varnothing$ é fechado e $A_n \supset A_{n+1}$. Para cada n escolha $x_n \in A_n$. isso fornece uma sequência $\{x_j\}_{j\in\mathbb{N}}$ que pelo item b) da proposição anterior tem um ponto de acumulação x. Visto que $\{x_j\}_{j\in\mathbb{N}} \subset A_n$ para todo n, e cada A_n é fechado, concluímos que $x \in \bigcap_{n=1}^{\infty} A_n = \varnothing$, absurdo. Logo $\{U_n\}_{n\in\mathbb{N}}$ admite uma subcobertura finita. □

EXERCÍCIO 13.4. Considere o cubo de Hilbert $C = [0,1]^{\mathbb{N}}$ equipado com a métrica produto, i.e, $d(x,y) = \sup_{n\in\mathbb{N}}\{|x_n - y_n|/n\}$.

a) Mostre que nessa topologia bolas $B_d(x,r)$ são conjuntos da forma $\prod_{n\in\mathbb{N}} B(x_n, nr)$ (aqui B sem o índice indica a bolas de $[0,1]$ relativas ao valor absoluto);
b) Mostre que C é completo;
c) Verifique que C é totalmente limitado e conclua que C é compacto.

DEFINIÇÃO 13.12. Seja Ω um espaço topológico, diremos que $A \subset \Omega$ é pré-compacto quando $\overline{A}$ for compacto.

PROPOSIÇÃO 13.13. *Um subconjunto $A \subset \mathbb{R}^n$ é compacto (na topologia usual) se e somente se é fechado e limitado.*

DEMONSTRAÇÃO. Sendo A compacto, então pela proposição 13.6 A é fechado, e pela proposição 13.11 A é totalmente limitado. Portanto existe um conjunto finito $\{x_1, \ldots, x_N\}$ tal que $A \subset \bigcup_{i=1}^{N} B(x_i, 1)$. Assim sendo $A \subset B(0,r)$ onde $r = 1 + \max\{|x_1|, \ldots, |x_N|\}$. Portando A é limitado e fechado.

Reciprocamente, sendo A fechado e limitado, temos em particular que A é completo (todo subconjunto fechado de um espaço métrico completo é ainda completo com a topologia induzida.)

O resultado estará provado se mostrarmos que A é totalmente limitado. Pois bem, como A é limitado existe $r > 0$ tal que $A \subset B(0,r)$, em particular A está contido no cubo C de centro 0 e lado $2r$. Então $\epsilon > 0$ escolha $\delta > 0$ tal que $\delta\sqrt{n} < \epsilon$. Considere o conjunto $\mathcal{L}_\delta = \{\delta \boldsymbol{m} : \boldsymbol{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^m\}$, note que $\mathcal{L}_\delta$ tem máximo $\lfloor \frac{2r+1}{\delta} \rfloor^n$ pontos, e que centrando em cada um desses pontos um cubo de lado δ obtemos uma cobertura de C. Para encerrar note que cada um desses cubos está contido numa bola de raio ϵ, assim A é seguramente coberto por $\lfloor \frac{2r+1}{\delta} \rfloor^n$ bolas de raio ϵ. Portanto A é totalmente limitado, assim, segue do teorema 13.11 que A é compacto.

□

DEFINIÇÃO 13.14. Sejam (Ω_1, d_1) e (Ω_2, d_2) espaços métricos. Diremos que uma função $f : \Omega_1 \to \Omega_2$ é *uniformemente contínua* quando dado $\epsilon > 0$ existe $\delta > 0$ tal que

$$\forall x, y \in \Omega_1, d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y) < \epsilon.$$

PROPOSIÇÃO 13.15. *Sejam Ω_1 e Ω_2 espaços métricos com Ω_1 compacto. Então qualquer função contínua $f : \Omega_1 \to \Omega_2$ é uniformemente contínua.*

TEOREMA 13.16. *Sejam Ω_1 e Ω_2 espaços topológicos com Ω_1 compacto. Então temos o seguinte:*

a) *Se $f : \Omega_1 \to \Omega_2$ é contínua então $f(\Omega_1)$ é compacto em Ω_2;*
b) *Se Ω_2 é Hausdorff e $f : \Omega_1 \to \Omega_2$ é contínua e bijetiva, então f é um homemorfismo.*

DEMONSTRAÇÃO. Prova de a): Seja $\{V_\alpha\}_{\alpha\in I}$ uma cobertura aberta de $f(\Omega_1)$. Como cada $V_\alpha = U_\alpha \cap f(\Omega_1)$, onde U_α é aberto em Ω_2, tem-se pela continuidade de f que $\{f^{-1}(U_\alpha)\}_{\alpha\in I}$ é uma cobertura aberta do compacto Ω_1. Extraindo uma subcobertura finita $\{f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})\}$ de Ω_1 concluímos que $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$ é uma cobertura de $f(\Omega_1)$. Portanto $f(\Omega_1)$ é compacto.

Prova de b): É suficiente mostrarmos que f envia fechados em fechados. Seja F fechado em Ω_1, temos pela proposição 13.6 item a) que F é compacto, então pelo item anterior temos que $f(F)$ é compacto em Ω_2. Como Ω_2 é Hausdorff segue da proposição 13.6 item b) que $f(F)$ é fechado. □

TEOREMA 13.17. *Seja $f : \Omega \to \mathbb{R}$ uma função contínua onde Ω é um espaço topológico compacto. Então f tem máximo e mínimo.*

DEMONSTRAÇÃO. Pela teorema anterior, $f(\Omega)$ é compacto em $\mathbb{R}$ e portanto é fechado e limitado. Logo $f(\Omega)$ contém seu supremo e seu ínfimo. □

EXERCÍCIO 13.5. Seja Ω um espaço compacto Hausdorff. Seja $f : \Omega \cup \{\infty\}$ uma função semicontínua inferiormente. Então f é limitada inferiormente, i.e, $\inf_{x\in\Omega} f(x) > -\infty$, e existe $x \in \Omega$ tal que $f(x) = \inf_{x\in\Omega} f(x)$.

1. Lista de Exercícios

1. Considere $[0,1]^{\mathbb{N}}$ com a topologia uniforme. Encontre nesse espaço um subconjunto infinito sem pontos de acumulação.
2. Mostre que $[0,1]$ como subespaço de $\mathbb{R}_\ell$ não é f.s.c.
3. Mostre que o círculo $S^1 = \{(x,y) : x^2 + y^2 = 1\}$ com a topologia induzida de $\mathbb{R}^2$ é compacto.
4. Mostre que $[0,1]$ não é compacto como subespaço de $\mathbb{R}_K$.
5. Seja $\{x_n\} \subset \mathbb{R}$, uma sequência convergente com limite x, mostre que $\{x, x_n, n \in \mathbb{N}\}$ é compacto.
5. Qualquer espaço métrico compacto Ω é homemorfo a algum subconjunto do cubo de Hilbert. (Sugestão: Ω é separável (justifique), então seja, $\{x_1, x_2, \ldots\}$ um subconjunto denso em Ω. Defina $F : \Omega \to C$ pondo $F(x) = (d(x,x_1), d(x,x_2), \ldots)$, mostre que F é o homeomorfismo desejado).

Universidade de Brasília

Topologia Geral 0º/2019 — Lista de Exercícios 1

Turma A — 24/01/2019

Problema 1 (infinitude dos primos): Considere o conjunto dos números inteiros $\mathbb{Z}$. Defina para cada par de números inteiros m, n, onde $m \neq 0$ o conjunto

$$A_{m,k} = \{mn + k, n \in \mathbb{Z}\}.$$

a) Prove que intersecções de dois conjuntos da forma $A_{m,k}$ é um conjunto da forma $A_{m',k'}$. Conclua que a família $\{A_{m,k},\ m, n \in \mathbb{Z}, m \neq 0\}$ é uma base para uma topologia em $\mathbb{Z}$ (chamada de topologia de Furstenberg);

b) Prove que cada $A_{m,k}$ é simultaneamente fechado e aberto (clopen) na topologia de Furstemberg;

c) Prove que cada conjunto aberto não vazio na topologia de Furstenberg é infinito;

d) Considere $P \subset Z$ o conjunto dos números primos (1 não é primo!). Seja $B = \bigcup_{p \in P} A_{p,0}$, quem é B?

e) Mostre que se P for finito então B é aberto

f) Conclua que P é infinito.

Problema 2(Axiomas de Kuratowski):

a) Seja (Ω, τ) um espaço topológico. Prove que o *operador fecho* $A \mapsto \overline{A}$ tem as seguintes propriedades:

i) $\overline{\varnothing} = \varnothing$, ii) $A \subset \overline{A}$ iii) $\overline{\overline{A}} = \overline{A}$ e iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b) Seja Ω um conjunto. Suponha que uma aplicação $F : \mathscr{P}(\Omega) \to \mathscr{P}(\Omega)$ tem as propriedades i)-iv) listadas acima, i.e: i) $F(\varnothing) = \varnothing$, ii) $A \subset F(A)$ iii) $F(F(A)) = F(A)$ e iv) $F(A \cup B) = F(A) \cup F(B)$. Mostre que existe uma única topologia τ tal que F seja o operador fecho para τ.

c) Considere $\Omega = \mathbb{N}$ e $F : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ dada por $F(A) = \overline{A} = \{k \cdot A, k \in \mathbb{N}\}$. Mostre que F satisfaz as propriedades i)-iv) listadas no item anterior. Sendo τ a também a topologia referida no item anterior, descreva os conjuntos fechados e os conjuntos abertos de $(\mathbb{N}, \tau)$.

d) Considere em $\mathbb{N}$ a topologia dada pelo item anterior. Mostre que uma função $f : \mathbb{N} \to \mathbb{N}$ é contínua, se e somente se, cada vez que $m|n$ tem-se que $f(m)|f(n)$.

Problema 3(Números de Liouville): Diremos que um número real ξ é de Liouville quando, para cada $n \geq 1$ existe um racional $p/q,\ q > 1$ tal que

$$0 < \left| \frac{p}{q} - \xi \right| < \frac{1}{q^n}.$$

a) Mostre que o conjunto L dos números de Liouville é um G_δ;

b) Conclua que L é não enumerável.

Universidade de Brasília

Topologia Geral 0º/2019 — Lista de Exercícios 2

Turma A — 07/02/2019

Problema 1: Seja $K : [0,1] \times [0,1] \to \mathbb{R}$ uma função contínua. Para cada $f \in C([0,1], \mathbb{R})$ defina

$$T(f) = \int_0^1 K(x,y) f(y) dy.$$

Mostre que $T(f) \in C([0,1], \mathbb{R})$ e que $\{T(f), \|f\| \leq 1\}$ é precompacto em $C([0,1], \mathbb{R})$.

