Topological classifications revealed by the Ricci flow

Matheus Horácio
May 27, 2026

English translation by Claude Opus 5.5 (Anthropic) of the original Portuguese text by Matheus Horácio, translated from the author's LaTeX source. The Portuguese original is the authoritative version.

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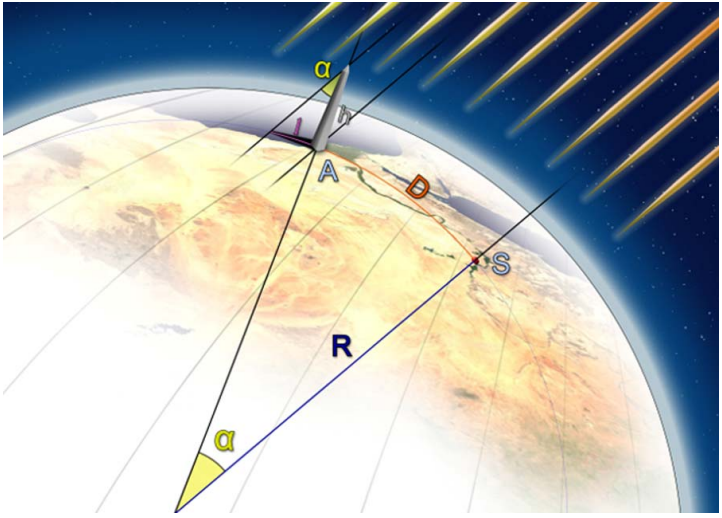
4. References

Geometry and cosmology: the beginning

Eratosthenes: the man who measured the world

- "Geometry" comes from Greek and means to "measure" the Earth. The first person on record to have done this was the great Greek astronomer and mathematician Eratosthenes, more than 2200 years ago.

Eratosthenes: the man who measured the world



The first cosmologists

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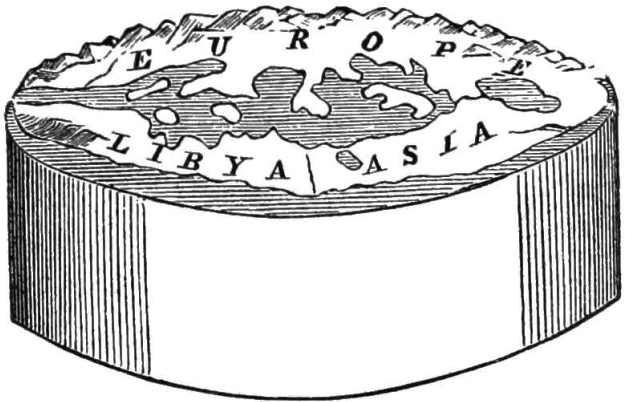


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Question

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- By "closed", we mean *compact and without boundary.*

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- We will see that these two questions are intimately related.

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- Einstein's theory of general relativity models the Cosmos as a four-dimensional pseudo-Riemannian manifold, whose local geometry is described by the following equation:

$$\mathrm{Ric} - \frac{1}{2} \cdot \mathrm{Scal} \cdot g + \Lambda g = \frac{8\pi G}{c^4} T$$

- That being so, we may well ask: given a fibration of the Cosmos by three-dimensional “time slices”, what are all the possible topologies and geometries of such slices?

Important remarks

- In dimension ≤ 3,

$$\text{topological classifications} \iff \text{differentiable classifications}$$

For more details, see [1]. We can then reformulate the previous problems as follows:

Topological classifications

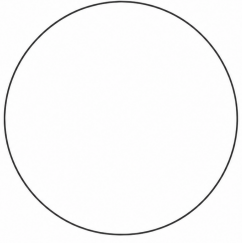
Warm-up in dimension one: curves

Compactness and the presence of boundary are topological invariants that together completely determine all manifolds of dimension 1.

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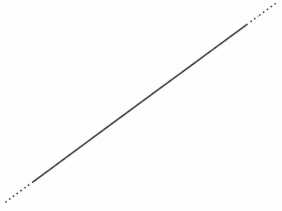
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O círculo $\mathbb{S}^1$.



(Fechado)

A reta $\mathbb{R}$.



(Aberta)

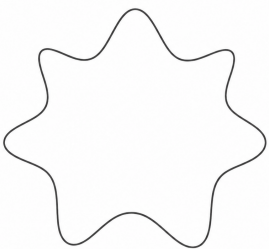
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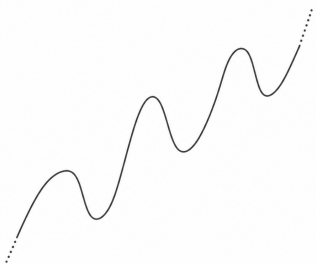
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A very old and natural question

What are all the closed surfaces?

