Local structure of almost Ricci solitons with harmonic Weyl curvature

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June 3, 2026

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English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio, translated from the author's LaTeX source. The Portuguese original is the authoritative version.

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Distinguished metrics and curvature conditions

The search for canonical metrics

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- The natural answer is to impose **conditions on the curvature** and ask which manifolds satisfy them, and how *rigid* they are.
- We know that very strong conditions (such as, for example, constant sectional curvature) leave very little freedom; weaker ones admit rich families and have too much flexibility.

The algebraic approach: holonomy

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- **Cartan** classified the simply connected *symmetric spaces* using this algebraic approach: a Riemannian symmetric space is of the form $(N = G/H, h)$, with $\nabla \mathrm{Rm} = 0$, and its restricted holonomy is determined by the isotropy representation of H^0.

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- Thus, the question "what are the best metrics?" is related to another algebraic problem: **which subgroups of** $\mathrm{O}(n)$ **can be holonomy groups?**

The de Rham theorem and Berger's list

- **De Rham** (1952), a student of Cartan, proved that, on a complete and simply connected manifold, a decomposition of the holonomy representation induces a Riemannian product decomposition: if $\mathrm{Hol}^0(g)$ acts reducibly, then

$$(\mathcal{M}, g) \cong (\mathcal{M}_1, g_1) \times \cdots \times (\mathcal{M}_k, g_k), \qquad \mathrm{Hol}(g) \cong \mathrm{Hol}(g_1) \times \cdots \times \mathrm{Hol}(g_k).$$

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- Hence it suffices to classify the *irreducible* holonomy groups: those that do not decompose in this way.
- **Berger** (1955) did exactly that: he classified the possibilities for the restricted holonomy of an irreducible, non-locally symmetric Riemannian manifold.

Berger's list and Einstein metrics

Berger's classification

If $(\mathcal{M}^n, g)$ is irreducible and not locally symmetric, then $\mathrm{Hol}^0(g)$ is one of the following:

$$\mathrm{SO}(n),\ \ \mathrm{U}(m),\ \ \mathrm{SU}(m),\ \ \mathrm{Sp}(m),\ \ \mathrm{Sp}(m)\mathrm{Sp}(1),\ \ G_2,\ \ \mathrm{Spin}(7).$$

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  - $\mathrm{Sp}(m)\mathrm{Sp}(1)$: quaternionic-Kähler and **Einstein**.
- **Moral:** restricting the holonomy is an algebraic way of imposing conditions on the curvature. Several items in Berger's list force the metric to be Einstein, or even Ricci-flat.

Table of Berger's restricted holonomies

Restricted holonomy	**Dim.**	**Geometry**	**Comment**
$\mathrm{SO}(n)$	n	orientable Riemannian	generic Riemannian
$\mathrm{U}(m)$	$2m$	Kähler	generic Kähler
$\mathrm{SU}(m)$	$2m$	Calabi–Yau	Ricci-flat Kähler
$\mathrm{Sp}(m)\mathrm{Sp}(1)$	$4m$	quaternionic-Kähler	Einstein
$\mathrm{Sp}(m)$	$4m$	hyper-Kähler	Ricci-flat
G_2	7	G_2-manifold	Ricci-flat
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)$-manifold	Ricci-flat

Einstein metrics: the central condition

Definition

$(\mathcal{M}^n, g)$ is *Einstein* if $\mathrm{Ric} = \lambda\, g$ for some constant $\lambda \in \mathbb{R}$.

- For $n \geq 3$ constancy is automatic (Schur's lemma): if $\mathrm{Ric} = \varphi\, g$ pointwise with $\varphi \in \mathscr{C}^\infty(\mathcal{M})$, then φ is constant.

Einstein metrics: existence

Existence question

Given a closed manifold $\mathcal{M}^n$, is there *any* Einstein metric on $\mathcal{M}$? The answer changes radically with n.

Dimensions 2 and 3: existence under control

- $n = 2$ **(Uniformization, Poincaré–Koebe).** Every closed surface admits a metric of *constant* Gaussian curvature $+1$, 0 or -1, according to the sign of χ. Existence is **total** and completely classified.

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Moral

Up to dimension 3, the existence of Einstein metrics is *completely understood*.

Dimension 4: the invariants χ and τ

- **Euler characteristic:** $\chi(\mathcal{M}) = \sum_{i=0}^{n} (-1)^i b_i$, with $b_i = \dim H^i(\mathcal{M}; \mathbb{R})$.

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- **Signature:** for closed oriented $\mathcal{M}^4$, the intersection form $H^2(\mathcal{M}; \mathbb{R}) \times H^2(\mathcal{M}; \mathbb{R}) \to \mathbb{R}$, $(\alpha, \beta) \mapsto \int_{\mathcal{M}} \alpha \wedge \beta$, is symmetric and nondegenerate; its signature is

$$\tau(\mathcal{M}) = b_2^+ - b_2^- .$$

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- **Chern–Gauss–Bonnet** and the **Hirzebruch signature** theorem express both invariants as curvature integrals:
$$\chi = \frac{1}{8\pi^2}\int_{\mathcal{M}} \Big(|W|^2 + \tfrac{\mathrm{Scal}^2}{24} - \tfrac{1}{2}|\mathring{\mathrm{Ric}}|^2 \Big)\,\mathrm{dVol}_g, \qquad \tau = \frac{1}{12\pi^2}\int_{\mathcal{M}} \big(|W^+|^2 - |W^-|^2 \big)\,\mathrm{dVol}_g,$$
where $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{\mathrm{Scal}}{4}\, g$ is the traceless Ricci tensor.

Dimension 4: the Hitchin–Thorpe inequality

- If g is Einstein, then $\mathring{\mathrm{Ric}} = 0$; combining the two formulas,

$$2\chi \pm 3\tau = \frac{1}{4\pi^2}\int_{\mathcal{M}} \left(2\,|W^{\pm}|^2 + \tfrac{\mathrm{Scal}^2}{24}\right) \mathrm{dVol}_g \;\geq\; 0.$$

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Theorem (Thorpe 1969; Hitchin 1974) [33]

If a closed oriented manifold $\mathcal{M}^4$ admits an Einstein metric, then

$$\boxed{2\,\chi(\mathcal{M}) \geq 3\,|\tau(\mathcal{M})|}.$$

Hitchin–Thorpe: consequences

- It is the **first genuine obstruction** to existence beyond dimension 3: there are 4-manifolds (e.g. suitable connected sums) that do *not* admit an Einstein metric.

Dimensions ≥ 4: unknown territory

Open problem (every $n \geq 4$)

Does the sphere $\mathbb{S}^n$ admit an Einstein metric of *negative* scalar curvature?

- The round metric is Einstein with Scal > 0; whether there is *another* Einstein metric on $\mathbb{S}^n$ with Scal < 0 remains **open** in *every* dimension $n \geq 4$.

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- Striking contrast (**Lohkamp** [34]): *every* manifold of dimension $n \geq 3$ admits a metric of *strictly negative* scalar curvature (in fact, Ricci curvature). Negative scalar curvature is totally *flexible* and unobstructed; the *rigid*, Einstein version, on the sphere, is unknown.

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The contrast with dimension 4

For $n \geq 5$ **no counterexample is known** to the statement "every closed manifold admits an Einstein metric"; unlike dimension 4, where Hitchin–Thorpe effectively *obstructs*. Existence in high dimension is an **open** problem [32].

Gromov: the search for the "best" metric

(e) **Einstein and the forlorn quest for the best metric.** It is geometers' dream (first articulated by Heinz Hopf, I believe) to find a canonical metric g_{best} on a given smooth manifold V so that all topology of V will be captured by geometry. This happened to come true for surfaces as all of them carry (almost unique) metrics of constant curvature and is predicted for $n = 3$ by Thurston's geometrization conjecture. Also, there is a glimpse of hope for $n = 4$ (Einstein, self-dual) but no trace of g_{best} has ever been found for $n \geq 5$. What is the reason for this? Let us take some (energy) function E on the space $\mathcal{G}$ of metrics, say built of the full curvature tensor, something like $\int (\text{Curv}(g))^{\frac{n}{2}} dv$ (where the exponent $n/2$ makes the integral scale invariant). Imagine, the gradient flow of E brings all of $\mathcal{G}$ to a "nice" subspace $\mathcal{G}_{\text{best}} \subset \mathcal{G}$ (ideally, a single point or something not very large anyway). Then the group Diff V would act on $\mathcal{G}_{\text{best}}$ (as all we do should be Diff-invariant) with compact isotropy subgroups (we assume V is compact at the moment), e.g. if $\mathcal{G}_{\text{best}}$ consisted of a single point, then Diff V would isometrically act on $(V, \mathcal{G}_{\text{best}})$. But the high dimensional topology (unknown to Hopf) tells us that the space Diff V is too vast, soft and unruly to be contained in something nice and cosy like the desired $\mathcal{G}_{\text{best}}$. (Diff is governed

Source: Gromov, *Spaces and Questions* [47].

Gromov: pessimism in dimension $n \geq 5$

Following Alex we (I speak for myself) are lead to the pessimistic conclusion that there is no chance for a distinguished g_{best} (or even $\mathcal{G}_{\text{best}} \subset \mathcal{G}$) for $n \geq 5$ and that "natural" metrics, e.g. Einstein $\mathcal{G}$ with $\text{Ri}(g) = \lambda g$ for $\lambda < 0$, must be chaotically scattered in the vastness of $\mathcal{G}$ with no meaningful link between geometry and topology. (This does not preclude, but rather predicts, the existence of such metrics, e.g. Einstein, on all V of dimension ≥ 5: the problem is there may be too many of them.)

- According to Gromov, in dimension $n \geq 5$ one should not expect a small distinguished class $\mathcal{G}_{\text{best}}$ concentrating the natural metrics.

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- According to Gromov, in dimension $n \geq 5$ one should not expect a small distinguished class $\mathcal{G}_{\text{best}}$ concentrating the natural metrics.
- Einstein metrics with negative constant may exist in abundance, but be scattered chaotically in the vast space $\mathcal{G}$ of metrics.