Problema 2: Seja (Ω, d) um espaço métrico. Uma função $f \in C(\Omega, \mathbb{R})$ é dita *α-Hölder contínua* ($\alpha > 0$) quando a quantidade abaixo é finita

$$\mathrm{Hol}_\alpha(f) = \sup_{x \neq y} \frac{f(x) - f(y)}{d(x,y)^\alpha}.$$

Mostre que se o espaço métrico Ω é compacto, então o conjunto $\{f \in C(\Omega, \mathbb{R}) : \|f\| \leq 1,\ \mathrm{Hol}_\alpha(f) \leq 1\}$ é compacto em $C(\Omega, \mathbb{R})$ na métrica uniforme.

1.2 Auxiliary results

Remark (O.1). We will implicitly use several times here that union is distributive over intersection and that intersection is distributive over union.

Definition (D.1). A nontrivial separation of Ω is a pair U, V of nonempty disjoint open sets of Ω whose union is equal to Ω. A trivial separation of Ω is a pair of disjoint open sets where one of them is empty and the other is Ω. Ω is said to be connected if there is no nontrivial separation of Ω, and disconnected otherwise.

Definition (D.2). We will say that Ω is path connected if for all $x, y \in \Omega$ there exists a continuous function $f : [a, b] \to \Omega$ (which we will call a path) going from a closed interval of the line into Ω such that $f(a) = x$ and $f(b) = y$.

Definition (D.3). We will say that $p \in \Omega$ is an accumulation point of the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$ if for every open neighborhood U of p and for every $n_0 \in \mathbb{N}$ there exists $n_1 > n_0$ such that $x_{n_1} \in U$.

Remark (O.2). We will assume that a sequence $\{x_n\}_{n\in\mathbf{N}}$ of Ω is a function $f : \mathbb{N} \to \Omega$ such that $f(1) = x_1$ and so on.

Remark (O.3). With this definition, an accumulation point of a sequence $\{x_n\}_{n\in\mathbb{N}}$ need not be an accumulation point of $X = \{x_1, x_2, \cdots\}$ (consider the sequence that alternates between 0 and 1 in the compact set $[0, 1]$), and the converse does not hold either (see the next remark).

Remark (O.4). **Munkres' definition**: We will say that a topological space Ω is limit point compact if every infinite subset of Ω has an accumulation point.

Remark (O.5). **Definition from the lecture notes**: We will say that a topological space Ω is weakly sequentially compact if every sequence $f : \mathbb{N} \to \Omega$ has an accumulation point.

Remark (O.6). Munkres' definition is weaker than that of the lecture notes, which requires every sequence in Ω to have an accumulation point: consider the sequence in $\mathbb{N} \times \{0, 1\}$ (where the first has the discrete topology and the second the indiscrete one) given by $x_1 = (1, 0), x_2 = (1, 1), x_3 = (2, 0), x_4 = (2, 1), \cdots$, which has no accumulation point, but every nonempty $S \subset \Omega$ (in particular if S is infinite) has an accumulation point (the proof of this is the same argument used in exercise 3 of the compactness boxes). However, note that the two definitions are equivalent if Ω satisfies the first countability axiom (or even if it satisfies at least that finite subsets are closed).

Remark (O.7). Being weakly sequentially compact is equivalent to being countably compact (i.e, every countable open cover has a finite open subcover) and also implies being limit point compact (but the converse only holds if the space satisfies at least that finite subsets are closed).

Lemma (L.1). *If $C \cup D = \Omega$ is a nontrivial separation of Ω and Y is a connected subspace of Ω, then either $Y \subset C$ or $Y \subset D$.*

Proof: Note that $C \cap Y$ and $D \cap Y$ are disjoint open sets of Y whose union is all of Y, so by the connectedness hypothesis on Y one of them is empty and the other is all of Y.

■

Lemma (L.2). *Any path connected set is connected.*

Proof: Suppose, for a contradiction, that Ω is path connected but not connected. Indeed, if $\Omega = A \cup B$ is a separation of Ω, then if $f : [a, b] \to \Omega$ is an arbitrary path, since $f([a, b])$ is the continuous image of a connected set, it is also connected, and using lemma **(L.1)**, we have that $f([a, b])$ is contained either in A or in B, so that there is no path from a point of A to a point of B, a contradiction.

■

Lemma (L.3). *Let $\mathscr{A}$ be an open cover of the metric space (Ω, d). If Ω is sequentially compact, there exists $\delta > 0$ such that every subset of Ω with diameter less than δ is contained in some element of $\mathscr{A}$.*

Proof: Suppose, for a contradiction, that Ω is sequentially compact but there is no δ under the conditions of the lemma. Then, in particular, for each $n \in \mathbb{N}$, there exists a set C_n of diameter less than $\frac{1}{n}$ that is not contained in any element of $\mathscr{A}$. For each n, choose a point $x_n \in C_n$. By hypothesis, some subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to some point a. Since $\mathscr{A}$ is a cover, there exists $A \in \mathscr{A}$ such that $a \in A$. Since A is open, there exists $\varepsilon > 0$ such that $B_d(a, \epsilon) \subset A$. Taking i large enough so that $\operatorname{diam}(C_{n_i}) = \frac{1}{n_i} < \frac{\varepsilon}{2}$ and $d(x_{n_i}, a) < \frac{\varepsilon}{2}$, we have (by the triangle inequality) that $C_{n_i} \subset B_d(x_{n_i}, \frac{\varepsilon}{2}) \subset B_d(a, \varepsilon) \subset A$, a contradiction.

■

Lemma (L.4). *Let (Ω, d) be a sequentially compact metric space. Then, $\forall \varepsilon > 0$, there exists a finite cover consisting of open balls of radius ε.*

Proof: We will prove the contrapositive of the statement, that is, if there exists $\varepsilon > 0$ such that Ω admits no finite cover by open balls of radius ε, then Ω is not sequentially compact. Indeed, if such an ε exists, we can construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ that has no convergent subsequence in the following way: take $x_1 \in \Omega$ arbitrary and choose x_2 so that $x_2 \notin B_d(x_1, \varepsilon)$ (by hypothesis we can do this, otherwise $B_d(x_1, \varepsilon)$ would be a finite cover of Ω). In general, given $x_1, \cdots, x_n$, take $x_{n+1} \in \Omega$ so that:

$$x_{n+1} \notin B_d(x_1, \varepsilon) \cup B_d(x_2, \varepsilon) \cup \cdots \cup B_d(x_n, \varepsilon)$$

(and again, the hypothesis guarantees that we can do this). It is obvious that, by construction, $d(x_{n+1}, x_i) \geq \varepsilon$ for all $i \in \{1, \cdots, n\}$, and in fact no subsequence of this sequence can converge: given any x_n, $B_d(x_n, \frac{\varepsilon}{2})$ contains at most one term of the sequence, namely x_n itself.

■

Lemma (L.5). *Let (Ω, d) be a metric space and $S \subset \Omega$. Then $x \in \Omega$ is an accumulation point of S if, and only if, for every $\varepsilon > 0$, $S \setminus \{x\} \cap B(x, \varepsilon) \neq \emptyset$, that is, for every $\varepsilon > 0$, there exists $y \neq x \in S$ such that $d(x, y) < \varepsilon$.*

Proof: The forward direction follows directly from the definition of accumulation point (since $B(x, \epsilon)$ is an open set of Ω containing x) and the backward direction directly from the fact that, given $x \in \Omega$, any open set $U \ni x$ of the topology generated by d contains some ball of radius ϵ, and therefore $U \cap S \setminus \{x\} \neq \emptyset$.

Remark (O.8). It is easy to see that this lemma is also equivalent to: x is an accumulation point of $S \iff$ every open neighborhood of x intersects S in infinitely many distinct points.

■

Lemma (L.6). *Any compact metric space (Ω, d) is separable.*

Proof: We know that every compact metric space (Ω, d) is totally bounded, so that for every $n \in \mathbb{N}$ the cover $C_n = \left\{ B_d\left(x, \frac{1}{n}\right) \mid x \in \Omega \right\}$ admits a finite subcover

$$F_n = \left\{ B_d\left(x_1, \frac{1}{n}\right), B_d\left(x_2, \frac{1}{n}\right), B_d\left(x_3, \frac{1}{n}\right), \cdots, B_d\left(x_k, \frac{1}{n}\right) \right\}$$

for some $x_1, \cdots, x_k \in \Omega$. For each $n \in \mathbb{N}$, let $A_n = \{x_1, x_2, \cdots, x_k\}$ be the set consisting of the centers of the balls of F_n. We claim that:

$$D = \bigcup_{n \in \mathbb{N}} A_n$$

is a countable subset of Ω. Indeed, Ω is countable, since it is a countable union of finite sets, and it is also dense, since given any nonempty open set U of Ω and any $x \in U$, there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$, and taking $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$, then $B_d\left(x, \frac{1}{m}\right) \subset B_d(x, \varepsilon)$. Since F_m is a finite subcover of Ω, there exists $y \in A_m$ such that $x \in B_d\left(y, \frac{1}{m}\right)$, hence $y \in B_d\left(x, \frac{1}{m}\right)$. Thus:

$$y \in B_d\left(x, \frac{1}{m}\right) \bigcap B_d(x, \varepsilon) \subset D \cap U \neq \emptyset$$

i.e, any nonempty open set of Ω intersects D, as desired.

■

Lemma (L.7). *If (Ω, d) is a metric space, then $d : \Omega \times \Omega \to \mathbb{R}$ is continuous.*

Proof: We will show that the inverse image of every open set U of the line is open in $\Omega \times \Omega$. Indeed, if $U \subset \mathbb{R}$ is open, we want to show that $d^{-1}(U) = \mathscr{O}$ is open, that is, for every $\mathfrak{o} \in \mathscr{O}$, there exists a basic open set B of $\Omega \times \Omega$ such that $\mathfrak{o} \in B \subset \mathscr{O}$. Indeed, note that if $(x, y) \in d^{-1}(U)$, then $d(x, y) = c \in U$, hence there exists $\varepsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset U$. We claim that $B = B_d\left(x, \frac{\varepsilon}{2}\right) \times B_d\left(y, \frac{\varepsilon}{2}\right)$ satisfies what is desired. It is obvious that $(x, y) \in B$; it remains to show that $B \subset \mathscr{O}$. Indeed, if $(x', y') \in B$, then note that:

$$\begin{aligned}
|d(x', y') - d(x, y)| &= |d(x', y') - d(x', y) + d(x', y) - d(x, y)| && \textbf{(1)} \\
&\leq |d(x', y') - d(x', y)| + |d(x', y) - d(x, y)| && \textbf{(2)} \\
&\leq d(y, y') + d(x, x') && \textbf{(3)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \textbf{(4)} \\
&= \varepsilon && \textbf{(5)}
\end{aligned}$$

hence $d(x', y') \in (c - \varepsilon, c + \varepsilon) \subset U$ and it follows that $(x', y') \in \mathscr{O}$. Since (x, y) and (x', y') were taken arbitrarily, the result follows.