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What are all the closed surfaces?

Answer

There are two topological invariants that completely parametrize the list of all closed surfaces: orientability and genus (or, equivalently, the Euler characteristic).

Dimension two: surfaces

A very old and natural question

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Answer

The building blocks for closed orientable surfaces are ***tori*** and we can put them together by taking ***connected sums***.

- Given two closed surfaces $\mathcal{M}$ and $\mathcal{N}$, there is a way to construct a new surface called the ***connected sum*** of $\mathcal{M}$ and $\mathcal{N}$, which we will denote by $\mathcal{M}\#\mathcal{N}$.

Dimension two: surfaces

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- We choose two balls V_1, V_2, in $\mathcal{M}$ and $\mathcal{N}$ respectively, small enough to be homeomorphic to disks. Next we remove the interior of these two balls, resulting in two boundaries ∂V_1 and ∂V_2 homeomorphic to $\mathbb{S}^1$. By identifying these boundaries, we obtain a new connected surface.

The connected sum of two surfaces

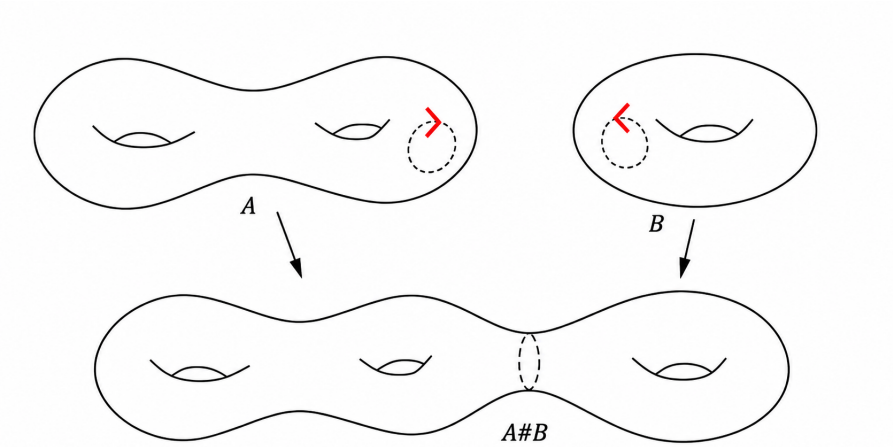


Figure 1: the connected sum of a torus with a double torus

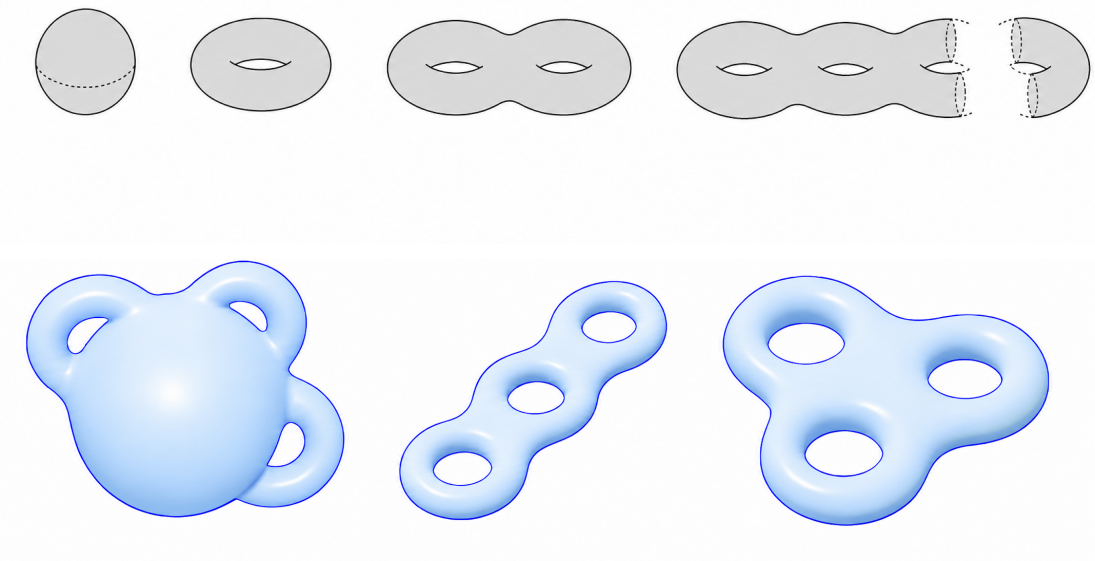


Figure: examples of surfaces of genus ≤ 3.