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The flexible invariant: the Yamabe problem

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Yamabe problem (Yamabe 1960; Trudinger 1968; Aubin 1976; Schoen 1984) [39]

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On every closed manifold $\mathcal{M}^n$, $n \geq 3$, **each conformal class** contains a metric of **constant scalar curvature**.

- In particular, **every** closed manifold admits a metric of constant scalar curvature, with no topological obstruction whatsoever.

The setting where existence is understood: Kähler–Einstein

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$$[\rho] = 2\pi\, c_1(\mathcal{M}).$$
- Hence being *Kähler–Einstein* ($\rho = \lambda\,\omega$) **requires** c_1 to have a definite sign:
$$\lambda > 0 \Rightarrow c_1 > 0 \text{ (Fano)}, \quad \lambda = 0 \Rightarrow c_1 = 0 \text{ (Calabi–Yau)}, \quad \lambda < 0 \Rightarrow c_1 < 0.$$

Kähler–Einstein: the Calabi conjecture

Calabi conjecture (1954) [35]

In a fixed Kähler class, *every* closed form representing $c_1(\mathcal{M})$ is the Ricci form of a **unique** Kähler metric in that class.

Analytically, this amounts to solving a **complex Monge–Ampère** equation $(\omega + i\partial\bar{\partial}\varphi)^n = e^F \omega^n$.

Yau and the resolution of the Calabi conjecture

- **Famous episode:** Yau spent years *trying to disprove* the conjecture; he even announced a **counterexample** at a conference in 1973. When trying to reconstruct the argument at Calabi's request, he realized it was *wrong*; then convinced that the conjecture was true, he turned all his efforts to proving it.

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- **Yau (1976/78)** [36] proved the Calabi conjecture (part of the work for his Fields Medal). The case $c_1 < 0$ was obtained independently by **Aubin** [37].

Consequences of the Calabi conjecture

Consequences for the Einstein problem

- $c_1 < 0$: there is **always** a (unique) Kähler–Einstein metric (Aubin–Yau).
- $c_1 = 0$: there are **Ricci-flat** Kähler metrics (*Calabi–Yau*), the first compact non-flat Ricci-flat examples, realizing $\mathrm{SU}(m)$ in Berger's list.
- $c_1 > 0$ (Fano): there are obstructions (Matsushima, Futaki); existence is equivalent to **K-stability** (Yau–Tian–Donaldson, **Chen–Donaldson–Sun** [38]).

Moral: in the Kähler world, the existence problem for Einstein metrics is *essentially solved*: it is the best understood broad setting.

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- **Example (LeBrun)** [11]: the moduli space of Einstein metrics on $\mathbb{S}^5$ has **infinitely many** connected components; there are families of unit volume with Einstein constant $\lambda_j \to 0^+$.
- In dimension 4 existence depends even on the *smooth structure*, and the curvature decomposes via $\mathscr{W} = \mathscr{W}^+ \oplus \mathscr{W}^-$, the root of special geometry in dimension 4.

“Canonical” does not mean “unique”

- Even when they exist, canonical metrics can be **abundant**.
- **Example (LeBrun)** [11]: the moduli space of Einstein metrics on $\mathbb{S}^5$ has **infinitely many** connected components; there are families of unit volume with Einstein constant $\lambda_j \to 0^+$.
- In dimension 4 existence depends even on the *smooth structure*, and the curvature decomposes via $\mathscr{W} = \mathscr{W}^+ \oplus \mathscr{W}^-$, the root of special geometry in dimension 4.
- **Moral:** to obtain classifications one must **add structural hypotheses**. Ours will be the combination of an (almost) Ricci soliton structure together with $\operatorname{div} \mathscr{W} = 0$.

Space forms

- With the convention of this talk, a metric of constant sectional curvature c satisfies

$$\mathrm{Rm}(X,Y,Z,T) = c\big(g(X,Z)g(Y,T) - g(X,T)g(Y,Z)\big).$$

Space forms

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$$\mathrm{Rm}(X,Y,Z,T) = c\big(g(X,Z)g(Y,T) - g(X,T)g(Y,Z)\big).$$

- The algebraic tensor naturally associated with the metric is $g \owedge g$. Since

$$(g \owedge g)(X,Y,Z,T) = 2\big(g(X,Z)g(Y,T) - g(X,T)g(Y,Z)\big),$$

we have

$$\boxed{\mathrm{Rm} = \frac{c}{2}\, g \owedge g}.$$

The Kulkarni–Nomizu product

- The map $g \mapsto g \owedge g$ is quadratic.

$$\begin{aligned}(h \owedge k)(X,Y,Z,T) = {} & h(X,Z)k(Y,T) + h(Y,T)k(X,Z) \\ & - h(X,T)k(Y,Z) - h(Y,Z)k(X,T).\end{aligned}$$

The Kulkarni–Nomizu product

- The map $g \mapsto g \owedge g$ is quadratic.
- Polarizing it yields the Kulkarni–Nomizu product of symmetric tensors:

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What the Ricci curvature determines

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- The curvature decomposition is

$$\boxed{\mathrm{Rm} = W + g \owedge \mathrm{Sch}}$$

and, equivalently,

$$W = \mathrm{Rm} - \frac{1}{n-2} g \owedge \mathrm{Ric} + \frac{\mathrm{Scal}}{2(n-1)(n-2)} g \owedge g.$$

The Weyl tensor

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- In dimension 3, $W \equiv 0$; the conformal curvature is then measured by the Cotton tensor,

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  It measures exactly the obstruction for the Schouten tensor to be Codazzi.
- If g is Einstein and $n \geq 3$, then $W = 0$ is equivalent to constant sectional curvature.

Harmonic curvature and Ricci-Codazzi

Definition

We say that g has *harmonic Riemannian curvature* if

$$\delta \operatorname{Rm} = 0, \qquad \nabla^i R_{ijk\ell} = 0.$$

- By the contracted second Bianchi identity,

$$\nabla^i R_{ijk\ell} = \nabla_k \operatorname{Ric}_{j\ell} - \nabla_\ell \operatorname{Ric}_{jk}\,.$$

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- Therefore

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- A further contraction gives $2 \operatorname{div} \operatorname{Ric} = \mathrm{d} \operatorname{Scal}$. Hence, if $\delta \operatorname{Rm} = 0$, then Scal is constant.

Harmonic Weyl: the weakened condition

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$(\mathcal{M}^n, g)$ has *harmonic Weyl curvature* if $\delta W = 0$; in this talk we also write $\operatorname{div} \mathscr{W} = 0$.

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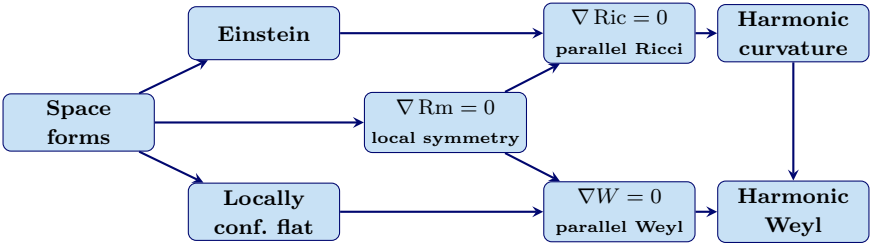
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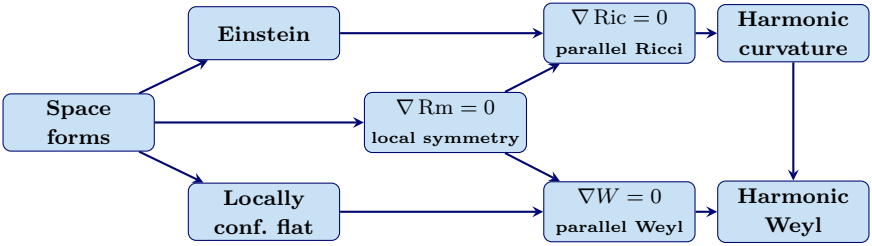
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where $\mathrm{C}_{ijk} = \nabla_i \mathrm{Sch}_{jk} - \nabla_j \mathrm{Sch}_{ik}$.
- The full harmonic curvature condition is stronger:
$$\delta \operatorname{Rm} = 0 \quad \Longleftrightarrow \quad \delta W = 0 \text{ and } \mathrm{d}\operatorname{Scal} = 0 \quad \Longleftrightarrow \quad \mathrm{Ric} \text{ is Codazzi}.$$

The inclusion diagram



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- Harmonic curvature = harmonic Weyl + constant Scal.
- Higher-order conditions add no new nodes to the Riemannian diagram: for $k > 1$, imposing $\nabla^k \operatorname{Rm} = 0$, $\nabla^k \operatorname{Ric} = 0$ or $\nabla^k W = 0$ collapses to the corresponding parallel conditions $\nabla \operatorname{Rm} = 0$, $\nabla \operatorname{Ric} = 0$ or $\nabla W = 0$ [21], [22]; $\nabla W = 0$ is the conformally symmetric condition [23].

1974: Tachibana

Theorem (Tachibana) [12]

If (M, g) is compact, has harmonic curvature and positive curvature operator, then g has constant positive sectional curvature. If the curvature operator is only nonnegative, one obtains local symmetry ($\nabla \operatorname{Rm} = 0$).

- In the positive case, the conclusion is that of a spherical space form, possibly after a finite quotient.

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- In the positive case, the conclusion is that of a spherical space form, possibly after a finite quotient.
- The positivity hypothesis on the curvature operator is relatively strong.
- The first result of the theory is therefore one of **rigidity under positivity**, and not a general classification.

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Bourguignon's question

Does every compact Riemannian manifold with $\delta \operatorname{Rm} = 0$ have $\nabla \operatorname{Ric} = 0$?

1980: Derdziński and the negative answer

Theorem and examples [14]

Derdziński constructed compact examples with

$$\delta \operatorname{Rm} = 0, \qquad \nabla \operatorname{Ric} \neq 0,$$

answering Bourguignon's question in the negative.

1980: Derdziński and local structure

Local theorem [14]

If $n > 3$, $\delta \operatorname{Rm} = 0$, $\nabla \operatorname{Ric} \neq 0$ and Ric has exactly two distinct eigenvalues, then

$$g = \mathrm{d}s^2 + F(s)\, h$$

is a *warped* product with one-dimensional base and Einstein fiber.