■

Remark (O.9). To go from **(2)** to **(3)** note that, again by the triangle inequality:

$$\begin{aligned}
d(x', y) - d(y, y') \leq d(x', y') \leq d(x', y) + d(y, y') \\
d(x, y) - d(x, x') \leq d(x', y) \leq d(x, x') + d(x, y)
\end{aligned}$$

Lemma (L.8). *If $\{A_\alpha\}_{\alpha\in I}$ is a collection of connected subspaces of Ω and A is a connected subspace of Ω such that $A\cap A_\alpha\neq\emptyset$ for all $\alpha\in I$, then $A\cup\left(\bigcup_{\alpha\in I}A_\alpha\right)$ is connected.*

Proof: Note that $A\cup A_\alpha$ is connected by proposition **(P.1)**. Then, again by proposition **(P.1)**, $\bigcup_{\alpha\in I}\left(A\cup\left(\bigcup_{\alpha\in I}A_\alpha\right)\right)$ $A\cup\left(\bigcup_{\alpha\in I}A_\alpha\right)$ is connected. Note that we could also have used lemma **(L.1)** to show that the only separation is the trivial one.

■

Lemma (L.9). *If A and B are proper subsets of X and Y, respectively, then $C=X\times Y\setminus A\times B$ is connected.*

Proof: Fix some $(a_1,a_2)\in C$ such that $a_1\notin A$ and $a_2\notin B$. For each $x\notin A$ define $V_x=\{x\}\times Y$ and for each $y\notin B$ define $H_y=X\times\{y\}$. It is clear that both are connected since they are homeomorphic to X and Y, respectively. Define $I=\{(x,y)\in X\times Y\mid x\notin A \text{ and } y\notin B\}$, $D=V_{a_1}\cup H_{a_2}$ and for each $\alpha\in I$ also define $A_\alpha=V_{\pi_1(\alpha)}\cup H_{\pi_2(\alpha)}$. Note that:

- **(i)** $D\cap A_\alpha=\{(\pi_1(\alpha),a_2),(a_1,\pi_2(\alpha))\}\neq\emptyset\ \forall\alpha\in I$
- **(ii)** D is obviously the union of connected, non-disjoint subspaces, whence it follows from proposition **(P.1)** that D is connected. Analogously, A_α is also connected for each $\alpha\in I$.

and the result follows from the previous lemma.

■

2 Connectedness exercises

Question 1. *Prove that the continuous image of a connected space is a connected space.*

Solution: Let $f:X\to Y$ be continuous and suppose that X is connected. Without loss of generality, we may suppose that f is surjective (if it were not, it would suffice to restrict the codomain to the image space $Z=f(X)$, which preserves continuity). Thus, suppose, for a contradiction, that $Z=f(X)=Y=A\cup B$, i.e, it is disconnected. Then, if we take $x\in X$, either $f(x)\in A$ or $f(x)\in B$, whence we conclude that $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint open sets (by the continuity hypothesis) whose union $f^{-1}(A)\cup f^{-1}(B)=X$, that is, X is disconnected, an absurdity.

■

Question 2. *Show that:*

- **(a)** *if S is connected, then so is $\overline{S}$.*
- **(b)** *if S is connected and $S\subset D\subset\overline{S}$, then D is connected.*

Solution:

(a) Suppose that S is connected. We will show that the only separation of $\overline{S}$ is the trivial one, that is, if $\overline{S} = A \cup B$ and $A \neq \emptyset$, then $B = \emptyset$. Indeed, we have $A \cap \overline{B} = \overline{A} \cap B = \emptyset$, and taking $a \in A$, since $a \notin \overline{B}$, there exists an open set $U \ni a$ such that $U \cap B = \emptyset$, and since $a \in \overline{S}$, there exists $x \in U \cap S \neq \emptyset$ with $x \notin B$, thus $x \in S \cap A \neq \emptyset$. Note that $S = (A \cap S) \cup (B \cap S)$, and by the connectedness hypothesis $B \cap S = \emptyset \implies B = \emptyset$.

(b) Suppose, for a contradiction, that $D = A \cup B$ with A, B disjoint nonempty open sets. By lemma **(L.1)**, we may assume without loss of generality that $S \subset A$, thus $\overline{S} \subset \overline{A}$. Since $\overline{A}$ and B are disjoint, we conclude that $B = \emptyset$, a contradiction, hence D is connected.

■

Question 3. *Show that:*

(a) *with the usual topology, $\mathbb{R}$ is connected.*

(b) *with the usual topology, $\mathbb{R}^n$ is connected.*

Solution: First, we will prove the following:

Proposition (P.1). Let (Ω, τ) be a topological space. Suppose that

$$\Omega = \bigcup_{\alpha \in I} S_\alpha,$$

where each S_α is connected and $\bigcap_{\alpha \in I} S_\alpha \neq \emptyset$. Then Ω is connected.

Proof: Suppose that $\Omega = A \cup B$, where A and B are disjoint open sets of Ω. Note that for each $\alpha \in I$, we have $S_\alpha \subset A$ or $S_\alpha \subset B$, since otherwise S_α would not be connected. Now let $x \in \bigcap_{\alpha \in I} S_\alpha$. If $x \in A$, it follows from the remark we have just made that $S_\alpha \subset A$ for every $\alpha \in I$, whence it follows that $B = \emptyset$. *Mutatis mutandis*, we see that if $x \in B$ then $A = \emptyset$. Therefore Ω is connected.

■

(a) We have $\mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n, n]$, a union of connected, non-disjoint subspaces of the line. The result follows from proposition **(P.1)**.

(b) Take $v = (x_1, \cdots, x_n) \in \mathbb{R}^n$ with $||v|| = 1$ and let L_v be the line of $\mathbb{R}^n$ passing through the origin with direction vector v. It is clear that $\mathbb{R}^n = \bigcup_{||v||=1} L_v$, a union of non-disjoint, connected subspaces of $\mathbb{R}^n$ (since they are the image of $\mathbb{R}$ under the continuous function $f : \mathbb{R} \to \mathbb{R}^n$ given by $f(t) = tv$), and the result follows from proposition **(P.1)**.

■

Question 4. *Let (Ω, τ) be a topological space. Suppose that each pair of points $x, y \in \Omega$ belongs to a connected set $S_{xy} \subset \Omega$. Then Ω is connected.*

Solution: Fix some $x \in \Omega$. Then it is clear that $\Omega = \bigcup_{x \neq y \in \Omega} S_{xy}$, and the result follows from proposition **(P.1)**.

■

Question 5. *Let (Ω_1, τ_1) and (Ω_2, τ_2) be nonempty topological spaces. Fix arbitrarily $x \in \Omega_1$ and $y \in \Omega_2$. Show that $\{x\} \times \Omega_2$ is homeomorphic to Ω_2 and that $\Omega_1 \times \{y\}$ is homeomorphic to Ω_1.*

Solution: It is clear that $f : \Omega_2 \to \{x\} \times \Omega_2$ defined by $f(t) = (x, t) \in \{x\} \times \Omega_2$ and $g : \Omega_1 \to \Omega_1 \times \{y\}$ defined by $g(t) = (t, y) \in \Omega_1 \times \{y\}$ are continuous bijections with continuous inverses, as desired.

■

Question 6. *Show that $\mathbb{R}^{\mathbb{N}}$ with the box topology is not connected. Hint: decompose $\mathbb{R}^{\mathbb{N}}$ into the set of bounded sequences and the set of unbounded sequences.*

Solution: Let $A = \{x = (x_1, x_2, \cdots) \in \mathbb{R}^{\mathbb{N}} \mid \exists K \in \mathbb{R}$ such that $x_n \leq K \ \forall n \in \mathbb{N}$ or $x_n \geq K \ \forall n \in \mathbb{N}\}$ and $B = \{x = (x_1, x_2, \cdots) \in \mathbb{R}^{\mathbb{N}} \mid \forall M \in \mathbb{R} \ \exists n_M \in \mathbb{N}$ such that $x_{n_M} > M$ or $\forall M \in \mathbb{R} \ \exists n_M \in \mathbb{N}$ such that $x_{n_M} < M\}$. It is clear that A and B are nonempty subsets of $\mathbb{R}^{\mathbb{N}}$ whose union is $\mathbb{R}^{\mathbb{N}}$ itself. In fact they are also open, since given $x = (x_1, x_2, \cdots) \in \mathbb{R}^{\mathbb{N}}$, either $x \in A$ or $x \in B$, and letting U be a basic open set of $\mathbb{R}^{\mathbb{N}}$ in the box topology given by:

$$U = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \cdots$$

we have either $x \in U \subset A$ or $x \in U \subset B$.

■

Question 7. *A topological space Ω is said to be* totally disconnected *when the only connected subsets of Ω are the singletons. Show that if Ω is equipped with the discrete topology then Ω is totally disconnected. Does the converse hold?*

Solution: Considering the discrete topology, any $V \subset \Omega$ with more than two elements is disconnected, since, fixing $x \in V$, $\{x\} \cup V \setminus \{x\} = V$ is a union of disjoint open sets whose union is V. The converse does not hold; a counterexample is example 4 in §23 of Munkres: the rationals with the topology induced from the line are also totally disconnected.