The connected sum of two surfaces

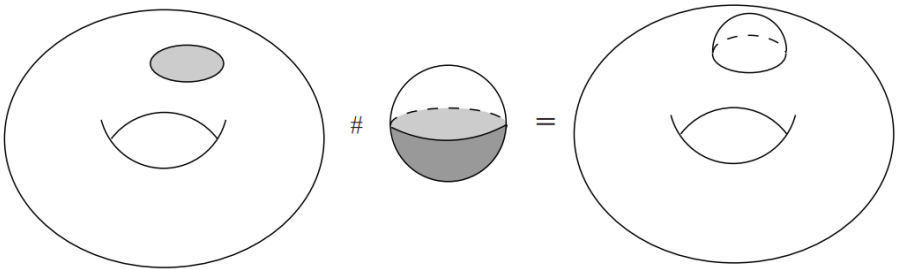


Figure 2: the connected sum with a sphere

Polygonal representations

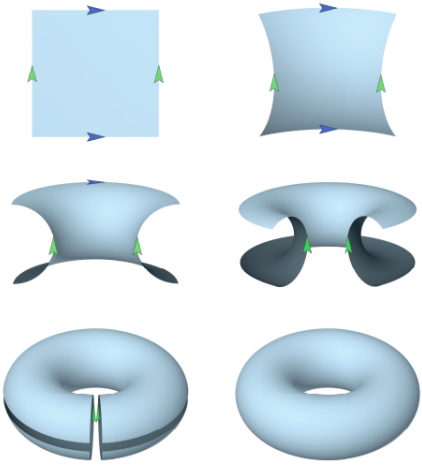
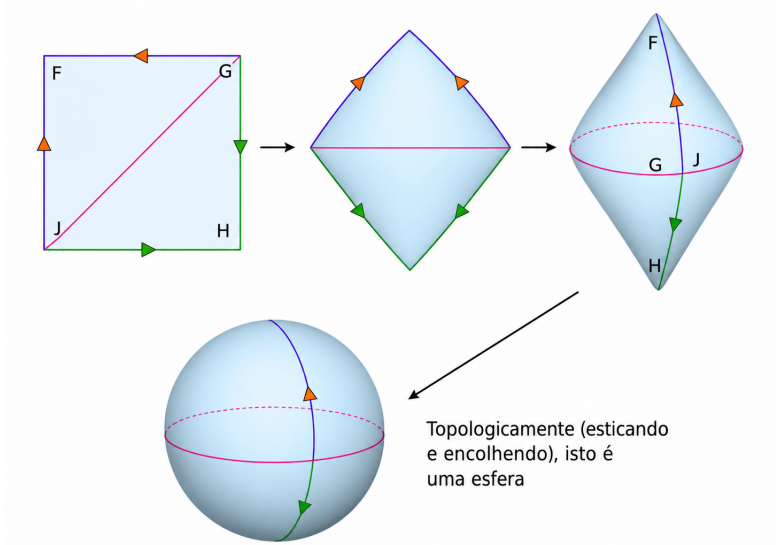


Figure 3: the polygonal representation of a torus

Polygonal representations



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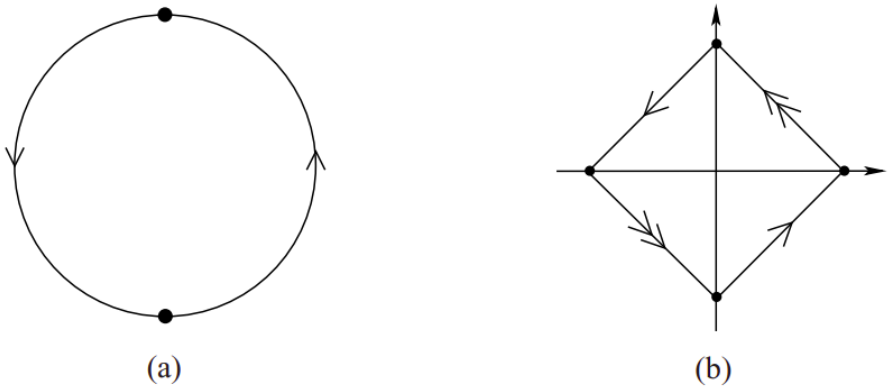


Figure 4: representations of the projective plane $\mathbb{RP}^2$

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Dimension two: surfaces

"The classification of surfaces theorem is a top-class mathematical achievement, comparable with the discovery of America or X-rays. It is a genuine discovery of mathematical natural science, and it is even difficult to say whether the fact itself is more attributable to physics or to mathematics. In its significance both for the applications and for the development of a correct Weltanschauung (worldview), it surpasses by far such "achievements" of mathematics as the proof of Fermat's last theorem or the proof that every sufficiently large integer can be represented as a sum of three prime numbers."