1981: the Codazzi mechanism

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- For a Codazzi tensor A, on the open set where the multiplicities of the eigenvalues are locally constant **[15]**:
- each eigendistribution is integrable and its leaves are totally umbilical;
- eigenvalues of multiplicity > 1 are constant along their own eigendistribution.
- If $\mathrm{tr}A$ is constant, A is not parallel and there are exactly two eigenvalues, the smaller block has dimension 1.

The question for solitons

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1. Gradient solitons already carry a potential function f, foliations by the level sets of f, and differential identities relating Ric, ∇f and Scal.
2. The natural question is: how does $\delta W = 0$ restrict the Ricci eigendistributions, and what does this condition force on the geometry of $\mathcal{M}$?

Ricci solitons and the Ricci flow

Richard Hamilton and the Ricci flow



"[. . .] we want a second-order, parabolic, quasi-linear flow that diffuses equally in all directions. There is only one flow with this property: the Ricci flow

$$\frac{\partial}{\partial t}\, g(t) = -2\ \mathrm{Ric}_{g(t)}\,.$$

"

Self-similar solutions = solitons

- The simplest solutions of the flow are the **self-similar** ones: those that evolve only by diffeomorphisms and rescalings,

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- Twofold importance: besides providing the self-similar solutions, solitons appear as **models of the singularities** formed by the flow.

Gradient Ricci solitons

Definition

$(\mathcal{M}^n, g, f)$ is a *gradient Ricci soliton* if $\mathrm{Ric} + \nabla^2 f = \lambda\, g$ for some constant λ. It is called *shrinking* if $\lambda > 0$, *steady* if $\lambda = 0$, *expanding* if $\lambda < 0$.

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- In the sense of **Petersen–Wylie**, a soliton is **rigid** if its universal cover is a product $\mathcal{N}^{n-k} \times \mathbb{R}^k$, with $\mathcal{N}$ Einstein, and $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor.

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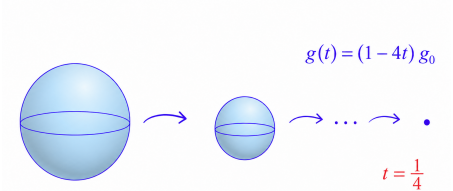
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- **Natural question:** under curvature hypotheses (harmonic Weyl!), what can be said about the local structure of a soliton?

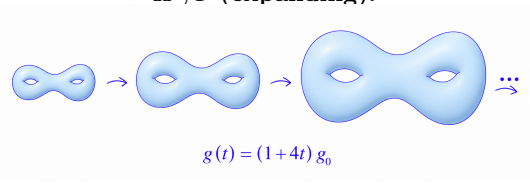
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Sphere (shrinking):



shrinks to a point; curvature blows up.

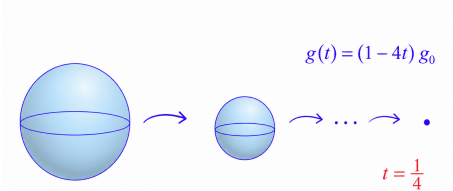
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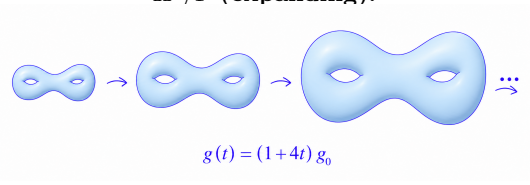
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For an Einstein metric, $g(t) = (1 - 2\lambda t)\, g_0$ solves the flow: the prototype of a trivial soliton.

Petersen–Wylie rigidity

Petersen–Wylie

A complete gradient Ricci soliton is rigid when, up to a quotient by isometries, it has the form

$$(\mathcal{N}^{n-k}, g_{\mathcal{N}}) \times (\mathbb{R}^k, g_{\mathrm{eucl}}), \qquad \mathrm{Ric}_{\mathcal{N}} = \lambda g_{\mathcal{N}}, \qquad f = \frac{\lambda}{2}|x|^2 + c.$$

The Petersen–Wylie criterion

Petersen–Wylie [7]

A complete gradient Ricci soliton is rigid if and only if

$$\text{Scal is constant} \qquad \text{and} \qquad \mathrm{Rm}(\cdot, \nabla f, \nabla f, \cdot) = 0.$$

- The second condition is **vanishing radial sectional curvature**: at the regular points of f,

$$\sec(E, \nabla f) = 0 \qquad \text{for every } E \perp \nabla f.$$

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- A distinct condition that has also been studied is

$$\mathscr{W}(\nabla f, \cdot, \cdot, \cdot) = 0,$$

that is, radially vanishing Weyl.

Constant scalar curvature: quantized values [43]

Fernández-López–García-Río (2014)

If $(\mathcal{M}^n, g, f)$ is a complete gradient Ricci soliton with constant scalar curvature, then

$$\mathrm{Scal} \in \{0, \lambda, 2\lambda, \ldots, (n-1)\lambda, n\lambda\}.$$

- When Scal is constant, the soliton identities give that f is isoparametric:

$$|\nabla f|^2 = 2\lambda f - \mathrm{Scal}, \qquad \Delta f = n\lambda - \mathrm{Scal}\,.$$

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- On a nonempty focal submanifold of f,

$$\mathrm{Ric}\,|_{M_\pm} = \begin{pmatrix} \lambda I_k & 0 \\ 0 & 0_{n-k} \end{pmatrix},$$

and taking the trace gives $\mathrm{Scal} = k\lambda$.

Few Ricci eigenvalues force rigidity [43]

Fernández-López–García-Río (2014)

If $(\mathcal{M}^n, g, f)$ is a complete gradient Ricci soliton with constant Scal and Ric has at most three distinct eigenvalues, then the soliton is rigid.

Further consequences of $\mathrm{Scal} = k\lambda$ [43]

- The extreme values have a rigid interpretation: $\mathrm{Scal} = 0$ gives the Ricci-flat metric, while $\mathrm{Scal} = n\lambda$ gives the trivial Einstein soliton.

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Open

We do not know the geometry of the intermediate cases.

Cao's conjecture for *shrinkers* [44]

Cao's conjecture

If $(\mathcal{M}^n, g, f)$, $n \geq 4$, is a complete *shrinking* gradient Ricci soliton with constant scalar curvature, then it is rigid. Equivalently, up to a finite quotient,

$$\mathcal{M}^n \cong \mathcal{N}^k \times \mathbb{R}^{n-k},$$

where $\mathcal{N}^k$ is Einstein with positive scalar curvature.

- Petersen–Wylie characterize rigidity by

$$\text{Scal constant} \qquad \text{and} \qquad \mathrm{Rm}(\cdot, \nabla f, \nabla f, \cdot) = 0.$$

The conjecture asserts that, for *shrinkers*, the first condition should already force the second.

Progress on Cao's conjecture

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- The conjecture remains open in full generality for $n \geq 5$.
- The bound of three eigenvalues reappears naturally: under $\operatorname{div} \mathscr{W} = 0$, the Codazzi structure of the Schouten tensor strongly restricts the eigendistributions of Ric. This is the mechanism that will be explored below.

Almost Ricci solitons

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$(\mathcal{M}^n, g, f)$ is a *gradient almost Ricci soliton* if

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Almost Ricci solitons: why they are delicate

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- They appear naturally in flows with reaction terms and in conformal variational problems.
- The extra freedom in λ makes the analysis of the local structure **considerably more delicate**.

The compact case: Barros–Batista–Ribeiro rigidity

Theorem (Barros–Batista–Ribeiro Jr.) [25]

Let $(\mathcal{M}^n, g, X, \lambda)$, $n \geq 3$, be a **compact**, oriented and **nontrivial** almost Ricci soliton. If the scalar curvature Scal is **constant**, then $\mathcal{M}^n$ is isometric to the round sphere $\mathbb{S}^n$.

Schouten solitons

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$$\mathrm{Ric} + \nabla^2 f = \big(\rho\ \mathrm{Scal} + \lambda\big)\, g.$$

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- For $\rho = \dfrac{1}{2(n-1)}$ we obtain the **Schouten soliton**:

$$\mathrm{Ric} + \nabla^2 f = \left(\frac{\mathrm{Scal}}{2(n-1)} + \lambda\right) g, \qquad \nabla^2 f = \lambda g - \big(\mathrm{Ric} - \tfrac{\mathrm{Scal}}{2(n-1)} g\big).$$

Rigidity in the Schouten case

In the Schouten case

For

$$\mathrm{Ric} + \nabla^2 f = \left(\frac{\mathrm{Scal}}{2(n-1)} + \tau \right) g,$$

with τ constant, we say that the soliton is rigid if its universal cover splits as

$$\mathcal{N}^{n-k} \times \mathbb{R}^k, \qquad \mathrm{Ric}_{\mathcal{N}} = \mu g_{\mathcal{N}}, \qquad f = \frac{\mu}{2}|x|^2 + c,$$

where the constant μ is compatible with the Schouten equation.

Rigidity in the Schouten case: compatible constant

Compatibility

The constant μ of the Einstein factor satisfies

$$\mu = \frac{\mathrm{Scal}_{\mathcal{N}}}{2(n-1)} + \tau \quad \left(= \frac{2(n-1)}{n+k-2}\tau \text{ if } \dim \mathcal{N} = n-k \right).$$

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We will prove in **Theorem 3** that the only gradient Schouten solitons with harmonic Weyl curvature are rigid.

Harmonic Weyl curvature

Definition

$(\mathcal{M}^n, g)$ has *harmonic Weyl curvature* if $\operatorname{div} \mathscr{W} = 0$.

- For $n \geq 4$, this is equivalent to

$$\operatorname{div} \mathscr{W} = 0 \iff \mathrm{C} = 0 \iff \nabla_i \operatorname{Sch}_{jk} = \nabla_j \operatorname{Sch}_{ik} \iff \operatorname{Sch} \text{ is Codazzi.}$$

Harmonic Weyl curvature: revisited

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- It simultaneously generalizes **Einstein** (Sch parallel $\Rightarrow$ Codazzi) and **LCF** ($\mathscr{W} = 0 \Rightarrow \operatorname{div} \mathscr{W} = 0$).