■

Question 8. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of connected subspaces of a topological space Ω satisfying $A_n \cap A_{n+1} \neq \emptyset$ for every $n \in \mathbb{N}$. Show that $\bigcup_{n\in\mathbb{N}} A_n$ is connected.*

Solution: First we will prove that $\bigcup_{i=1}^{n} A_i$ is connected for every $n \in \mathbb{N}$. Indeed, by proposition **(P.1)**, we have that $A_1 \cup A_2$ is connected, thus $A_1 \cup A_2 \cup A_3$ is also connected, also by proposition **(P.1)**. Repeating this process, we have that the finite union is connected. Now, we claim that

$$B = \bigcup_{n\in\mathbb{N}} B_n$$

is connected, where $B_n = \bigcup_{i=1}^{n} A_i$. Indeed, using lemma 23.2 of Munkres and supposing that there is a separation $B = U \cup V$, we will show that either $U = \emptyset$ or $V = \emptyset$, that is, B is connected. Indeed, since $B_1 \in B$ is connected and $B_1 \subset B_n \ \forall n \in \mathbb{N}$, then either $B_1 \subset U$ and $B_n \subset U \ \forall n \in \mathbb{N} \implies U = B$ and $V = \emptyset$ or $B_1 \subset V$ and $B_n \subset V \ \forall n \in \mathbb{N} \implies V = B$ and $U = \emptyset$.

■

Question 9. *Is the topological space $\mathbb{R}_\ell$ (id est, $\mathbb{R}$ with the lower limit topology) connected?*

Solution: $\mathbb{R}_\ell$ is not connected, since $\mathbb{R} = (-\infty, 0) \cup [0, \infty)$, a union of disjoint open sets in the lower limit topology that gives all of $\mathbb{R}$.

■

Question 10. *Show that a topological space Ω is connected if, and only if, the only continuous functions $f : \Omega \to \{0, 1\}$ are the constant ones.*

Solution: First we will prove the contrapositive of connected $\implies$ the only continuous functions are constant. Indeed, if $f : \Omega \to \{0, 1\}$ is nonconstant and continuous, then, since $\{0, 1\}$ is clopen regardless of its topology, by hypothesis $f^{-1}(\{0, 1\})$ is a nonempty clopen set of Ω, i.e, Ω is disconnected. Conversely, if $\Omega = A \cup B$ with A and B nonempty disjoint open sets, note that $f : \Omega \to \{0, 1\}$ defined by setting $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$ is nonconstant and continuous.

■

Question 11. *Show that for each $n \geq 2$, $\mathbb{R}^n \setminus \{x_1, \ldots, x_n\}$ is connected.*

Solution: Without using path connectedness: Take:

$$A = \{(\pi_1(x_1), \cdots, \pi_{n-1}(x_1)), (\pi_1(x_2), \cdots, \pi_{n-1}(x_2)), \cdots (\pi_1(x_k), \cdots, \pi_{n-1}(x_k))\}$$
$$B = \{\pi_n(x_1), \cdots, \pi_n(x_k)\}$$

and note that

$$(\mathbb{R}^{n-1} \times \mathbb{R}) \setminus (A \times B) \subset \mathbb{R}^n \setminus \{x_1, \cdots, x_k\} \subset \underbrace{\overline{(\mathbb{R}^{n-1} \times \mathbb{R}) \setminus (A \times B)}}_{= \mathbb{R}^n, \text{ since } A \times B \text{ is finite}}$$

therefore the result follows from lemma 8 and from b) of the second exercise of the connectedness boxes.

Using path connectedness: We will show the stronger condition that $\mathbb{R}^n \setminus F$, where $F \subset \mathbb{R}^n$ is finite, is path connected (and it is easy to see that every path connected space is connected). Indeed, given $x, y \in \mathbb{R}^n$, there are infinitely many (*and uncountably many*) lines passing through x that do not intersect F; choose one of these lines at random and let α be its slope. It is clear that there are also infinitely many (*and uncountably many*) lines passing through y with slope $\beta \in \mathbb{R} \setminus \{\alpha\}$ that do not intersect F; also choose one of these lines and let β be its slope. Since L_α and L_β have different slopes, their intersection is nonempty. It then suffices to take some p in this intersection and note that the path that goes from x to p contained in L_α and then from p to y contained in L_β is continuous. Note that we could weaken the condition that F be finite; the proof also works if F is only countable.

■

Question 12. *Show that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected.*

Solution: Observe that $\mathbb{Q}^2$ is countable. The proof is identical to that of the previous exercise (an easier proof is to note that given two points p, q with irrational coordinates, they can be connected by the path passing through the intermediate point $(\sqrt{2}, \sqrt{2})$ by horizontal and vertical line segments).

■

Question 13. *Show that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic for all $n \geq 2$.*

Solution: $\mathbb{R} \setminus \{0\}$ is disconnected, but for $n \geq 2$ removing any point from $\mathbb{R}^n$ still leaves it connected. Since connectedness is a topological invariant (that is, if two spaces are homeomorphic either both are connected or both are disconnected), the result follows.

■

Question 14. *Let $E \subset \mathbb{R}^n$ be a subspace of codimension ≥ 2. Show that $\mathbb{R}^n \setminus E$ is connected.*

Solution:
Without using path connectedness: Suppose that $\text{codim}(E) \geq 2$, or, equivalently, that $\dim(E) = j \leq n-2$. Make a change of basis* so that:

$$v \in E \iff v = \left(\sum_{i=1}^{j} \phi_i e_i\right) + 0e_{j+1} + \cdots + 0e_n$$

for some $\phi_i, \cdots, \phi_j \in \mathbb{R}$, where $\{e_1, \cdots, e_j\}$ is a basis of E and $\{e_{j+1}, \cdots, e_n\}$ is a basis of its orthogonal complement. Define the family $\{A_{\alpha_1,\cdots,\alpha_j}\}_{\alpha_1,\cdots,\alpha_j \in \mathbb{R}}$ in the following way:

$$\textit{Having fixed } \alpha_1, \cdots, \alpha_j \in \mathbb{R}, \textit{ then } v \in A_{\alpha_1,\cdots,\alpha_j} \iff v = \left(\sum_{i=1}^{j} \alpha_i e_i\right) + \beta_1 e_{j+1} + \cdots + \beta_{n-j} e_n$$

where we will require that $\beta_i \neq 0$ for at least some $i \in \{1, \cdots, n-j\}$. It is clear that $A_{\alpha_1,\cdots,\alpha_j} \cong \mathbb{R}^{n-j} \setminus \{\mathbf{0}\}$, which is connected (*note that it is precisely here that we use the hypothesis that the codimension is at least 2!*). Now, fix $\mathbf{0} \neq a = (a_{j+1}, \cdots, a_n) \in \mathbb{R}^{n-j}$ and define another subspace A homeomorphic to $\mathbb{R}^j$ in the following way:

$$v \in A \iff v = \left(\sum_{i=1}^{j} \gamma_i e_i\right) + a_{j+1}e_{j+1} + \cdots a_n e_n$$

(note that here we require nothing of the γ). It is clear that $A \cong \mathbb{R}^j$ and that* $A \cap A_{\alpha_1,\cdots,\alpha_j} \neq \emptyset$ for all $\alpha_1, \cdots, \alpha_j \in \mathbb{R}$. Note also that*:

$$A \cup \left(\bigcup_{\alpha_1,\cdots,\alpha_j \in \mathbb{R}} A_{\alpha_1,\cdots,\alpha_j}\right) = \mathbb{R}^n \setminus E$$

and the result follows from lemma 7.

Remark (O.10). *This is always possible since $E \cong \mathbb{R}^j$.*

Remark (O.11). *Note that if $(\alpha_1, \cdots, \alpha_j) \neq \mathbf{0}$ then $(\alpha_1, \cdots, \alpha_j, a_{j+1}, \cdots, a_n)$ witnesses this fact.*

Remark (O.12). *The first inclusion is proved in the following way: if $v \in A$ or $v \in \bigcup_{\alpha_1,\cdots,\alpha_j \in \mathbb{R}} A_{\alpha_1,\cdots,\alpha_j}$, then its first j coordinates are linear combinations of vectors of the basis of E but there is at least one coordinate among the remaining $n-j$ that is not zero, so that $v \in \mathbb{R}^n \setminus E$. Conversely, if $v \in \mathbb{R}^n \setminus E$, then either all the coordinates from the $(j+1)$-th onward are equal to those of a or this does not happen. In the first case we have $v \in A$ and in the second we have $v \in \bigcup_{\alpha_1,\cdots,\alpha_j \in \mathbb{R}} A_{\alpha_1,\cdots,\alpha_j}$.*

Using path connectedness: Let F be a complementary subspace of E, that is, $\mathbb{R}^n = E \oplus F$. Take $x, y \in \mathbb{R}^n \setminus E$ and denote by $\bar{x}, \bar{y}$ their projections onto F. We claim that the following path joins x to y:

$$x \to \bar{x} \to_* \bar{y} \to y$$

where each $\to$ denotes a line segment, and we must take care not to pass through the origin going from $\bar{x}$ to $\bar{y}$ (which is easily accomplished: choose lines passing through $\bar{x}$ and through $\bar{y}$ that do not pass through the origin and intersect at p, and take the path $\bar{x} \to p \to \bar{y}$).

Remark (O.13). The converse of this exercise also holds (and is easier to prove), i.e, if E is a vector subspace, $\mathbb{R}^n \setminus E$ being connected implies that $\dim(E) \leq n - 2$.

■

3 Compactness exercises

Question 15. *Let $K \subset \Omega$. Then K is compact if, and only if, every cover of K by open sets of Ω admits a finite subcover.*

Solution: Suppose that K is compact and $\mathscr{A} = \{A_\alpha\}_{\alpha \in J}$ is a cover of Y by open sets of Ω. Then it is clear that:

$$\{A_\alpha \cap K \mid \alpha \in J\}$$

is a cover of K by open sets of K, and by hypothesis a finite subcover of the form:

$$\{A_{\alpha_1} \cap K, \cdots, A_{\alpha_n} \cap K\}$$

covers K. It follows that $\{A_{\alpha_1}, \cdots, A_{\alpha_n}\}$ is a subcover of $\mathscr{A}$ that covers K.

Conversely, if every cover of K by open sets of Ω has a finite subcover that covers K, we will show that K is compact. Indeed, if $\mathscr{A}' = \{A'_\alpha\}$ is a cover of K by open sets of K, then, by definition of the induced topology, we have that for each A'_α, there exists an open set A_α of Ω such that:

$$A'_\alpha = A_\alpha \cap K$$

We have that $\mathscr{A} = \{A_\alpha\}$ is a cover of K by open sets of Ω, so by hypothesis there exists some finite subcover $\{A_{\alpha_1}, \cdots, A_{\alpha_n}\}$ that covers K; it follows that $\{A'_{\alpha_1}, \cdots, A'_{\alpha_n}\}$ is a finite subcover of $\mathscr{A}'$ that covers K.