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V. I. Arnold, *On teaching mathematics*

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- In the search for topological invariants in higher dimensions, Poincaré discovered the fundamental group.

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Consequently, every closed surface is diffeomorphic to exactly one quotient of one of the following three model geometries: $\mathbb{S}^2, \mathbb{R}^2$ and $\mathbb{H}^2$. Topology and geometry are related by the Gauss-Bonnet theorem:

$$\int_{\mathcal{M}^2} K \, \mathrm{d}A = 2\pi \cdot \chi(\mathcal{M}^2)$$

Dimension three: closed manifolds

A very old and natural question

What are all the closed three-dimensional manifolds?

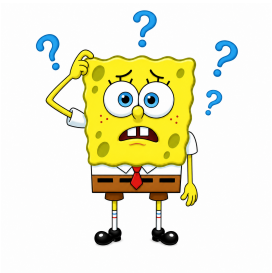
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The zoo

M/Γ

$\mathbb{S}^2 \times \mathbb{S}^1$



$\mathbb{S}^3$

$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$

$\mathbb{RP}^3$

? ? ? ?

$M_1 \# M_2 \# \cdots \# M_k$

$\Sigma_g \times \mathbb{S}^1$

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- One of his main results in this area was the uniformization theorem (which is geometrization for surfaces).
- But (possibly even more importantly) he also introduced concepts such as the **fundamental group** and the **homology groups** (using algebra to distinguish topologies).

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CINQUIÈME COMPLÉMENT À L'ANALYSIS SITUS.

Par M. H. Poincaré, à Paris.

Adunanza del 22 novembre 1903.

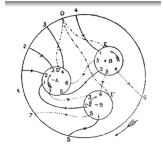


Fig. 4.

The conjecture is ***false***. There is a 3-manifold S with the homology of $\mathbb{S}^3$, but not homeomorphic to $\mathbb{S}^3$ (the first example of a homology sphere).

Il resterait une question à traiter :

Est-il possible que le groupe fondamental de V se réduise à la substitution identique, et que pourtant V ne soit pas simplement connexe?

En d'autres termes, peut-on tracer les cycles K_1'' et K_2'' de telle façon qu'ils ne soient pas bouclés et ne se coupent pas; que les équivalences

$$K_1' \equiv K_2' \equiv 0, \quad K_1'' \equiv K_2'' \equiv 0$$

entraînent les équivalences

$$C_1 \equiv C_2 \equiv C_3 \equiv C_4 \equiv 0$$

et que cependant la surface W ne puisse pas être regardée comme homéomorphe à elle-même de telle façon qu'aux cycles C_1, C_2, C_3, C_4 correspondent les cycles C_1', C_2', C_3', C_4'; que les équivalences

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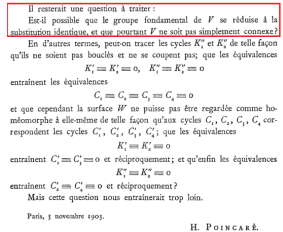
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Poincaré's question (1903).

One question remains to be dealt with:

Is it possible that the fundamental group of V is trivial and yet V is not homeomorphic to $\mathbb{S}^3$?

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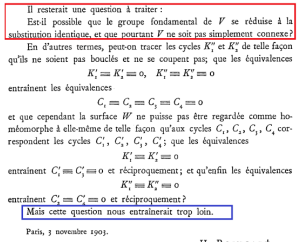
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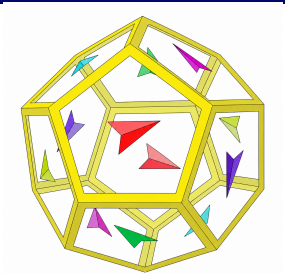
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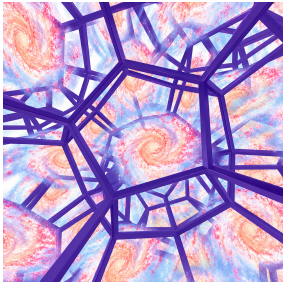
Poincaré's comment.

Mais cette question nous entraînerait trop loin
(But this question would take us too far.)