What if we required harmonic curvature?

- Recall: $\delta \operatorname{Rm} = 0 \iff \operatorname{div} \mathscr{W} = 0$ *and* Scal constant. Harmonic curvature is therefore **strictly stronger** than our hypothesis.

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Harmonic curvature ⇒ strong rigidity

A **proper** gradient almost soliton (f nonconstant) with **harmonic curvature** has constant Scal. In the **compact** case, Barros–Batista–Ribeiro [25] guarantee that $\mathcal{M}^n$ is isometric to the round sphere $\mathbb{S}^n$.

Harmonic curvature: consequence

- In dimension 4, Kim [26] obtains a **complete description** of gradient almost solitons with harmonic curvature.

The gear: Codazzi tensors

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Structure of a Codazzi tensor

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On the open set of constant multiplicities, Derdziński [15] guarantees integrability of the eigendistributions and totally umbilical leaves.

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Structure of a Codazzi tensor

On the open set of constant multiplicities, Derdziński [15] guarantees integrability of the eigendistributions and totally umbilical leaves.

- Since Sch is just a rescaling of Ric (plus a multiple of g), Ric **and** Sch **have the same eigenspaces**.
- The (almost) soliton equation couples Ric to $\nabla^2 f$, forcing ∇f to interact with this decomposition. This is where the control of the geometry comes from.

Main results

Logical order of the results

The recurring mechanism

$$\begin{aligned} \operatorname{div} \mathscr{W} = 0 &\Rightarrow \text{Sch Codazzi} \\ &\Rightarrow \text{controlled eigendistributions} \\ &\Rightarrow \text{warped product and few eigenvalues of } \operatorname{Ric}. \end{aligned}$$

Logical order of the results: outline

- First, the **easier** route: a **purely algebraic** path to the eigenvalue bound, valid when a warped structure is already available (locally decomposable net).

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- Next, the **general engine** (the hard route): for **almost solitons** (variable λ), the *refined adapted frame* method gives the same bound and, **without** assuming decomposability, the local warped product structure with ≤ 2 fibers.

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- First, the **easier** route: a **purely algebraic** path to the eigenvalue bound, valid when a warped structure is already available (locally decomposable net).
- Next, the **general engine** (the hard route): for **almost solitons** (variable λ), the *refined adapted frame* method gives the same bound and, **without** assuming decomposability, the local warped product structure with ≤ 2 fibers.
- Finally, we use this structure for **Schouten solitons**: in this case, the special equation forces the warping functions to be linear, and completeness gives rigidity.

Common lemmas: what $\operatorname{div}\mathscr{W} = 0$ buys

Radial vanishing Cotton lemma

For an almost soliton $\operatorname{Ric} + \nabla^2 f = \lambda g$, $C = 0$ is equivalent to

$$\operatorname{Rm}(\nabla f, X, Y, Z) = \frac{1}{n-1}\{\operatorname{Ric}(\nabla f, Y)g(X, Z) - \operatorname{Ric}(\nabla f, Z)g(X, Y)\}.$$

In dimension $n \geq 4$, this is the same as harmonic Weyl.

- Consequences on a regular level: $E_1 = \nabla f/|\nabla f|$ is an eigenvector of Ric, $|\nabla f|$ is constant on Σ_c, there is a coordinate s with $E_1 = \nabla s$, and $\nabla_{E_1} E_1 = 0$.

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- Since Sch is Codazzi, Derdziński gives integrability of the eigendistributions and totally umbilical leaves on the open set of constant multiplicities.

Theorem 2: almost solitons [2]

Theorem (Borges–Horácio–Dos Santos)

Let $\mathcal{M}^{n\geq 4}$ be a **nontrivial gradient almost Ricci soliton** with harmonic Weyl curvature. Then:

1. the Ricci tensor admits **at most three distinct eigenvalues**;

Theorem (continued)

Around any regular point of f, $\mathcal{M}$ is **locally isometric** to a multiply *warped* product

$$(I,\, \mathrm{d}s^2) \times_{\varphi_1} \mathcal{N}_1^{k_1} \times_{\varphi_2} \mathcal{N}_2^{k_2},$$

with one-dimensional base and **at most two Einstein fibers** $\mathcal{N}_1, \mathcal{N}_2$.

Density of the regular set [2]

Lemma (density–dichotomy; Lemma 3.1 of [2])

Let $\mathcal{M}^n$ be a gradient almost Ricci soliton with harmonic Weyl curvature, f nonconstant, and $\mathcal{R} = \{\, \nabla f \neq 0 \,\}$ the regular set. If W is a connected component of $M_{\mathcal{A}}$, then $W \cap \mathcal{R}$ is **dense in** W or **empty**.

- **Why it is needed.** In the Ricci soliton case, g and f are analytic in harmonic coordinates and $\mathcal{R}$ is controlled by analyticity. For almost solitons this *fails*: g, f need not be analytic. The lemma replaces that argument.

Lemmas on dependence on s

- We work on $M_{\mathcal{A}} \cap \mathcal{R}$, where the multiplicities of Ric are constant; by the density lemma, if f is not constant this set is dense in each relevant component of $M_{\mathcal{A}}$.

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- **Lemma.** On each connected component of a regular level Σ_c, the functions

$$\text{Scal}, \quad \lambda, \quad \lambda_1 = \text{Ric}(E_1, E_1), \quad \lambda_2, \ldots, \lambda_n$$

are constant. Hence they all depend only on the parameter s.

The technical core: the eigenvalue estimate

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- Given this bound, the following are consequences (Thm. 3.6 of [2]):
  - the multiply warped product structure with ≤ 2 fibers (item 2 of Theorem 2);
  - the rigidity of Schouten solitons (Theorem 3, below).

The technical core: two routes

- Kim follows the same order: he first proves the estimate ([5]) and only then deduces the local structure. **We will prove the bound in two ways:** first a short *algebraic shortcut* (under decomposability of the net) and then the *general method* via the refined frame, which dispenses with this hypothesis. The local structure is a corollary of the bound.

Algebraic route: Ricci solitons [1]

- We begin with the **easier** route: a **purely algebraic shortcut** to the bound, available *when a warped structure is already at hand* (locally decomposable net).

Algebraic route: Ricci solitons [1]

- We begin with the **easier** route: a **purely algebraic shortcut** to the bound, available *when a warped structure is already at hand* (locally decomposable net).
- It is short and transparent; the price is the decomposability hypothesis, whose subtlety we will explain below. The *general* route (refined frame) will come later, precisely to **dispense with it**.

Algebraic route: theorem [1]

Theorem (Borges–Horácio–Dos Santos)

Let $\mathcal{M}^{n\geq 4}$ be a **nontrivial gradient Ricci soliton** with harmonic Weyl curvature. Suppose that the net of eigendistributions of the Ricci tensor of $\mathcal{M}$ is **locally decomposable**. Then the Ricci tensor of $\mathcal{M}$ admits **at most three distinct eigenvalues**.

Algebraic route: geometric reading

- The hypothesis of **local decomposability of the net** *avoids* the refined frame (the general route, below): it immediately delivers the adapted coordinates.

Algebraic route: ingredients (already seen)

1. The same ingredients as before: since $\operatorname{div} \mathscr{W} = 0$, the Schouten tensor is Codazzi and **Derdziński** gives integrability of the eigendistributions of Ric.

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1. The same ingredients as before: since $\operatorname{div} \mathscr{W} = 0$, the Schouten tensor is Codazzi and **Derdziński** gives integrability of the eigendistributions of Ric.
2. By the radial lemma, $E_1 = \nabla f / |\nabla f|$ is an eigenvector of Ric; writing $\mathrm{Ric}(E_i, \cdot) = \lambda_i g(E_i, \cdot)$,

$$\nabla_{E_a} E_1 = \xi_a E_a, \qquad \xi_a = \frac{\lambda - \lambda_a}{f'} \quad (a \geq 2).$$

Here s is the arc-length parameter of E_1 and $f' = \mathrm{d}f/\mathrm{d}s$.

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   Here s is the arc-length parameter of E_1 and $f' = \mathrm{d}f/\mathrm{d}s$.
3. The quantities Scal, λ_i and ξ_i depend only on s (constant on each regular level $\Sigma_c = f^{-1}(c)$).

Algebraic route: multiply warped product

- *Assuming* that the net is **locally decomposable** (Reckziegel–Schaaf), a point we will discuss below, one obtains coordinates adapted simultaneously to all eigenvalue blocks.

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- In these coordinates,

$$\partial_1 g_{ab} = 2\xi_a g_{ab}, \qquad \partial_\alpha g_{ab} = 0 \quad ([\alpha] \neq [a]).$$

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- In these coordinates,

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- Integrating the first equation in s one obtains, with fibers $\mathcal{N}_i^{r_i}$ of dimension r_i,

$$g = \mathrm{d}s^2 + h_1(s)^2 g_{\mathcal{N}_1} + \cdots + h_k(s)^2 g_{\mathcal{N}_k}, \qquad \xi_i = h_i'/h_i.$$

Algebraic route: structural conclusion

Structural conclusion

The Ricci formulas for warped products, combined with the soliton equation, show that every fiber $\mathcal{N}_i$ with $r_i \geq 2$ is Einstein, with constant μ_i.

Where do the coordinates come from? Local decomposability [27]

Definition (locally decomposable net, Reckziegel–Schaaf)

A *net* $\mathcal{E} = (D_0, D_1, \dots, D_k)$ of mutually orthogonal distributions is **locally decomposable** if, around each point, there is an *adapted* coordinate system: each D_i is spanned by a block of the coordinate fields ∂_x.

- These are exactly the coordinates we use to write the warped product.

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Theorem (Reckziegel–Schaaf, *Results in Math.* 35, 1999)

The net is locally decomposable **if and only if every partial sum** $D_I := \bigoplus_{i \in I} D_i$ $(I \subseteq \{0, \dots, k\})$ **is integrable**.

An aside: quasi-Einstein manifolds

- The broader natural setting encompassing solitons and almost solitons is that of *quasi-Einstein manifolds*.