■

Question 16. *Show that a metric space is weakly sequentially compact if, and only if, it is sequentially compact. If Ω is only a topological space, which of these notions implies the other? Give an example of a non-metrizable space in which these notions do not coincide.*

Solution: If every sequence in a metric space has an accumulation point, then it is clear that every sequence has a convergent subsequence; it suffices to use the definition for open balls of smaller and smaller radii all centered at the accumulation point, identically to what is done below. For the converse it suffices to use theorem 13.11 and proposition 13.8 of the lecture notes. A counterexample that satisfies what is asked at the end of the exercise is the following:

Take $I = [0,1]$ and consider I^I with the product topology. We have that it is compact by Tychonoff's theorem and therefore weakly sequentially compact, but the sequence of functions $\alpha_n \in I^I$ defined by:

$$\alpha_n(x) \doteq \text{ the } n\text{-th digit in the binary expansion of } x$$

has no convergent subsequence. Indeed, suppose, for a contradiction, that $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence that converges to some $\alpha \in I^I$; then for each $x \in I$, $\alpha_{n_k}(x)$ converges to $\alpha(x) \in I$. Define $p \in I$ with the property that $\alpha_{n_k}(p) = 0$ or 1 depending on the parity of k. Then the sequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ is given by $0,1,0,1,\cdots$, which obviously cannot converge.

Proof that in metric spaces being limit point compact is equivalent to being sequentially compact, but that in general topological spaces the converse does not hold: *Suppose that (Ω, d) is weakly sequentially compact, i.e, every infinite subset has an accumulation point. Given a sequence $\{x_n\}_{n\in\mathbb{N}}$, if it has only a finite number of distinct terms, then it trivially contains a constant subsequence, which is therefore convergent. If the sequence contains infinitely many distinct terms, by hypothesis it has an accumulation point x; then define the following subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$, choosing $x_{n_1}, x_{n_2}, \cdots$ so that:*

$$x_{n_1} \in B_d(x, 1)$$

and define n_i inductively, in terms of n_{i-1}, such that $n_i > n_{i-1}$ and:

$$x_{n_i} \in B_d\left(x, \frac{1}{i}\right)$$

which we can always do, since $B_d\left(x, \frac{1}{i}\right)$ intersects A in infinitely many distinct points. Then it is clear that $x_{n_i} \to x$. Since $\{x_n\}_{n\in\mathbb{N}}$ was chosen arbitrarily, it follows that Ω is sequentially compact.*

Conversely, let us prove that if Ω is sequentially compact, then it is compact (and we know that this will imply that Ω is weakly sequentially compact, as desired). By hypothesis lemmas 2 and 3 hold, therefore there exists a finite cover by open balls of radius $\varepsilon = \frac{\delta}{3}$ (where δ is as in lemma **(L.3)**) *and diameter $\frac{2\delta}{3}$, so that each of them is contained in some $A \in \mathscr{A}$. The set consisting of all the A of this form is obviously a finite subcover of Ω, as desired.*

An example of a space that is limit point compact but not sequentially compact (and therefore not metrizable) is $\mathbb{R}$ with the topology generated by $\{(a,\infty) \mid a \in \mathbb{R}\}$. Any nonempty subset $A \subset \mathbb{R}$ has accumulation points, since given $a \in A$, $a - \epsilon$ is an accumulation point of A, since $(a-\epsilon, \infty) \cap A \setminus \{a-\epsilon\} = a$. Note that the sequence defined by $x_n = -n$ has no convergent subsequence in this topology.

■

Question 17. *Let $\Omega = \{0,1\}$ and consider on Ω the topology $\tau = \{\emptyset, \Omega\}$. Equip $\mathbb{N}$ with the discrete topology and consider the product $\mathbb{N} \times \Omega$. Use this example to show that not every weakly sequentially compact space is compact.*

Solution: We claim that under the conditions of the exercise, any nonempty set of $\mathbb{N}\times\{0,1\}$ has accumulation points. Indeed, if $S \subset \mathbb{N} \times \{0,1\}$ is nonempty, then there exists $n \in \mathbb{N}$ such that either $(n,0) \in S$ and clearly $(n,1)$ is an accumulation point (any basic open set of the product topology containing $(n,1)$ intersects S at $(n,0) \neq (n,1)$) or, analogously, $(n,1) \in S$ and $(n,0)$ is an accumulation point. Finally, note that $\mathscr{U} = \{U_n\}_{n\in\mathbb{N}} = \{\{n\} \times (0,1)\}_{n\in\mathbb{N}}$ is an infinite open cover that admits no finite open subcover.

■

Question 18. *Consider the Hilbert cube* $C = [0,1]^{\mathbb{N}}$ *equipped with the product metric, i.e,* $d(x,y) = \sup_{n\in\mathbb{N}}\left\{\frac{|x_n - y_n|}{n}\right\}$.

(a) *Show that in this topology, balls* $B_d(x,r)$ *are sets of the form* $\prod_{n\in\mathbb{N}} B(x_n, nr)$ *(here* B *without the subscript denotes the balls of* $[0,1]$ *relative to the absolute value);*

(b) *Show that* C *is complete;*

(c) *Verify that* C *is totally bounded and conclude that* C *is compact.*

Solution:

(a) Note that, given $r > 0$ and taking $N \in \mathbb{N}$ such that $\frac{1}{N} < r$, we have:

$$y \in \prod_{n\in\mathbb{N}} B(x_n, nr) \iff \frac{|x_n - y_n|}{n} < r \ \forall n \in \mathbb{N} \tag{6}$$

$$\iff \sup_{n\in\mathbb{N}}\left\{\frac{|x_n - y_n|}{n}\right\} = \max\left\{|x_1 - y_1|, \frac{|x_2 - y_2|}{2}, \cdots, \frac{|x_{N-1} - y_{N-1}|}{N-1}, \sup_{n\geq N}\left\{\frac{|x_n - y_n|}{n}\right\}\right\} < r \tag{7}$$

$$\iff y \in B_d(x,r) \tag{8}$$

where in the penultimate equality each of the terms inside the braces is less than r (see the following remark), and therefore their maximum is also less than r.

Remark (O.14). *For* $n < N$ *this is trivial; note that if* $n \geq N$, *we have* $\frac{|x_n - y_n|}{n} \leq \frac{1}{n} \leq \frac{1}{N} \implies \sup_{n\geq N}\left\{\frac{|x_n - y_n|}{n}\right\} \leq \frac{1}{N} < r$

(b) Let $\{x_k\}_{k\in\mathbb{N}} \subset [0,1]^{\mathbb{N}}$ be a Cauchy sequence, with $x_k = (x_k^{(1)}, x_k^{(2)}, \cdots) \in [0,1]^{\mathbb{N}}$. We want to show that $\{x_k\}_{k\in\mathbb{N}}$ converges, that is, to find a sequence $y = (y^{(1)}, y^{(2)}, \cdots) = (y^{(j)})_{j\in\mathbb{N}}$ such that $d(x_k, y) \to 0$. We will do this in the following way:

(i) We will show that if $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy, then $\{x_k^{(j)}\}_{k\in\mathbb{N}} = \{\pi_j(x_k)\}_{k\in\mathbb{N}} \subset [0,1]$ converges for each $j \in \mathbb{N}$.

(ii) Using an exercise from a previous problem set (question 6 of §19 of Munkres), we will conclude that $\{x_k\}_{k\in\mathbb{N}}$ converges. The completeness of $[0,1]^{\mathbb{N}}$ follows immediately.

Indeed, given arbitrary $\varepsilon > 0$ and $j_0 \in \mathbb{N}$, then, by hypothesis, there exists $K \in \mathbb{N}$ such that $k_1, k_2 \geq K$ implies that:

$$d(x_{k_1}, x_{k_2}) = \sup_{n\in\mathbb{N}}\left\{\frac{\left|x_{k_1}^{(n)} - x_{k_2}^{(n)}\right|}{n}\right\} < \frac{\varepsilon}{j_0}$$

But it is also obvious that:

$$\frac{\left|x_{k_1}^{(j_0)} - x_{k_2}^{(j_0)}\right|}{j_0} \leq \sup_{n\in\mathbb{N}}\left\{\frac{\left|x_{k_1}^{(n)} - x_{k_2}^{(n)}\right|}{n}\right\} < \frac{\varepsilon}{j_0} \implies \left|x_{k_1}^{(j_0)} - x_{k_2}^{(j_0)}\right| < \varepsilon$$

whence it follows that $\{x_k^{(j)}\}_{k\in\mathbb{N}}$ is Cauchy and therefore converges to some $y^{(j)}$ (hence step **(i)** has been carried out). As desired, we conclude that $\pi_j(x_k) \to \pi_j(y)$ for each $j \in \mathbb{N}$, where $y = (y^{(1)}, y^{(2)}, \cdots) = (y^{(j)})_{j\in\mathbb{N}}$.

(c) Let $\varepsilon > 0$ be arbitrary and take $n \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. For each $i > N$, take $p_i = 1$. Fix $i \in \{1, \cdots, N\}$. Since $[0,1]$ is totally bounded, there exists a finite number of points $\{x_1, x_2, \cdots, x_M\} = A_0$ such that:

$$x \in [0,1] \implies |x - x_j| < \frac{\varepsilon}{2} \text{ for some } j \in \{1, \cdots, M\}$$

Now, define $A = \{x = (x_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}} \mid x_1, x_2, \cdots, x_N \in A_0 \text{ and } x_{N+1} = x_{N+2} = \cdots = 1\}$. There are M^N points in A, therefore A is finite. Moreover, if $x = (x_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$, then for each $i \in \{1, \cdots, N\}$, there exist $x_j^{(i)} \in A_0$ (which means that all the x_j depend on the i) such that $|x_i - x_j^{(i)}| < \frac{\varepsilon}{2}$. Now, if we define $y_i = x_j^{(i)}$ for $i \in \{1, \cdots, N\}$ and $y_i = 1$ otherwise, then by construction $y = (y_n)_{n\in\mathbb{N}} \in A$, and for each $i \in \{1, \cdots, N\}$, we have that $\frac{|x_i - y_i|}{n} \leq |x_i - y_i| < \frac{\varepsilon}{2}$; also for each $i > N$, it holds that $\frac{|x_i - y_i|}{i} \leq \frac{1}{i} < \frac{1}{N} < \frac{\varepsilon}{2}$, hence $\sup_{n\in\mathbb{N}} \left\{\frac{|x_n - y_n|}{n}\right\} \leq \frac{\varepsilon}{2} < \varepsilon$ and it follows that $[0,1]^{\mathbb{N}} = \bigcup_{a\in A} B(a, \varepsilon)$.