What is the geometry of the Universe?

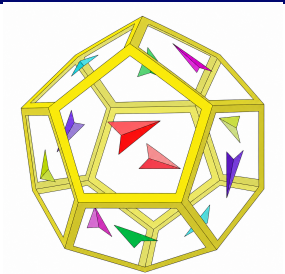


Visualization of the Poincaré dodecahedron: glue opposite faces after a tenth of a turn counterclockwise (rotation by $2\pi/10 = \pi/5$ radians).

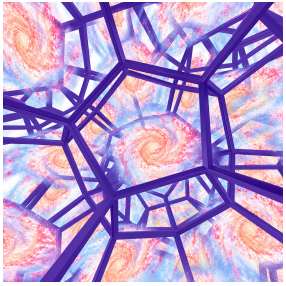


View inside a single-galaxy universe modeled by the Poincaré dodecahedron.

What is the geometry of the Universe?



Visualization of the Poincaré dodecahedron: glue opposite faces after a tenth of a turn counterclockwise (rotation by $2\pi/10 = \pi/5$ radians).

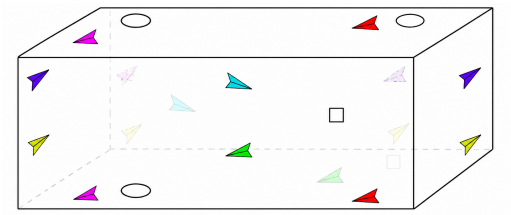


View inside a single-galaxy universe modeled by the Poincaré dodecahedron.

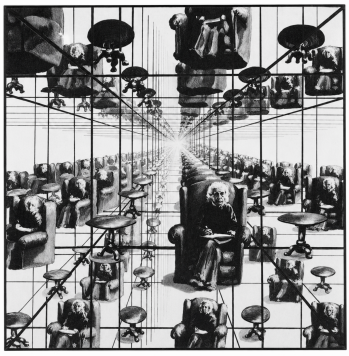
One can show that the Poincaré dodecahedral space is a quotient of the 3-sphere $\mathbb{S}^3$ by a group of isometries. Although far from obvious, it is possible to verify that $\mathbb{S}^3$ is tiled by 120 copies of this dodecahedron!

In [13], the Poincaré dodecahedron was proposed as a possible shape of the Universe.

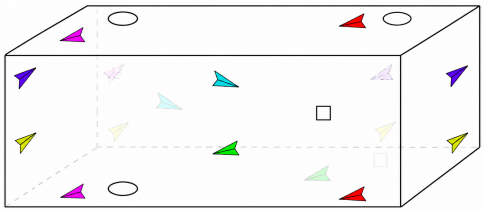
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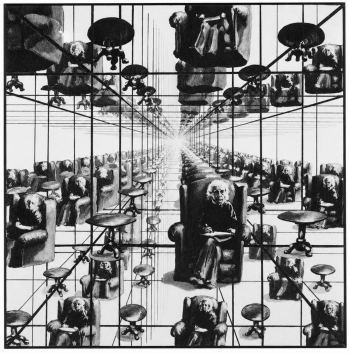
Three-dimensional torus $\mathbb{T}^3$: opposite faces of the cube are identified.



What is the geometry of the Universe?



Three-dimensional torus $\mathbb{T}^3$: opposite faces of the cube are identified.



Experimental cosmological data indicate that the observable Universe is spatially flat. Under the additional hypothesis of compactness, the three-dimensional torus $\mathbb{T}^3$ is one of only 10 compact Euclidean space forms that could model its global shape.

William Thurston.



"And in dimension three? Is it possible to find ***model geometries*** *for all three-dimensional manifolds?"*

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 3, May 1982

THREE DIMENSIONAL MANIFOLDS, KLEINIAN GROUPS AND HYPERBOLIC GEOMETRY

BY WILLIAM P. THURSTON

1. A conjectural picture of 3-manifolds. A major thrust of mathematics in the late 19th century, in which Poincaré had a large role, was the uniformization theory for Riemann surfaces: that every conformal structure on a closed oriented surface is represented by a Riemannian metric of constant curvature. For the typical case of negative Euler characteristic (genus greater than 1) such a metric gives a hyperbolic structure: any small neighborhood in the surface is isometric to a neighborhood in the hyperbolic plane, and the surface itself is the quotient of the hyperbolic plane by a discrete group of motions. The exceptional cases, the sphere and the torus, have spherical and Euclidean structures.