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Definition (m-quasi-Einstein) [41]

$(\mathcal{M}^n, g, f, \lambda)$ is *m-quasi-Einstein* $(0 < m \leq \infty)$ if

$$\mathrm{Ric} + \nabla^2 f - \tfrac{1}{m}\, \mathrm{d}f \otimes \mathrm{d}f = \lambda\, g, \qquad \lambda \in \mathscr{C}^\infty(\mathcal{M}).$$

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- $m = \infty$ recovers the **gradient almost Ricci soliton** ($\mathrm{Ric} + \nabla^2 f = \lambda g$); with λ constant, the usual Ricci soliton.
- For $m \in \mathbb{Z}_{>0}$, the equation (with λ constant) is *exactly* the condition for the **warped** product $\mathcal{M}^n \times_{e^{-f/m}} \mathcal{F}^m$ to be **Einstein** over a fiber of dimension m; hence the name.

An integrability conjecture

- In our setting, each *individual* eigendistribution D_{λ_i} is integrable (Derdziński). But the theorem above requires the integrability of **all partial sums**, and this is *not* automatic.

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Conjecture

For a **gradient Ricci soliton** (and, more generally, a quasi-Einstein manifold) with **harmonic Weyl curvature**, the direct sum of any distinct eigendistributions of the Ricci tensor is **integrable**.

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Conjecture

For a **gradient Ricci soliton** (and, more generally, a quasi-Einstein manifold) with **harmonic Weyl curvature**, the direct sum of any distinct eigendistributions of the Ricci tensor is **integrable**.

- **If the conjecture holds** $\Rightarrow$ (Reckziegel–Schaaf) decomposable net $\Rightarrow$ adapted coordinates $\Rightarrow$ multiply warped product. This is exactly the hypothesis under which the alternative route works.

Integrability is far from automatic [28]

- In a question we asked on **MathOverflow**, **Robert Bryant** constructed counterexamples to the conjecture in the *conformally flat* case once the Ricci soliton structure hypothesis is removed.

Integrability is not automatic!

Moral

The conformal hypothesis $\mathscr{W} = 0$ **alone** does *not* give the conjecture.

It is the soliton structure that saves the day (LCF case) [29]

- Compare with Fernández-López–García-Río (*Geom. Dedicata* **168**, 2014): vanishing Weyl by itself does not force the integrability of the mixed sums (Bryant), but

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Fernández-López–García-Río

A **conformally flat gradient Ricci soliton** is locally isometric to a warped product $I \times_h N$ of an interval with a **space form**; hence with *a single* fiber.

The importance of the soliton structure

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- That is: it is the **addition of the soliton equation**, even when $\mathscr{W} = 0$, that forces the direct sum to be integrable.
- We believe that our situation is analogous: $\operatorname{div} \mathscr{W} = 0$ alone is insufficient; one must use the soliton equations in a finer way.

The conjecture in the literature: Cao–Li–Siene [30]

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- The proof of the multiply warped product structure (their central theorem) starts **exactly from these coordinates adapted** to all Ricci eigendistributions; that is, it **presupposes the conjecture**.

The delicate point

The existence of these coordinates is *equivalent* to a nontrivial statement of simultaneous integrability (of partial sums); it is not a matter of applying Frobenius distribution by distribution.

Delicate points

- **Sphericity (Hiepko)** [31]: to obtain the warped product from the distributions, orthogonality + integrability + umbilicity is not enough; the leaves must be **spherical**, with parallel mean curvature vector H_i *and* totally geodesic orthogonal complement.

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- **Low-dimensional phenomenon:** with only three distributions (one of them one-dimensional), the two transversal classes are autoparallel in Σ_c, hence parallel, and the de Rham theorem closes the argument immediately. This does *not* generalize to more than two transversal classes.

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- **Low-dimensional phenomenon:** with only three distributions (one of them one-dimensional), the two transversal classes are autoparallel in Σ_c, hence parallel, and the de Rham theorem closes the argument immediately. This does *not* generalize to more than two transversal classes.
- In fact, the eigenvalue bound is equivalent to the **parallelism** of the eigendistributions; the only way parallelism can fail appears only with more than two transversal classes.

Algebraic route: Li's system

- Having the warped structure, it remains to show that there are **at most two** factors $\mathcal{N}_i$; this becomes pure algebra.

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- Having the warped structure, it remains to show that there are **at most two** factors $\mathcal{N}_i$; this becomes pure algebra.
- **Lemma (F. Li).** The functions $\xi_i = h_i'/h_i$ satisfy

$$\xi_i' + \xi_i^2 = -\frac{\mathrm{Scal}'}{2(n-1)f'}, \qquad -f'\xi_i + \lambda = -\xi_i' - \xi_i \sum_{j=2}^{n} \xi_j + (r_i - 1)\frac{\mu_i}{h_i^2}.$$

Algebraic route: Li's system (continued)

- With
$$B = \frac{(n-1)\lambda - \mathrm{Scal} + \lambda_1 - (f')^2}{f'}, \qquad C = -\frac{\mathrm{Scal}'}{2(n-1)f'} + \lambda,$$
one obtains
$$\xi_i^2 - B\,\xi_i - C = -(r_i - 1)\frac{\mu_i}{h_i^2}.$$

Algebraic route: the polynomial

- Differentiating and using $h_i'/h_i = \xi_i$, we eliminate h_i and obtain the polynomial

$$B\,\xi_i^2 + (B' + 2\lambda)\,\xi_i + (C - \lambda)B + C' = 0,$$

which is nontrivial; hence there are at most two distinct values of ξ_i.

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- Therefore $k \leq 2$ and Ric has at most three distinct eigenvalues, recovering the bound by a short algebraic path.

The general route: refined adapted frame [5]

- **Now the general route.** The algebraic shortcut required decomposability of the net. The *refined frame* method obtains the same bound **without** assuming it, at the price of Kim's machinery [5].

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- **Now the general route.** The algebraic shortcut required decomposability of the net. The *refined frame* method obtains the same bound **without** assuming it, at the price of Kim's machinery [5].
- Each E_i ($i \geq 2$) is transported along the flow of E_1 and renormalized:

$$F_1 = E_1, \qquad \nabla_{F_1} F_i = 0 \quad (i > 1).$$

This kills the radial derivatives, controlling the connection coefficients in dimension ≥ 5.

Riccati linearization and the common variable h

- $\lambda = \lambda(s)$, and each $\xi_i := \langle \nabla_{E_i} E_1, E_i \rangle = (\lambda - \lambda_i)/f'$ $(i \geq 2)$ satisfies the **Riccati** equation $\xi_i' + \xi_i^2 = -\lambda_1/(n-1)$, **with no** λ'.

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- **Hypothesis, essential here:** $\xi_i \not\equiv 0$ for every $i \geq 2$. Only then does the linearization $\xi_i = u_i'/u_i$ hold, which transforms the Riccati equation into $u'' + \frac{\lambda_1}{n-1} u = 0$ and produces the structural form below. If some $\xi_{i_0} \equiv 0$, the Riccati equation forces $\lambda_1 = 0$: this is the **degenerate** case, treated separately below.

Riccati linearization and the common variable h

- $\lambda = \lambda(s)$, and each $\xi_i := \langle \nabla_{E_i} E_1, E_i \rangle = (\lambda - \lambda_i)/f'$ $(i \geq 2)$ satisfies the **Riccati** equation $\xi_i' + \xi_i^2 = -\lambda_1/(n-1)$, **with no** λ'.
- **Hypothesis, essential here:** $\xi_i \not\equiv 0$ for every $i \geq 2$. Only then does the linearization $\xi_i = u_i'/u_i$ hold, which transforms the Riccati equation into $u'' + \frac{\lambda_1}{n-1} u = 0$ and produces the structural form below. If some $\xi_{i_0} \equiv 0$, the Riccati equation forces $\lambda_1 = 0$: this is the **degenerate** case, treated separately below.
- Taking [2] as the distinguished class, u_2 solves that linear ODE and defines the **common variable** $h(s) := \int_{s_0}^{s} u_2(v)^{-2}\, \mathrm{d}v$ (hence $h' = u_2^{-2} > 0$). All the other classes inherit

$$\xi_i = \frac{u_2'}{u_2} + \frac{h'}{c_i + h}, \qquad [i] \neq [2],$$

with c_i distinct for each class; from this, Kim's algebraic reduction carries over entirely.

The master identity and the breaking point

- Subtracting $\lambda_i = \lambda - \xi_i f'$ between classes, λ **cancels**. Kim's reduction ends with constants a, b, c such that

$$h' = ch^2 + ah + b, \quad \xi_2 = -ch - \tfrac{a}{2}, \quad \xi_{s_\alpha} = \tfrac{a}{2} - cc_\alpha + \tfrac{P_\alpha}{h + c_\alpha}.$$

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- For each class $[s_\alpha]$, the **master identity** ([5], Eq. 46) expresses $\lambda - C_0$ as a rational function of h:

$$\boxed{\lambda - C_0 = L\,h' + D_\alpha\,Q_\alpha(h) - \sum_{\beta \neq \alpha} E_{\beta,\alpha}\,Q_\beta(h)}$$

with $L := A_2 - c(n_2 - 1)$, $Q_\beta(h) := c + \dfrac{a - 2cc_\beta}{h + c_\beta} + \dfrac{P_\beta}{(h + c_\beta)^2} = \dfrac{h'}{(h + c_\beta)^2}$, and constants $A_2, D_\alpha, E_{\beta,\alpha}$.

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- **Kim:** λ constant $\Rightarrow$ Lemma 6.1 (partial fractions) reads off the coefficients and closes the argument.
- **Our case:** $\lambda = C_0 + f''$ is a function of s; Lemma 6.1 **does not apply** directly. It is here, and *only* here, that constant λ is used in an essential way in Kim.