■

Question 19. *Let Ω be a compact Hausdorff space. Let $f : \Omega \cup \{\infty\} \to \mathbb{R}$ be a lower semicontinuous function. Then f is bounded below, i.e, $\inf_{x\in\Omega} f(x) > -\infty$, and there exists $x \in \Omega$ such that $f(x) = \inf_{x\in\Omega} f(x)$.*

Solution:

Remark (O.15). There is a typo in the statement. Although it makes no difference, the question is: *Let Ω be a compact Hausdorff space. Let $f : \Omega \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous. Then f is bounded below, i.e, the infimum of the image of Ω_1 exists and f attains its infimum on Ω_1.*

Let* $m = \inf f(\Omega)$. For each n, define $C_n = f^{-1}(\left(-\infty, m + \frac{1}{n}\right])$. Note that $C = \{C_n\}$ satisfies the finite intersection property, since:

$$\begin{aligned}
&f^{-1}\left(\left(-\infty, m + \frac{1}{n_1}\right]\right) \bigcap f^{-1}\left(\left(-\infty, m + \frac{1}{n_2}\right]\right) \bigcap \cdots \bigcap f^{-1}\left(\left(-\infty, m + \frac{1}{n_k}\right]\right) \\
&= f^{-1}\left(\left(-\infty, m + \frac{1}{n_1}\right] \bigcap \left(-\infty, m + \frac{1}{n_2}\right] \cdots \bigcap \left(-\infty, m + \frac{1}{n_k}\right]\right) \\
&= f^{-1}\left(\left(-\infty, m + \frac{1}{n_j}\right]\right), \text{ where } n_j = \max\{n_1, \cdots, n_k\} \\
&\neq \emptyset \text{ by the definition of infimum}
\end{aligned}$$

Hence (since Ω is compact), $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$. But if $x \in \bigcap_{n\in\mathbb{N}} C_n$, then $f(x) \leq m$, and since m is the infimum, $f(x) = m$ and it follows that f attains its infimum, as desired.

Remark (O.16). *The infimum of $f(\Omega)$ exists, since $\{U_\alpha\}_{\alpha\in\mathbb{R}}$, with $U_\alpha = f^{-1}((\alpha, \infty))$, is an open cover that admits a finite subcover and therefore $f(\Omega)$ is bounded below.*

Remark (O.17). *Note that we proved something stronger, since at no point did we use the hypothesis that Ω is Hausdorff (being compact alone is enough). Note also that the ∞ was placed in the exercise unnecessarily. If the typo were not a typo, we would have that the ∞ was introduced for no purpose at all, only to be thrown away afterwards. If it had not been thrown away, the exercise would also be false (since it does not even specify the topology of $\Omega \cup \{\infty\}$): consider $\Omega = [0,1]$ and put on $\Omega \cup \{\infty\}$ the topology $\tau_{[0,1]} \cup \{\{\infty\}, \Omega \cup \{\infty\}\}$; then setting $f(x) = 2$ if $x \in [0,1]$ and $f(\infty) = 1$, $\inf f(\Omega \cup \{\infty\})$ is not attained in Ω.*

■

Question 20. *Consider $[0,1]^{\mathbb{N}}$ with the uniform topology. Find in this space an infinite subset without accumulation points.*

Solution:

Remark (O.18). *The metric of the uniform topology is given by:*

$$d(\mathbf{x}, \mathbf{y}) = \sup \left\{ \overline{d}\,(x_\alpha, y_\alpha) \mid \alpha \in J \right\}$$

where $\overline{d}$ is the standard bounded metric of $\mathbb{R}$.

Now, for each $j \in \mathbb{N}$ define $e_j \in [0,1]^{\mathbb{N}}$ by setting $\pi_i(e_j) = 1$ if $i = j$ and $\pi_i(e_j) = 0$ otherwise. We claim that $E = \bigcup_{j \in \mathbb{N}} \{e_j\} = \{e_j\}_{j \in \mathbb{N}} \subset [0,1]^{\mathbb{N}}$ has no accumulation points. Indeed, if $p \in [0,1]^{\mathbb{N}}$ is an accumulation point, then $B = B_d\left(p, \frac{1}{3}\right)$ (where d is the metric of the uniform topology), then by hypothesis there are infinitely many distinct points of E in B. This is an absurdity, since B cannot contain even two distinct points of E: if $i \neq k$, then $d(e_i, e_k) = 1 > \sup_{x,y \in B} d(x,y) = \operatorname{diam}(B) = \frac{2}{3}$.

Alternative proof: indeed, recalling lemma **(L.4)**, *if we set $\varepsilon = \frac{1}{2}$, for example, then given $e_k \in E$, for every $e_j \neq e_k$, we have $d(e_k, e_j) = 1 > \frac{1}{2}$, for example, then given $e_k \in E$, for every $e_j \neq e_k$, we have $d(e_k, e_j) = 1 > \frac{1}{2}$, and therefore no point of E is an accumulation point. We will now show that it also holds that no point outside E is an accumulation point, and therefore E' is empty:*
Given $x \in E^c$, we may suppose that there exists $j_0 \in \mathbb{N}$ such that $a = d(x, e_{j_0}) \in (0,1)$ (otherwise we would have $d(x, e_n) = 1$ for all $n \in \mathbb{N}$ and the proof would be over). By the triangle inequality, for all $i \neq j$, we have;

$$d(x, e_i) + d(x, e_j) = d(e_i, x) + d(x, e_j) \geq d(e_i, e_j) = 1 \implies d(x, e_i) \geq 1 - d(x, e_j)$$

for all $j \neq i$. In particular, $d(x, e_i) \geq 1 - a > \frac{1-a}{2}$ for all $i \neq j_0$. Thus, we have that $d(x, e_i) > \max\{\frac{1-a}{2}, \frac{a}{2}\}$ for all $i \in \mathbb{N}$ and it follows that x is not an accumulation point.

■

Question 21. *Show that $[0,1]$ as a subspace of $\mathbb{R}_\ell$ is not w.s.c.*

Solution: Here we will show that $[0,1]$ as a subspace of $\mathbb{R}_l$ is not limit point compact. The result will follow immediately, since if it were weakly sequentially compact, it would be limit point compact, as already observed previously.

We will show that $A = \left\{1 - \frac{1}{n}\right\}_{n\in\mathbb{N}}$ has no accumulation points. Indeed, if $x = 1 - \frac{1}{n} \in A$, then we have

$$1 - 1/n \in \left[1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right)$$

but $\left[1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right) \cap A \setminus \{x\} = \emptyset$, hence x is not an accumulation point. If $x \in (0,1)$ with $x \notin A$, then $x \in \left[1 - \frac{1}{j}, 1 - \frac{1}{j+1}\right) = U_x$ for some $j \in \mathbb{N}$ but $\left[x, 1 - \frac{1}{j+1}\right) \cap A = \emptyset$ and so x is not an accumulation point. Trivially, if $x > 1$ or $x < 0$ then x is not an accumulation point, so it remains to show that 1 is not an accumulation point. This is clear, since $[1,2) \cap [0,1] = \{1\}$.

Remark (O.19). *A more direct and elegant solution is to note that every accumulation point in a finer topology is necessarily an accumulation point in the coarser topology, so if A had accumulation points in $\mathbb{R}_l$ they would have to be accumulation points of A in $\mathbb{R}$ as well, but the only accumulation point of A in $\mathbb{R}$ is 1, which is not an accumulation point of A in $\mathbb{R}_l$ since $\{1\} = [0,1] \cap [1,2)$ is an open set of $[0,1]$ that does not intersect A.*

■

Question 22. *Show that the circle $\mathbb{S}^1$ with the topology induced from $\mathbb{R}^2$ is compact.*

Solution: We will show that $\mathbb{S}^1$ is closed and bounded; it will follow that it is compact. Indeed, $\mathbb{S}^1$ is closed, since $\mathbb{S}^1 = f^{-1}(\{1\})$, the inverse image of a closed set and therefore closed, where $f(x,y) = x^2 + y^2$. It is bounded since it is contained in the open ball (with the standard topology of $\mathbb{R}^2$) with center $(0,0)$ and radius 2.

■

Question 23. *Show that $[0,1]$ is not compact as a subspace of $\mathbb{R}_K$.*

Solution: It is easy to see that $\{U_n\}_{n\in\mathbb{N}}$, where:

$$U_n = \left\{\left(\frac{1}{n}, 2\right)\right\}_{n\in\mathbb{N}} \bigcup (-1,1) - K$$

is an open cover that admits no finite open subcover.

■

Question 24. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a convergent sequence with limit x. Show that $\{x\} \cup \{x_n\}_{n\in\mathbb{N}}$ is compact.*

Solution: Let $\mathscr{A}$ be an open cover of $\Omega = \{x\} \cup \{x_n \mid n \in \mathbb{N}\}$. Since $\mathscr{A}$ is a cover, there exists $A_0 \in \mathscr{A}$ containing x. Since A_0 is an open neighborhood of x and by hypothesis $x_n \to x$, A_0 contains all terms of the sequence from a certain $N \in \mathbb{N}$ onward, that is, there exists $N \in \mathbb{N}$ such that $\{x_N, x_{N+1}, \cdots\} \in A_0$. For each x_i with $1 \leq i \leq N-1$, we will use that $\mathscr{A}$ is a cover and choose $A_1, \cdots, A_{N-1} \in \mathscr{A}$ such that $x_i \in A_i$ for all $1 \leq i \leq N-1$. Then it is clear that:

$$\{A_0, A_1, \cdots, A_{N-1}\}$$

is a finite open subcover of Ω.