Three-manifolds are greatly more complicated than surfaces, and I think it is fair to say that until recently there was little reason to expect any analogous theory for manifolds of dimension 3 (or more)—except perhaps for the fact that so many 3-manifolds are beautiful. The situation has changed, so that I feel fairly confident in proposing the

1.1. CONJECTURE. *The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures.*



- In dimension three, we still have the three model geometries of constant curvature: $\mathbb{S}^3$, $\mathbb{R}^3$ and $\mathbb{H}^3$. But there is no hope of a classification as nice as in dimension two: in fact, there are infinitely many three-dimensional manifolds that do not admit any metric of constant sectional curvature.

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Faithful global visualizations of these geometries are impossible. A good deal of imagination gets around this problem to some extent, as can be seen in [12].

Geometrization in dimension three: preliminaries

- We say that $\mathcal{M}^3$ is *prime* if, whenever $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$, then $\mathcal{M}_1$ or $\mathcal{M}_2$ is homeomorphic to $\mathbb{S}^3$. The Kneser-Milnor theorem guarantees that every closed orientable 3-manifold is a finite, essentially unique, connected sum of prime manifolds.

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$$\pi_1(\mathbb{T}^2) \hookrightarrow \pi_1(\mathcal{M}^3).$$
- With the sole exception of $\mathbb{RP}^3 \# \mathbb{RP}^3$ (which admits a metric locally isometric to that of $\mathbb{S}^2 \times \mathbb{R}$), no nontrivial connected sum $\mathcal{M}_1 \# \mathcal{M}_2$ admits any geometrization. In fact, there are infinitely many examples of closed orientable 3-manifolds that do not admit a single geometric structure modeled on one of the eight geometries (that is, they only become geometric after the decomposition into connected sums and, in the prime factors, after cutting along incompressible tori).

Thurston's geometrization conjecture

Let $\mathcal{M}^3$ be closed, orientable and prime. Then there exists a finite collection of pairwise disjoint incompressible embedded tori,

$$\mathcal{T} = \{\mathcal{T}_1, \ldots, \mathcal{T}_k\}$$

such that the interior of each component C of $\mathcal{M} \setminus \bigsqcup_i \mathcal{T}_i$ admits a complete metric of finite volume, modeled on one of the eight Thurston geometries (that is, isometric to a quotient of one of the model geometries by a discrete group of isometries acting freely and properly discontinuously).

Geometrization $\Longrightarrow$ Poincaré conjecture

Suppose $\mathcal{M}^3$ is closed and simply connected.

- By Kneser-Milnor, $\mathcal{M} = N_1 \# \cdots \# N_k$ with each N_i prime; and by van Kampen, $\pi_1(N_i) = 1, \ \forall i$.

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- Let us apply geometrization to each N_i: an incompressible torus $T \subset N_i$ would provide an injection $\mathbb{Z}^2 = \pi_1(T) \hookrightarrow \pi_1(N_i) = 1$, which is evidently absurd. Therefore the collection of tori is empty, and N_i itself can be geometrized by one of the eight model geometries.

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- The fact that N_i is simply connected guarantees that it is its own universal cover, which is also compact. Of the eight Thurston geometries, only $\mathbb{S}^3$ is compact. Hence every N_i is $\mathbb{S}^3$, so that

$$\mathcal{M}^3 = \mathbb{S}^3 \# \cdots \# \mathbb{S}^3 = \mathbb{S}^3.$$

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The importance of Thurston's work

"*In my view, before Thurston's work on hyperbolic 3-manifolds and his formulation of the Geometrization Conjecture, there was no consensus among experts as to the validity of the Poincaré conjecture. After Thurston's work (notwithstanding the fact that it had no direct bearing on the Poincaré Conjecture), a consensus developed that both the Poincaré Conjecture and the Geometrization Conjecture were true.*"

The Ricci flow

Richard Hamilton.



"If one stops to think about what we should expect from a flow, we want it to be second order, parabolic and quasilinear, and to diffuse equally in all directions. It turns out that there is only one flow with this property: the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2 \, \mathrm{Ric}_{g(t)}.$$

"

The plan of attack

- Start with an arbitrary closed 3-manifold equipped with any metric. Where the curvature is large, deform the metric so that it decreases, and where it is small, deform it so that it increases. In principle, the best one can hope for is that the deformation leaves the initial manifold with a "uniform" geometry, of constant curvature. But which curvature should we consider?

$$\frac{\partial}{\partial t} g = ?$$

The plan of attack

- In dimension three, Ric determines Rm. It is then natural to consider

$$\frac{\partial}{\partial t} g = c \cdot \mathrm{Ric}$$

for some constant $c \neq 0$.