The fix: the contracted Bianchi identity (1/2)

- We import an equation **absent** from Kim. Taking the divergence of the almost soliton equation, using the **contracted second Bianchi identity** and differentiating the trace $\mathrm{Scal} + \Delta f = n\lambda$:

$$\mathrm{Ric}(\nabla f, \cdot) = (1-n)\,\mathrm{d}\lambda + \tfrac{1}{2}\,\mathrm{d}\,\mathrm{Scal}\,.$$

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- Evaluating at $E_1 = \nabla f/|\nabla f|$ and writing $\Delta f = f'' + f'S$, with $S := \sum_{i>1} \xi_i$, this becomes a **3rd-order** ODE for f:

$$\boxed{(n-1)f''' + f''S + f'S' + 2C_0 f' = 0}\,.$$

The fix: partial fractions and Laurent (2/2)

- Since $\lambda = C_0 + f''$, the master identity becomes $f'' = L\,h' + D_\alpha Q_\alpha(h) - \sum_{\beta \neq \alpha} E_{\beta,\alpha} Q_\beta(h)$. Subtracting the instances for two labels α, γ and applying partial fractions (Lemma 6.1) in the variable $x = h(s)$:

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- **Laurent in the Bianchi ODE:** setting $x := h + c_\alpha \to 0$, the leading term x^{-3} has coefficient $2\big(n_{s_\alpha} - (n-1)\big) D_\alpha P_\alpha^2$. Since $n_{s_\alpha} \le n - 3 < n - 1$, we force $\boxed{D_\alpha = 0}$ (the subcase $P_\alpha = 0$ is analogous, via x^{-1}).

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- Integrating, $f' = Lh + M$; substituting into the Bianchi ODE and comparing the highest powers of h gives $\boxed{L = 0}$, hence $\lambda \equiv C_0$ is **constant**. Kim's final contradiction ([5], Section 7) excludes ≥ 3 classes. □

Degenerate case and summary of the adaptation

Prop. 3.5: case $\xi_{i_0} \equiv 0$

In Kim, $\xi_{i_0} = 0$ gives $\lambda_1 = 0$ and $\mathrm{Scal}' = 2\lambda_1 f'$ closes the argument. For the almost soliton, $\mathrm{Scal}' = 2\lambda_1 f' + 2(n-1)\lambda'$: it does not close. **Way out:** apply the reduction with the null class as the distinguished one ($c = a = 0$, $b = 1$, $C_0 = 0$); Laurent in the Bianchi ODE forces $D_\alpha = 0$ and λ constant, giving ≤ 2 eigenvalues.

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- **Result:** constant λ **emerges** from the structure. ≥ 3 transversal classes are excluded.
- The three new points vs. Kim: (i) identifying the master identity as the exact breaking point; (ii) importing the contracted Bianchi identity (3rd order); (iii) redoing the degenerate case $a = c = 0$.

Theorem 2: from the bound to the local structure

Structure theorem (Thm. 3.6 of [2])

A nontrivial gradient almost soliton with harmonic Weyl is, near a regular point of f,

$$g = \mathrm{d}s^2 + h_1(s)^2 g_{N_1} + \cdots + h_k(s)^2 g_{N_k}, \qquad k \leq 2,$$

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- The transversal estimate gives ≤ 2 classes tangent to the levels of f.
- The rest of the proof is geometric: integrate these classes, write the metric in adapted coordinates, and tie the almost soliton equation to the Ricci formulas for warped products.

Theorem 2: how the structure appears

- **One transversal class:** the eigendistributions add up to the tangent space of the level, and integrating ξ along E_1 gives

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- In both cases, the Ricci formulas for warped products ([2], Lemma 2.4) show that the fibers of dimension ≥ 2 are Einstein.

Convention: counting fibers

For a multiply warped product

$$M = B \times_{h_1} N_1 \times \cdots \times_{h_k} N_k,$$

we use the conventions of the Remark in [2].

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- Proportional warping functions, $h_i = c\,h_j$ with $c > 0$, are identified after rescaling the fiber metric.
- Therefore we assume the h_i nonconstant and pairwise nonproportional.
- The **number of fibers** is the number of classes for the relation

$$h \sim c\,\tilde{h}, \qquad c > 0.$$

Comparison: vanishing Weyl vs harmonic Weyl

Brozos–García-Río–Vázquez-Lorenzo [24]

For a multiply warped product with one-dimensional base

$$M = I \times_{h_1} N_1 \times \cdots \times_{h_k} N_k,$$

if $\mathscr{W} = 0$ and the fibers have dimension at least 2, then $k \leq 3$ and each fiber N_i has constant sectional curvature.

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- In our theorem, the conformal hypothesis is weaker, $\operatorname{div} \mathscr{W} = 0$, but it is coupled to the almost soliton equation; the result gives $k \leq 2$ and Einstein fibers.
- Thus, $\mathscr{W} = 0$ forces the *fibers to be space forms*; whereas $\operatorname{div} \mathscr{W} = 0$ coupled to the almost soliton imposes *fewer fibers*, but also imposes that the fibers be Einstein manifolds.

Example from the paper: many fibers

- The bound $k \leq 2$ essentially uses $\operatorname{div} \mathscr{W} = 0$. Without this hypothesis, there are multiply warped gradient almost solitons with **any number** of fibers.

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- Fix $k \geq 2$. For $1 \leq i \leq k$, take $(\mathcal{N}_i^{r_i}, g_i)$ Ricci-flat and define

$$n = 1 + \sum_{j=1}^{k} r_j, \qquad C = \sum_{j=1}^{k} r_j j, \qquad L^2 = (n-1) \sum_{j=1}^{k} r_j j^2 - C^2 > 0.$$

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- On $I = (-\varepsilon, \varepsilon)$, with $\varepsilon = \pi/(2L)$, consider

$$g = \mathrm{d}s^2 + \sum_{i=1}^{k} h_i(s)^2 g_i.$$

Example from the paper: the explicit choice

$$
\begin{aligned}
h_i(s) &= \exp\left(\left(i - \frac{C}{n-1}\right) s\right) \cos(Ls)^{-1/(n-1)}, \\
f(s) &= -\log(\cos(Ls)), \\
\lambda(s) &= -\frac{L^2}{(n-1)\cos^2(Ls)}.
\end{aligned}
$$

- Since $s \in (-\pi/(2L), \pi/(2L))$, we have $h_i(s) > 0$.

Example from the paper: verification

- For every i,

$$f'' - \sum_{j=1}^{k} r_j \frac{h_j''}{h_j} = \lambda,$$

$$-\frac{h_i''}{h_i} - (r_i - 1)\left(\frac{h_i'}{h_i}\right)^2 - \frac{h_i'}{h_i}\sum_{j \neq i} r_j \frac{h_j'}{h_j} + f'\frac{h_i'}{h_i} = \lambda.$$

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- Therefore $\left(I \times_{h_1} \mathcal{N}_1 \times \cdots \times_{h_k} \mathcal{N}_k, g, f, \lambda\right)$ is a gradient almost soliton.

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- Indeed, for $1 \leq i, j \leq k$,

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- Hence, in general, these quantities do not coincide; the Cotton tensor does not vanish.

Back to the literature: Catino's result

Definition (Catino) [42]

$(\mathcal{M}^n, g)$, $n \geq 3$, is *generalized quasi-Einstein* (GQE) if there are $f, \mu, \lambda \in \mathscr{C}^\infty(\mathcal{M})$ with

$$\mathrm{Ric} + \nabla^2 f - \mu \, \mathrm{d}f \otimes \mathrm{d}f = \lambda \, g.$$

- It encompasses: **Einstein** (f, λ constant), **Ricci solitons** (λ constant, $\mu = 0$), **gradient almost solitons** ($\mu = 0$) and m**-quasi-Einstein** manifolds ($\mu = 1/m$, λ constant), the definition from the earlier aside.

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- It is *trivial* when f is constant; then g is Einstein.

Catino's theorem [42]

Theorem (Catino, 2012)

Let $(\mathcal{M}^n, g)$, $n \geq 3$, be GQE with **harmonic Weyl curvature** and **radially vanishing Weyl**, $\mathscr{W}(\nabla f, \cdot, \cdot, \cdot) = 0$. Then, around each regular point of f, $\mathcal{M}$ is locally a *warped* product $I \times_h N^{n-1}$ with **Einstein** fiber of dimension $n-1$.

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- A **single** fiber ($k = 1$): more rigid than our $k \leq 2$, but at the cost of the extra radial hypothesis.
- **Radially vanishing Weyl is essential**: in the shrinking soliton $\mathbb{R}^k \times \mathbb{S}^{n-k}$ ($k \geq 2$) with $f = \frac{1}{2}(|x^1|^2 + \cdots + |x^k|^2)$, the metric has harmonic Weyl (product of Einstein manifolds), but $\mathscr{W}(\nabla f, \cdot, \cdot, \cdot) \neq 0$.

Catino: conformally flat corollaries

- **Corollary (LCF).** A *conformally flat* gradient almost Ricci soliton is, around regular points of f, a warped product $I \times_h N^{n-1}$ with fiber of **constant sectional curvature**.

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- **Corollary ($n = 4$).** A 4-dimensional GQE with harmonic Weyl and $\mathscr{W}(\nabla f, \cdot, \cdot, \cdot) = 0$, if nontrivial, is conformally flat (the 3-dimensional Einstein fiber is a space form).

Sketch of Catino's proof: chain of dependencies

1. **Harmonic Weyl $\Leftrightarrow$ vanishing Cotton**: for $n \geq 4$, $\nabla^d \mathscr{W}_{abcd} = -\frac{n-3}{n-2}\, \mathrm{C}_{abc}$, hence $\operatorname{div} \mathscr{W} = 0 \iff \mathrm{C} = 0$.

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2. **Conformal change**: radially vanishing Weyl, $\mathscr{W}(\nabla f, \cdot, \cdot, \cdot) = 0$, and the conformal transformation law of the Cotton tensor,
$$\widetilde{\mathrm{C}}_{abc} = \mathrm{C}_{abc} + \frac{1}{n-2}\,\mathscr{W}_{abcd}\nabla^d f,$$
give $\widetilde{\mathrm{C}} = 0$ for $\widetilde{g} = e^{-\frac{2}{n-2}f} g$; that is, the Schouten tensor $\underset{g}{\mathrm{Sch}}$ is **Codazzi**.

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3. **Two eigenvalues**: the GQE equation forces $\underset{\sim}{\mathrm{Ric}}_{g}$ (hence $\underset{\sim}{\mathrm{Sch}}_{g}$) to have *at most two* eigenvalues, of multiplicities 1 and $n-1$, with ∇f in the distinguished direction.