■

Question 25. *Show that any compact metric space Ω is homeomorphic to some subset of the Hilbert cube. (Hint: Ω is separable (justify), so let $\{x_1, x_2, \ldots\}$ be a dense subset of Ω. Define $F : \Omega \to C$ by setting $F(x) = (d(x, x_1), d(x, x_2), \ldots)$ and show that F is a homeomorphism).*

Solution: We have already proved that (Ω, d) is separable in lemma **(L.6)** and we know that any continuous bijective function from a compact space to a Hausdorff space is a homeomorphism. Note that, without loss of generality, we may assume that $d(x, y) \leq 1$ for all $x, y \in \Omega$ (if this did not happen we could simply use the standard bounded metric, which is equivalent). Letting $A = \{x_1, x_2, \cdots\}$ be a dense subset of Ω, we claim that the desired homeomorphism is:

$$\begin{aligned} F : (\Omega, d) &\to F(\Omega) \subset C = [0, 1]^{\mathbb{N}} \\ x &\mapsto (d(x, x_1), d(x, x_2), \cdots) \end{aligned}$$

where the topology on $F(\Omega)$ is the one induced from C. Indeed, since F is surjective by construction and $F(\Omega)$ is Hausdorff (since $[0, 1]^{\mathbb{N}}$ is Hausdorff and obviously any subspace of a Hausdorff space is Hausdorff as well), all that remains for us is to prove that F is injective (since, by lemma 6, the coordinate functions of F are continuous and therefore F is continuous). Now, if $f(x) = f(y)$ for $x, y \in \Omega$, then:

$$\begin{aligned} &d(x, x_1) = d(y, x_1) \\ &d(x, x_2) = d(y, x_2) \\ &\vdots \\ &d(x, x_n) = d(y, x_n)\ \forall n \in \mathbb{N} \end{aligned}$$

and note that, since A is dense, for every $\varepsilon > 0$ there exists $x_k \in A$ such that $d(x, x_k) = d(y, x_k) < \dfrac{\varepsilon}{1332}$. It follows that for every $\varepsilon > 0$, we have:

$$d(x, y) \leq d(x, x_k) + d(y, x_k) < \frac{\varepsilon}{1332} + \frac{\varepsilon}{1332} = \frac{\varepsilon}{666} < \varepsilon$$

and we conclude that $x = y$.

■

English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio. The Portuguese original is the authoritative version.

1 INFINITUDE OF THE PRIMES

a) Take $m, k, s, t \in \mathbb{Z}$ and note that if $A_{m,k} \cap A_{s,t} = (m\mathbb{Z} + k) \cap (s\mathbb{Z} + t) \neq \emptyset$, then it contains some $c \in \mathbb{Z}$, and so we claim that:

$$(m\mathbb{Z} + k) \cap (s\mathbb{Z} + t) = \text{lcm}(m, s)\mathbb{Z} + c$$

Indeed, if c belongs to the intersection, then c plus any common multiple of m and s also belongs to it (to make sure we do not leave any out, we start from the least common multiple). The reverse inclusion is a little less trivial: given $x \in (m\mathbb{Z} + k) \cap (s\mathbb{Z} + t)$, note that $x - c \equiv 0 \pmod{m}$ and $x - c \equiv 0 \pmod{s}$ (it is easy to see that the difference of any two elements of $m\mathbb{Z}$ or $s\mathbb{Z}$ is divisible by m or s, respectively; in particular, this also holds for x and c), whence it follows that $x - c \equiv 0 \pmod{\text{lcm}(m, s)}$, as desired.

Given any $x \in \mathbb{Z}$, we have $x \in A_{1,0}$, so that the first condition for being a basis is easily satisfied. Since we have already proved the second, we conclude that $\mathscr{B} = \{A_{m,k} \mid m, k \in \mathbb{Z}, m \neq 0\}$ is indeed a basis for a topology on $\mathbb{Z}$.

b) By definition, elements of the basis are open, so that we only need to prove now that $A_{m,k}$ is closed. Note that:

$$\mathbb{Z} \setminus A_{m,k} = \bigcup_{i=1}^{m-1} A_{m,k+i}$$

since $A_{m,k}$ is the union of all arithmetic progressions $\{a_n\}_{n \in \mathbb{N}}$ with common difference k such that a_1 is congruent to k modulo m. Its complement is, therefore, the union of all arithmetic progressions with common difference k such that a_1 is congruent to $k + 1, k + 2, \cdots, k + m - 1$ modulo m. Since the right-hand side of the equality is a union of open sets, it follows that $A_{m,k}$ is closed.

c) By definition, nonempty open sets of any topology on any set always contain elements of the basis. In this case, we note that, by construction, elements of $\mathscr{B}$ always contain an infinite arithmetic progression, so that no nonempty open set can be finite.

d) We know by the fundamental theorem of arithmetic that the only integers that are not integer multiples of some prime number are -1 and 1, whence it follows immediately that:

$$\mathbb{Z} \setminus \{-1, 1\} = B = \bigcup_{p \in P} A_{p,0} = 2\mathbb{Z} \cup 3\mathbb{Z} \cup 5\mathbb{Z} \cdots$$

e) The cardinality of P does not matter for B to be open, since B is, by definition, a union of open sets. The remark relevant to the proof of the infinitude of the primes that will be used in the next item is the following: if P is finite, the right-hand side of the equality above is a finite union of closed sets (we have already proved that all elements of $\mathscr{B}$ are clopen), that is, B is closed.

f) We proved in c) that no finite set can be open, or, equivalently, that the complement of a finite set cannot be closed. In e) we also noted that if P is finite, then B is closed, which is a contradiction, since B is the complement of $\{-1, 1\}$, a finite set! We conclude that P is infinite.

2 KURATOWSKI AXIOMS

a) First let us observe that a set F is closed if, and only if, $F = \overline{F}$. Indeed, if F is closed, F is necessarily the smallest closed set containing F, and, by definition, it follows that $F = \overline{F}$. Conversely, if $F = \overline{F}$, we have that F is closed, since $\overline{F}$ is, by definition, an intersection of closed sets and therefore also closed. With this in mind, note that:

i) $\Omega \setminus \emptyset = \Omega$, which is open by the definition of topology. Thus, $\emptyset$ is closed and it follows that $\overline{\emptyset} = \emptyset$.

ii) The closure of A is defined as the intersection of all closed sets that contain A, or, equivalently, as the smallest closed set that contains A. Therefore it follows directly from the definition that $A \subset \overline{A}$.

iii) The closure of A is closed, and by our initial remark, it follows that $\overline{\overline{A}} = \overline{A}$.

iv) Observe that $A \subset B \implies \overline{A} \subset \overline{B}$, since $\overline{B}$ is closed and contains B, which contains A. Since $\overline{B}$ is a closed set that contains A, it follows by definition that $\overline{A} \subset \overline{B}$. That said, let us note that $A \subset A \cup B$ and $B \subset A \cup B$, whence it follows that $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$, hence $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. To prove the reverse inclusion, note also that $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, and since we defined the closure as the *smallest* closed set that contains the set, it follows that $\overline{A} \cup \overline{B} \supset \overline{A \cup B}$. We conclude that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

b) We will use the following result (whose proof is very easy and not worth giving here; it only involves De Morgan's laws):

Let $\mathscr{C} \subset \mathscr{P}(\Omega)$ be such that:

i) $\emptyset$ and Ω are in $\mathscr{C}$

ii) finite unions of elements of $\mathscr{C}$ are still in $\mathscr{C}$

iii) arbitrary intersections of elements of $\mathscr{C}$ are still in $\mathscr{C}$

Then $\{\Omega \setminus C \mid C \in \mathscr{C}\}$ is a topology on Ω. Define $\mathscr{F} = \{A \subset X \mid A = F(A) = \overline{A}\}$. We claim that $\mathscr{F}$ satisfies i), ii) and iii). Indeed:

i) We have $F(\emptyset) = \emptyset$. Note that $F(\Omega) = \Omega$, since $\Omega \subset F(\Omega)$ by definition and $F(\Omega) \subset \Omega$ since F is a function from $\mathscr{P}(\Omega)$ to $\mathscr{P}(\Omega)$.

ii) If $A_1, A_2, \cdots, A_n \in \mathscr{F}$, then $A_1 \cup A_2 = \overline{A_1} \cup \overline{A_2} = \overline{A_1 \cup A_2} \in \mathscr{F}$ and it follows by induction that $A_1 \cup A_2 \cup \cdots \cup A_n = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} = \overline{A_1 \cup A_2 \cup \cdots \cup A_n} \in \mathscr{F}$.

iii) By the second axiom, letting $\mathcal{I}$ be an arbitrary index set, we have $\bigcap_{i \in \mathcal{I}} C_i \subset \mathrm{F}\left(\bigcap_{i \in \mathcal{I}} C_i\right)$.

Observe also that an immediate consequence of the third axiom is that F preserves inclusions, so that:

$$\bigcap_{i \in \mathcal{I}} C_i \subset C_i \forall i \in \mathcal{I} \Rightarrow \mathrm{F}\left(\bigcap_{i \in \mathcal{I}} C_i\right) \subset \mathrm{F}\left(C_i\right) = C_i \forall i \in \mathcal{I} \Rightarrow \mathrm{F}\left(\bigcap_{i \in \mathcal{I}} C_i\right) \subset \bigcap_{i \in \mathcal{I}} C_i$$

and, as desired, the arbitrary intersection is closed and therefore belongs to $\mathscr{F}$. The topology induced by F is, then, $\tau = \{\Omega \setminus A \mid A = F(A)\}$. It is also unique, since given any $A \subset \Omega$, the set $F(A)$ is the closure of A in the topological space (Ω, τ): indeed, given $A \subset \Omega$, since $F(F(A)) = F(A)$, we know that $F(A) \in \mathscr{F}$, and from the first axiom we know that $A \subset F(A)$. If K is any other element of $\mathscr{F}$ containing A, then $F(A) \subset F(K) = K$, and we conclude that

$F(A)$ is the smallest element of $\mathscr{F}$ containing A, as desired. It follows that every Kuratowski closure operator determines and is determined by a unique topology.

c) Let us observe that in this case $F(A) = \overline{A} = \bigcup_{n\in\mathbb{N}} nA$. Thus:

i) is trivially satisfied, since $\bigcup_{n\in\mathbb{N}} n\emptyset = \emptyset$

ii) follows directly from the fact that $A \subset \bigcup_{n\in\mathbb{N}} nA = A \cup 2A \cup \cdots$

iii) by definition, we have that $F(A) = A \iff A$ contains all multiples of elements of A. $F(A)$ satisfies this condition by its very definition, so that $F(F(A)) = F(A)$, as desired.

iv) $\overline{A} \cup \overline{B} = \bigcup_{n\in\mathbb{N}} nA \bigcup \bigcup_{n\in\mathbb{N}} nB = \bigcup_{n\in\mathbb{N}} nA \cup nB = \bigcup_{n\in\mathbb{N}} n(A \cup B) = \overline{A \cup B}$

By the remark in iii), the closed sets of this topology are the sets $k\mathbb{N}$ with $k \in \mathbb{N}$ or finite unions or intersections of them. The open sets are the complements of these closed sets. Define B_n in the following way: $a \in \mathbb{N} \in B_n \iff a|n$, that is, B_n is the set containing n and all its divisors. We claim that:

$$\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$$

is a basis for τ. Indeed, given any open set U of $\mathbb{N}$ and $x \in U$ we have $x \in B_x \subset U$, since if $B_x \not\subset U$, then $a \in B_x \implies a \in U^c$, which is closed; hence, since a divides x and U^c is closed and contains all multiples of its elements, it follows that $x \in U^c$, an absurdity! To verify the second basis condition, note that if $x \in B_{k_1} \cap B_{k_2}$, then $x|k_1$ and $x|k_2$, whence it follows that $x|\gcd(k_1, k_2) = k_3$ and therefore $x \in B_{k_3} \subset B_{k_1} \cap B_{k_2}$.

d) Suppose that f is continuous and $m|n$, that is, $n = km$ for some $k \in \mathbb{N}$. Then:

$$f(n) \in f(m\mathbb{N}) = f(\overline{\{m\}}) \subset \overline{f(\{m\})} = \overline{\{f(m)\}} = f(m)\mathbb{N}$$

and we conclude that $f(m)|f(n)$. Conversely, if $m|n \implies f(m)|f(n)$, we will show that given any $A \subset \mathbb{N}$, we have $f(\overline{A}) \subset \overline{f(A)}$, i.e, f is continuous. Indeed, if $y \in f(\overline{A}) = f\left(\bigcup_{n\in\mathbb{N}} nA\right)$, then $y \in f(n_0A)$ for some $n_0 \in \mathbb{N}$. Hence $y = f(n_0a)$ for some $a \in A$. Since $a|n_0a$, we have, by hypothesis, that $f(a)|f(n_0a)$, hence there exists $k \in \mathbb{N}$ such that $y = f(n_0a) = kf(a) \in kf(A) \subset \overline{f(A)}$, as desired.