- Since

$$\begin{aligned}
-2 \cdot \mathrm{Ric}_{jk} &= -2 \left\{ \mathrm{Rm}^{p}_{pjk} \right\} \\
&= -2 \left\{ \partial_p \Gamma^{p}_{jk} - \partial_j \Gamma^{p}_{pk} + \Gamma^{q}_{jk} \Gamma^{p}_{pq} - \Gamma^{q}_{pk} \Gamma^{p}_{jq} \right\} \\
&= -\partial_p \left\{ g^{pq} \left(\partial_j g_{kq} + \partial_k g_{jq} - \partial_q g_{jk} \right) \right\} \\
&+ \partial_j \left\{ g^{pq} \left(\partial_p g_{kq} + \partial_k g_{pq} - \partial_q g_{kp} \right) \right\} + \left\{ -2\Gamma^{q}_{jk} \Gamma^{p}_{pq} + 2\Gamma^{q}_{pk} \Gamma^{p}_{jq} \right\} \\
&= \cdots \\
&= \Delta g_{jk} + Q\left(\partial g, g^{-1} \right)
\end{aligned}$$

  the natural choice is $c = -2$.

The plan of attack

- We will then consider the equation

$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric}_{g(t)}$$

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$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric}_{g(t)}$$

- Hamilton showed the success of the strategy described above in dimension 2. Although his original proof used the uniformization theorem, this dependence was removed by Chen-Lu-Tian in [5], so that the flow can be used to prove the uniformization theorem.

Examples

- Suppose that $(\mathcal{M}, g_0)$ is Einstein, so that $\mathrm{Ric}_{ij}(p,0) = \lambda g_{ij}(p,0)$ for every $p \in \mathcal{M}$, where λ is a constant. It is easy to show that $g(t) = (1 - 2\lambda t)g_0$ is the desired solution of the Ricci flow.

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- Indeed, making the guess that $g_{ij}(x,t) = \rho^2(t) g_{ij}(x,0)$ we see that $\mathrm{Ric}_{ij}(p,t) = \mathrm{Ric}_{ij}(p,0) = \lambda g_{ij}(p,0)$.

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- In this case, the Ricci flow equation then reads:

$$\frac{\partial}{\partial t}\left(\rho^2(t) g_{ij}(p,0)\right) = -2\lambda g_{ij}(x,0)$$

which gives us the ODE

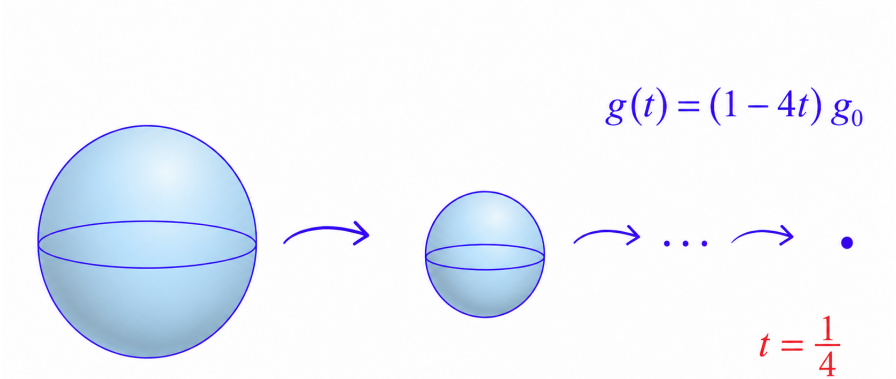
$$\frac{d\rho}{dt} = -\frac{\lambda}{\rho}$$

whose solution is given by

$$\rho^2(t) = 1 - 2\lambda t$$

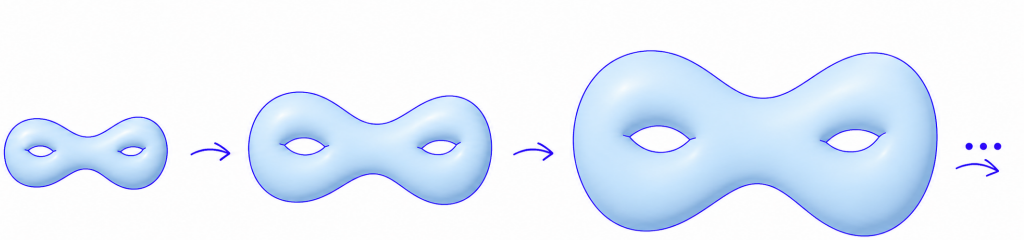
Examples

- In particular, a sphere has an eternal past but a finite future: under the flow, it shrinks to a point, while its curvature "blows up" as it approaches the singularity.