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give $\widetilde{\mathrm{C}} = 0$ for $\widetilde{g} = e^{-\frac{2}{n-2}f} g$; that is, the Schouten tensor $\underset{\sim}{\mathrm{Sch}}_{g}$ is **Codazzi**.
3. **Two eigenvalues**: the GQE equation forces $\underset{\sim}{\mathrm{Ric}}_{g}$ (hence $\underset{\sim}{\mathrm{Sch}}_{g}$) to have *at most two* eigenvalues, of multiplicities 1 and $n-1$, with ∇f in the distinguished direction.
4. **Codazzi splitting**: a Codazzi tensor with two eigenvalues $\Rightarrow$ (Derdziński [15]; Hiepko–Reckziegel [31]; Besse [32], Prop. 16.11) TM splits into a line V_{σ_1} and an umbilical distribution of codimension 1, V_{σ_2}.

Sketch of Catino's proof: chain of dependencies

1. **Harmonic Weyl $\Leftrightarrow$ vanishing Cotton**: for $n \geq 4$, $\nabla^d \mathscr{W}_{abcd} = -\frac{n-3}{n-2}\,\mathrm{C}_{abc}$, hence $\operatorname{div}\mathscr{W} = 0 \iff \mathrm{C} = 0$.
2. **Conformal change**: radially vanishing Weyl, $\mathscr{W}(\nabla f, \cdot, \cdot, \cdot) = 0$, and the conformal transformation law of the Cotton tensor,
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5. **Codazzi–Mainardi**: the mean curvature of the leaves is constant $\Rightarrow$ V_{σ_1} is geodesic $\Rightarrow$ **warped** product with one fiber; the harmonic Weyl tensor of g makes the fiber **Einstein**.

Our improvement: dropping the vanishing radial Weyl hypothesis [2]

Borges–Horácio–Dos Santos

Every **nontrivial** gradient almost Ricci soliton with $\operatorname{div} \mathscr{W} = 0$ is, around regular points of f, a multiply *warped* product with one-dimensional base and **at most two** fibers, those of dimension ≥ 2 being Einstein. In particular, Ric has at most **three** distinct eigenvalues.

- We **remove** Catino's hypothesis $\mathscr{W}(\nabla f, \cdot, \cdot, \cdot) = 0$; the price is going from $k = 1$ to $k \leq 2$ fibers.

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- The bound $k \leq 2$ is **optimal**: Kim's examples in dimension 4 [26] do realize two fibers.
- Imposing $\mathscr{W}(\nabla f, \cdot, \cdot, \cdot) = 0$ one falls into a single transversal class and recovers exactly Catino's theorem.

Our improvement: chain of dependencies

The radially vanishing Weyl hypothesis served Catino only to reduce to *two* eigenvalues via the conformal metric. We work directly with the Codazzi tensor Sch_g and *prove* the estimate of at most *three* distinct eigenvalues.

1. **Radial lemmas** (Kim 2.1/2.2): $\operatorname{div} \mathscr{W} = 0 \Leftrightarrow$ radial identity for Rm; $E_1 = \nabla f/|\nabla f|$ is an eigenvector of Ric, $|\nabla f|$ const. on Σ_c, there is a coordinate s with $E_1 = \nabla s$ and $\nabla_{E_1} E_1 = 0$.

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3. Define $\xi_i := (\lambda - \lambda_i)/f'$. The ODE $\xi_i' + \xi_i^2 = -\lambda_1/(n-1)$ is **free of** λ; solving it, $\xi_i = \frac{u_2'}{u_2} + \frac{h'}{c_i + h}$.

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4. **Transversal bound** (Props. 3.4/3.5): at most *two* transversal classes. Novelty vs. Kim: λ **cancels** in the identity for f'; combining the differentiated form of the master identity with the **contracted Bianchi identity**, one forces λ to be constant and reduces to the partial fractions argument (Kim, Lemma 6.1).

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5. **Structure** (Thm. 3.6): ≤ 2 classes + Codazzi (totally geodesic leaves in Σ_c, via item 3 of Derdziński) + de Rham $\Rightarrow$ multiply warped with $k \leq 2$; the warped Ricci formula (Dobarro–Ünal) makes the fibers of dimension ≥ 2 Einstein.

Conformally flat corollary: Catino *vs.* us [2]

Borges–Horácio–Dos Santos

A gradient almost Ricci soliton is **conformally flat** *if and only if*, around each regular point of f, it is a warped product $I \times_h N^{n-1}$ with N^{n-1} of constant sectional curvature and with f, λ **explicitly** determined by the warping function h.

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- Catino [42] gave only the implication "LCF $\Rightarrow$ warped"; we add the **converse** and the closed formulas for f and λ.
- Consequence: *every* warped product $I \times_h N^{n-1}$ over a space form can be endowed with a conformally flat almost soliton structure.

Refinement: exactly two eigenvalues [2]

Theorem (Borges–Horácio–Dos Santos)

Let $\mathcal{M}^{n\geq 4}$ be a gradient almost soliton with f nonconstant and $\operatorname{div}\mathscr{W} = 0$, whose Ric has **exactly two** eigenvalues. Then, locally, one of the following cases occurs:

1. $\mathcal{M}$ is a warped product $I \times_h \mathcal{N}^{n-1}$ with $\mathcal{N}$ Einstein (*one* fiber);
2. $\mathcal{M}$ is a product $B^k \times \mathcal{N}^{n-k}$ with B Ricci-flat, $\nabla_B \nabla_B f = \lambda g_B$, $\lambda \neq 0$ *constant*, and $\mathcal{N}$ Einstein with curvature λ.

Refinement: route of the proof [2]

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- **One fiber** $\Rightarrow$ case (1): it is the structure of the Corollary for the one-fiber case, with f, λ determined by h; immediate.
- **Two fibers** $\Rightarrow$ the delicate case: one must **show** that the second warping function h_2 cannot be nonconstant. The whole chain Lemma 5.1 $\rightarrow$ 5.2 $\rightarrow$ Prop. 5.3 (Appendix) serves this purpose.

The equations of the two factors [2]

Let $\mathcal{M} = I \times_{h_1} \mathcal{N}_1^{r_1} \times_{h_2} \mathcal{N}_2^{r_2}$, $r_1 + r_2 = n - 1$, $\mathcal{N}_i$ Einstein (const. μ_i) if $r_i \geq 2$. Set $a = h_1'/h_1$, $b = h_2'/h_2$.

Almost soliton equation and eigenvalues of Ric

$$f'' = r_1(a' + a^2) + r_2(b' + b^2) + \lambda,$$
$$f'a = a' + r_1 a^2 + r_2 ab - \mu_1/h_1^2 + \lambda, \quad f'b = b' + r_2 b^2 + r_1 ab - \mu_2/h_2^2 + \lambda.$$

- **Harmonic Weyl** $\iff a' + a^2 = b' + b^2 \iff h_1''/h_1 = h_2''/h_2$.

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- **Harmonic Weyl** $\iff a' + a^2 = b' + b^2 \iff h_1''/h_1 = h_2''/h_2$.
- **Exactly two eigenvalues:** without loss of generality, $\mathrm{Ric}_{11} = \mathrm{Ric}_{22}$ ($(1 + r_1)$-dim eigenspace).

Lemma 5.1: f determines h_1, h_2 and λ [2]

Lemma 5.1 (case $\mathrm{Ric}_{11} = \mathrm{Ric}_{22}$)

$\exists\, C_1 \neq 0 :\ h_1 = C_1^{-1} f', \quad \lambda = f'' - (n-1)\dfrac{f'''}{f'}, \quad$ and if h_2 is not constant, $\exists\, C_2, C_3 :\ h_2 = -(n-1)C_1^{-1} e^{-\frac{1}{n-1}(f - C_2)} + C_3$.

- $\mathrm{Ric}_{11} = \mathrm{Ric}_{22}$ gives $f'' = af'$, which integrates to $f' = C_1 h_1$. For λ, substitute into the first equation. For h_2, combine $\mathrm{Ric}_{11} = \mathrm{Ric}_{22}$ with the expression for λ, divide by b and integrate.

Lemma 5.2 and Prop. 5.3: the overdetermination collapses [2]

Lemma 5.2

Rewriting all quantities in terms of f and its derivatives, one obtains **two independent ODEs** for f:

$$(n-1)f'''\Lambda = (n-1)f'f'' - (f')^3,$$
$$\left[(n-2)f'f''' - (r_1-1)(f'')^2 + C_1^2\mu_1\right]\Lambda = r_2(f')^2 f'',$$

where $\Lambda = C_1C_3e^{\frac{1}{n-1}(f-C_2)} - (n-1)$.

Proposition 5.3

Every function satisfying *simultaneously* the two ODEs is **constant**. The proof takes up the Appendix: case $r_1 = 1$ (direct) and $r_1 > 1$ (via a polynomial of degree 12 in f').

Two fibers: the rigid product is born [2]

By Prop. 5.3, in the two-fiber branch necessarily $h_2 \equiv C$ is constant. Then $h_1''/h_1 = h_2''/h_2 = 0$, hence $h_1(s) = As + B$.

- The soliton equation gives $\lambda = f''$; the equation of $\mathcal{N}_2$ gives $\lambda = \mu_2/h_2^2 =$ **constant**. Hence $f(s) = \frac{\lambda}{2}s^2 + Ds + E$.

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- $\mathrm{Ric}_{11} = \mathrm{Ric}_{22} = 0$ forces $\mu_1 = (r_1 - 1)A^2$; the block $B^{1+r_1} := I \times_{h_1} \mathcal{N}_1$ is **Ricci-flat**.

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- $\mathrm{Ric}_{11} = \mathrm{Ric}_{22} = 0$ forces $\mu_1 = (r_1 - 1)A^2$; the block $B^{1+r_1} := I \times_{h_1} \mathcal{N}_1$ is **Ricci-flat**.
- Normalizing $C = 1$, $\mathcal{N}_2^{r_2}$ is Einstein with curvature $\lambda \neq 0$. On B, $\nabla_B \nabla_B f = \lambda g_B$: the **rigid product** of item (2). □

Appendix: case $r_1 = 1$ [2]

Proposition 6.2

If $r_1 = 1$, every solution of the two ODEs of Lemma 5.2 is constant.