3 LIOUVILLE NUMBERS

a) By the definition of Liouville numbers, note that we can write

$$\mathbb{L} = \{x \in \mathbb{R} \mid x \text{ is a Liouville number}\} = \bigcap_{n=1}^{\infty} U_n$$

where:

$$U_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left\{ x \in \mathbb{R} : 0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \right\} = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right) \setminus \left\{ \frac{p}{q} \right\}$$

Observe that $\overline{U_n} \supset \mathbb{Q}$ since the $\overline{U_n}$ contain each $\frac{p}{q} \in \mathbb{Q}$. Indeed, if $\frac{p}{q} \in \mathbb{Q}$, then we have

$$\frac{p}{q} \in \overline{\left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right) \setminus \left\{ \frac{p}{q} \right\}} \subset \overline{U_n} \implies \overline{\overline{U_n}} = \overline{U_n} \supset \overline{\mathbb{Q}} = \mathbb{R} \implies \overline{U_n} = \mathbb{R}$$

Thus, we write $\mathbb{L}$ (it is easy to see, by our very construction of the U_n and by the definition of Liouville numbers, that if x is a Liouville number, then $x \in U_n \; \forall n \in \mathbb{N}$ and, vice versa, if x is in the intersection, then by definition it is a Liouville number) as a countable intersection of dense open sets (since each U_n is a union of open sets) of $\mathbb{R}$. It follows that $\mathbb{L}$ is a G_δ.

b) We will prove the following corollary of the Baire theorem, from which the result follows immediately:

Let X be a (nonempty) complete metric space without isolated points and $D \subset X$ a G_δ subset of X. Then D is uncountable.

Suppose that $D = \bigcap_{n \in \mathbb{N}} U_n$, where each U_n is a dense open set. We can write $X \setminus D = \bigcap_{x \in D} X \setminus \{x\}$. Note that each $X \setminus \{x\}$ is a dense open set, since $x \in D$ is not an isolated point. Then, supposing, for a contradiction, that D is countable, we have that

$$\emptyset = D \cap (X \setminus D) = \left(\bigcap_{n \in \mathbb{N}} U_n \right) \bigcap \left(\bigcap_{x \in D} X \setminus \{x\} \right)$$

that is, we have exhibited a countable intersection of dense open sets that is empty, contradicting the Baire theorem. It follows that D is uncountable.

Indeed, we can conclude that $\mathbb{L}$ is uncountable, since, as we proved previously, $\mathbb{L}$ is a G_δ subset of $\mathbb{R}$, a complete metric space without isolated points.

English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio. The Portuguese original is the authoritative version.

Lemma 1. *Let (Ω_1, d_1) be a compact metric space and (Ω_2, d_2) an arbitrary metric space, and suppose that $f : \Omega_1 \to \Omega_2$ is continuous. Then f is uniformly continuous.*

Proof: Given $\varepsilon > 0$, by the continuity hypothesis on f, for each $x \in \Omega_1$ there exists δ_x such that $f(B_{d_1}(x, \delta_x)) \subset B_{d_2}\left(f(x), \frac{\varepsilon}{2}\right)$. Since Ω_1 is compact, the open cover $\left\{B_{d_1}\left(x, \frac{\delta_x}{2}\right)\right\}_{x\in\Omega_1}$ admits a finite subcover $\left\{B_{d_1}\left(x_i, \frac{\delta_{x_i}}{2}\right)\right\}_{i=1}^{n}$. Now, taking $\delta \doteq \min\limits_{1\leq i\leq n} \dfrac{\delta_{x_i}}{2}$, we claim that:

$$d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \varepsilon$$

Indeed, this happens because:

$$d_2(f(x), f(y)) \leq d_2(f(x), f(x_i)) + d_2(f(x_i), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

where we used that $x_i \in B_{d_1}\left(x_i, \frac{\delta_{x_i}}{2}\right) \subset B_{d_1}(x_i, \delta_{x_i}) \implies f(x_i) \in B_{d_2}\left(f(x), \frac{\varepsilon}{2}\right)$ and also that:

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i} \implies f(y) \in B_{d_2}\left(f(x_i), \frac{\varepsilon}{2}\right)$$

■

Notation. Setting $\|f\| = \sup\limits_{x\in\Omega} |f(x)|$ we induce a metric on $\mathscr{C}(\Omega, \mathbb{R})$ (the space of continuous functions from Ω to $\mathbb{R}$) given by $d(f, g) = \sup\limits_{x\in\Omega} |f(x) - g(x)|$.

1 PROBLEM 1

Using some well-known inequalities from Calculus and the definition of supremum, we have that:

$$|T(f(x))| = \left|\int_0^1 K(x, y)f(y)\ \mathrm{dy}\right| \leq \int_0^1 |K(x, y)||f(y)|\ \mathrm{dy} \leq \sup_{x,y\in[0,1]^2} |K(x, y)| \cdot \sup_{y\in[0,1]} |f(y)| = \|K\| \cdot \|f\| \leq \|K\|$$

so that $\{T(f) \mid \|f\| \leq 1\}$ is pointwise bounded. To apply Arzelà-Ascoli and finish the exercise, it remains to show that it is also equicontinuous. Indeed, by lemma 1, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, x' \in [0, 1]$ with $|x - x'| < \delta \implies |K(x, y) - K(x', y)| < \dfrac{\varepsilon}{2\|f\|}$ for all $y \in [0, 1]$, and thus:

$$\begin{aligned}
|T(f(x)) - T(f(x'))| &= \left|\int_0^1 K(x, y) f(y)\ \mathrm{dy} - \int_0^1 K(x', y) f(y)\ \mathrm{dy}\right| \\
&= \left|\int_0^1 (K(x, y) - K(x', y)) f(y)\ \mathrm{dy}\right| \\
&\leq \int_0^1 |K(x, y) - K(x', y)|\, |f(y)|\ \mathrm{dy} \\
&\leq \sup_{y\in[0,1]} |K(x, y) - K(x', y)| \cdot \|f\| \\
&\leq \frac{\varepsilon}{2\|f\|} \cdot \|f\| \\
&= \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

as desired (note that, in particular, this implies that $T(f) \in C([0, 1], \mathbb{R})$). The result follows by the Arzelà-Ascoli theorem.

2 PROBLEM 2

First, let us note that $\mathscr{F} = \{f \in \mathscr{C}(\Omega, \mathbb{R}) \mid \|f\| \leq 1 \text{ and } \mathrm{Hol}_\alpha(f) \leq 1\}$ is equicontinuous. Indeed, given $f \in \mathscr{F}$ and $\varepsilon > 0$, it is clear that for all $x, y \in \Omega$ with $d(x,y) < \delta \doteq \varepsilon^{\frac{1}{\alpha}}$, we have that:

$$\begin{aligned} |f(x) - f(y)| &\leq \mathrm{Hol}_\alpha(f) d(x,y)^\alpha \\ &\leq d(x,y)^\alpha \\ &< \delta^\alpha \\ &= \varepsilon \end{aligned}$$

It is also obvious that $\mathscr{F}$ is pointwise bounded, since given $x \in \Omega$ and $f \in \mathscr{F}$, we have $|f(x)| \leq \|f\| \leq 1$. By Arzelà-Ascoli it follows that $\overline{\mathscr{F}}$ is compact, so that it only remains for us to show that $\mathscr{F} = \overline{\mathscr{F}}$.

Indeed, note that it suffices to show that every limit of a sequence in $\mathscr{F}$ also belongs to $\mathscr{F}$ (this is a consequence of lemma 21.2 of Munkres, seen in class). Indeed, if $\{f_n\}_{n\in\mathbb{N}} \in \mathscr{F}^{\mathbb{N}}$ converges to some $f \in \mathscr{C}(\Omega, \mathbb{R})$, then given $\varepsilon > 0$ we can choose $n \in \mathbb{N}$ so that $\|f_n - f\| < \frac{\varepsilon}{2}$, and it follows that:

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \leq \frac{|f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|}{d(x,y)^\alpha} \tag{1}$$

$$\leq \frac{\varepsilon}{d(x,y)^\alpha} + \frac{|f_n(x) - f_n(y)|}{d(x,y)^\alpha} \tag{2}$$

$$\leq 1 + \frac{\varepsilon}{d(x,y)^\alpha} \tag{3}$$

and since this holds for every $\varepsilon > 0$, we have that:

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \leq 1$$

whence we conclude that $\mathrm{Hol}_\alpha(f) \leq 1$, as desired. It remains to show that $\|f\| \leq 1$. Indeed this is true, since given arbitrary $x \in \Omega$, we have that $\{f_n(x)\}_{n\in\mathbb{N}}$ is a sequence in $[-1, 1]$ that converges to $f(x)$, therefore $|f(x)| \leq 1$ and it follows that $\|f\| \leq 1$.

Remark. To go from (2) to (3) note that by hypothesis we have $\mathrm{Hol}_\alpha(f_n) \leq 1$. Also, the frequently used hypothesis that $x \neq y$ was not made explicit, since it is self-evident.