- $r_1 = 1 \Rightarrow \mu_1 = 0$. With $r_2 = n - 2$, the two ODEs collapse to

$$(n-2)(f')^3 f'' = 0, \qquad 4(n-1)(n-2)(f')^2 f'' = (n-2)(f')^4.$$

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- The first forces $f'f'' = 0$; if $f'' = 0$, the second gives $(f')^4 = 0$. In any case $f' \equiv 0$: contradiction.

Appendix: case $r_1 > 1$ and the polynomial of degree 12 [2]

- Dividing and combining the two ODEs of Lemma 5.2, one eliminates Λ and f''', obtaining three identities involving f', f'', f''' and the parameters n, r_1, μ_1 (Lemma 6.3).

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- Passing to $y = f'$, $u(y) = f''$: one of the identities becomes *quadratic* in u, whose solution is $u(y) = \eta_1 y^2 + \varepsilon\sqrt{\eta_2 y^4 - \eta_3}$.

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- Substituting into the first-order ODE and rationalizing the radical, a polynomial $P(x) = a_{12}x^{12} + a_8 x^8 + a_4 x^4 + a_0$ arises, with

$$\boxed{a_{12} = -\frac{(n-2)^2(4n-3r_1-5)^2}{81\,(n-1)^6(r_1-1)^4} \neq 0} \qquad (r_1 > 1).$$

Appendix: conclusion [2]

Proposition 6.5

If $r_1 > 1$, every solution of the ODEs of Lemma 5.2 is constant.

- $P\big(f'(s)\big) = 0$ for every s, with P **nonconstant** ($a_{12} \neq 0$). Since $s \mapsto f'(s)$ is continuous, f' takes finitely many values, hence is **constant**. The first ODE with $f'' = f''' = 0$ forces $f' = 0$: contradiction.

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Summary (Proposition 5.3)

$r_1 = 1$ (Prop. 6.2) and $r_1 > 1$ (Prop. 6.5) exhaust the cases $\Rightarrow$ Prop. 5.3. Hence the two-fiber branch survives only with h_2 constant, producing the dichotomy (1)/(2) above; the same machinery closes Theorem 3 (Schouten).

Theorem 3: Schouten solitons [2]

Theorem (Borges–Horácio–Dos Santos)

Every **complete gradient Schouten soliton** with harmonic Weyl curvature is **rigid in the sense above**.

- Hypothesis: $\operatorname{div} \mathscr{W} = 0$. Conclusion: global decomposition $\mathcal{N}^{n-k} \times \mathbb{R}^k$ with quadratic potential on the Euclidean factor.

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- Hypothesis: $\operatorname{div} \mathscr{W} = 0$. Conclusion: global decomposition $\mathcal{N}^{n-k} \times \mathbb{R}^k$ with quadratic potential on the Euclidean factor.
- We apply Theorem 2: around the regular points, g is a warped product with one or two Einstein fibers.

Theorem 3: strategy

- In the one-fiber case, the Schouten equation and the harmonic Weyl identity give $h'' = 0$, hence $h(s) = As + B$.

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- Completeness eliminates the local models that are incomplete and closes the global rigidity.

Theorem 3: the ODEs involved (one fiber)

- The Schouten soliton is the particular case

$$\mathrm{Ric} + \nabla^2 f = \left(\frac{\mathrm{Scal}}{2(n-1)} + \tau \right) g.$$

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- In the Schouten case, the radial identity gives $h'' = 0$, so $h = As + B$.

Theorem 3: the ODEs involved (two fibers)

- **Two fibers.** Writing $a_i = h_i'/h_i$, harmonic Weyl is equivalent to

$$a_1' + a_1^2 = a_2' + a_2^2 \quad \Longleftrightarrow \quad h_1''/h_1 = h_2''/h_2.$$

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- In the Schouten case this forces $h_i = A_i s + B_i$: each warping function is **affine**.

Theorem 3: closing the rigidity argument

- If $g = I \times_h N$ and $h = As + B$, then

$$f(s) = -\frac{\mu - (n-2)A^2}{2A^2} \log(As + B) + \frac{\tau}{2} s^2 + c_0 s + c_1.$$

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- With two fibers, subtracting the two fiber equations gives a rational expression for f' in terms of $(A_i s + B_i)^{-1}$. On the other hand, integrating $\lambda = f''$ using the formula for Scal produces, if $A_1 A_2 \neq 0$,

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- The factor with $A_i \neq 0$ is Ricci-flat and, by completeness, its universal cover is Euclidean; the other factor is Einstein. Therefore the soliton is rigid in the sense defined for the Schouten case.

New corollary – radial criterion for almost solitons

Corollary of Theorem 3 [2]

Let $(\mathcal{M}^n, g, f, \lambda)$, $n \geq 4$, be a **complete** and connected gradient almost Ricci soliton with harmonic Weyl curvature. Then:

$$\mathcal{M} \text{ is rigid} \quad \Longleftrightarrow \quad \mathrm{Rm}(\cdot, \nabla f, \nabla f, \cdot) = 0.$$

- Therefore, the formulation analogous to that of Petersen–Wylie also holds:

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- But, under $\operatorname{div} \mathscr{W} = 0$, the hypothesis "Scal constant" is **redundant**: vanishing radial sectional curvature already forces rigidity.

Why vanishing radial sectional curvatures force the Schouten case

Since $\operatorname{div} \mathscr{W} = 0$, the Cotton tensor vanishes and the radial lemma gives, with $q := \frac{\mathrm{Scal}}{2(n-1)} - \lambda$,

$$\mathrm{Rm}(\nabla f, X, Y, Z) = Y(q)g(X, Z) - Z(q)g(X, Y).$$

- At a regular point, $E_1 = \nabla f/|\nabla f|$ and $U \perp E_1$: with $(X, Y, Z) = (E_1, U, E_1)$ one obtains $U(q) = 0$; with $(X, Y, Z) = (U, E_1, U)$ and vanishing radial sectional curvature, $E_1(q) = 0$. Hence $dq = 0$ at the regular points.

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- At the critical points of f, $\nabla f = 0$ and the same identity gives $Y(q) = 0$ directly. Therefore $q \equiv -\tau$ is constant and the almost soliton is a **Schouten soliton**.

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- By Theorem 3, vanishing radial sectional curvature $\Rightarrow$ Schouten $\Rightarrow$ rigidity. Conversely, in the rigid model ∇f is tangent to the Euclidean factor, hence the radial sectional curvatures vanish.

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Important!

The proof uses $\operatorname{div} \mathscr{W} = 0$ in a crucial way. Without harmonic Weyl, this argument does not establish a rigidity criterion for arbitrary almost solitons.

Epilogue: local equivalence [1]

This result is not used in the previous proofs; it organizes, a posteriori, several equivalent ways of recognizing the same local structure.

Notation. $E_1 = \nabla f/|\nabla f|$; D_α are eigendistributions tangent to the regular level Σ_s; ξ_α are their principal curvatures; $k_\alpha = \mathrm{Ric}_{\Sigma_s}(E_\alpha, E_\alpha)$.

Theorem (Borges–Horácio–Dos Santos)

Let $\mathcal{M}^{n\geq 4}$ have harmonic Weyl curvature and f nonconstant. The following are equivalent:

(i) the net of eigendistributions $\langle E_1\rangle, D_1, \ldots, D_p$ is **locally decomposable**;

(ii) $\mathcal{M}$ is locally a multiply *warped* product $I \times_{h_1} \mathcal{N}_1^{r_1} \times \cdots \times_{h_k} \mathcal{N}_k^{r_k}$;

(iii) each foliation tangent to D_α is **spherical** in the sense of Hiepko;

(iv) $k'_\alpha + 2\xi_\alpha k_\alpha = 0$ for every D_α (the prime is the derivative in the direction E_1);

Epilogue: local equivalence (continued)

Theorem (continued)

Under the same hypotheses, the previous conditions are also equivalent to:

(v) every regular level is **isoparametric with at most two** distinct principal curvatures;

(vi) every regular level has parallel intrinsic Ricci tensor: $\nabla^{\Sigma_s} \mathrm{Ric}_{\Sigma_s} = 0$;

(vii) the Ricci tensor of $\mathcal{M}$ has **at most three** distinct eigenvalues.

Extensions

The same theorem holds for **gradient Ricci solitons**, **gradient almost Ricci solitons** and **quasi-Einstein manifolds**: the proof uses only the Codazzi structure of Sch (from $\operatorname{div} \mathscr{W} = 0$) and the fact that ∇f is an eigendirection of Ric, common to the three equations.

Summary and perspectives

- Under $\operatorname{div} \mathscr{W} = 0$, Ricci solitons and almost Ricci solitons have extremely controlled local geometry: Ric has ≤ 3 eigenvalues and the metric is a **warped product** with one-dimensional base.

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- As a corollary, under $\operatorname{div} \mathscr{W} = 0$, **vanishing radial sectional curvature** characterizes the rigidity of complete almost solitons; the constant scalar curvature hypothesis of the classical Petersen–Wylie criterion becomes automatic.

Summary and perspectives

- Under $\operatorname{div} \mathscr{W} = 0$, Ricci solitons and almost Ricci solitons have extremely controlled local geometry: Ric has ≤ 3 eigenvalues and the metric is a **warped product** with one-dimensional base.
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- As a corollary, under $\operatorname{div} \mathscr{W} = 0$, **vanishing radial sectional curvature** characterizes the rigidity of complete almost solitons; the constant scalar curvature hypothesis of the classical Petersen–Wylie criterion becomes automatic.

Open

Global (complete) classification of all complete cases; a proof of the estimate on distinct eigenvalues that does not use Kim's frame and with no additional hypothesis (or some hypothesis weaker than local decomposability).

Thank you

Joint work with **Valter Borges** (UFPA) and **João Paulo dos Santos** (UnB).

Slides available at:

`horatiomarm.com/meus_textos/seminario2705.pdf`

`horatiomarm.com/meus_textos/seminario0306.pdf`

Thank you!

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