Examples: Einstein manifolds

- On the other hand, a manifold $\mathbb{H}^3/\Gamma$ of constant negative curvature has a finite past and an eternal future: under the flow, it expands indefinitely without ever "blowing up", and its curvature becomes arbitrarily close to zero as $t \to +\infty$.



$$g(t) = (1+4t)\, g_0$$

Hamilton's first results

- **Short-time existence and uniqueness.** If $(\mathcal{M}, g_0)$ is a compact Riemannian manifold, there exist $\varepsilon > 0$ depending only on g_0 and a unique solution $g(t)$ of the Ricci flow defined for $t \in [0, \varepsilon)$ with $g(0) = g_0$.

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- **Characterization of singularity formation by the curvature.** If the solution of the flow exists on a time interval $[0, T)$ but does not extend to any larger interval $[0, T + \delta)$ with $\delta > 0$, then there exists a point $x \in \mathcal{M}$ such that the curvature tensor $\mathrm{Rm}(x, t)$ of the metric $g(t)$ "blows up", i.e.

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- The positivity of the curvature operator $\mathrm{Rm} : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M})$ is preserved by the flow.

Infinite speed of propagation

- The evolution of the scalar curvature along the flow is governed by the following **reaction-diffusion** equation: $\partial_t \mathrm{Scal} = \Delta \mathrm{Scal} + 2|\mathrm{Ric}|^2$.
- If $\mathcal{M}$ is closed, connected, $\mathrm{Scal}_{g_0} \geq 0$ and $\mathrm{Scal}_{g_0} \not\equiv 0$, then $\mathrm{Scal}_{g(t)}(x) > 0 \ \forall x \in \mathcal{M}, t > 0$.
- An initially flat region immediately feels the rest of the manifold: one can imagine that the flow "explores" the whole manifold before starting.



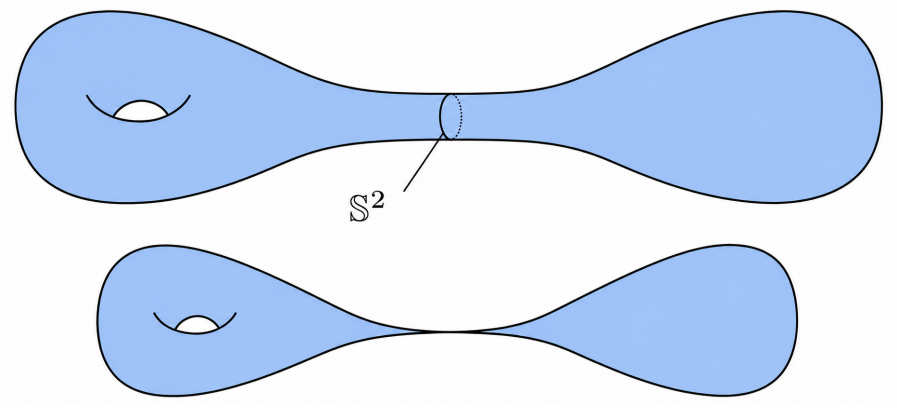
Figure: Flat cylinder with two rounded caps: the central region, initially of zero curvature, acquires positive curvature "instantaneously" under the flow.

Obstacles

- The notion that under the flow the metric should "converge" to a metric of constant curvature needs to be formalized.

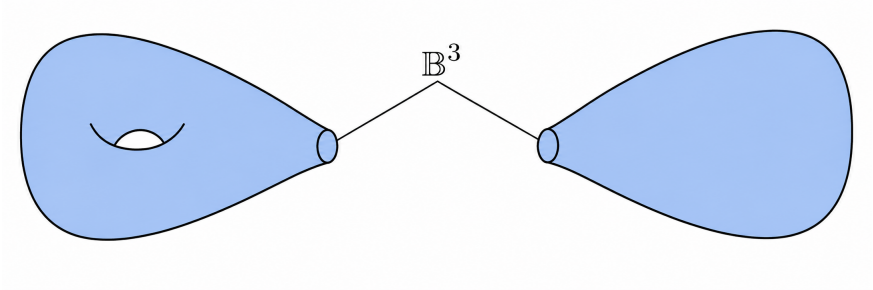
Obstacles

- The notion that under the flow the metric should "converge" to a metric of constant curvature needs to be formalized.
- The flow may encounter singularities of "neck pinch" type



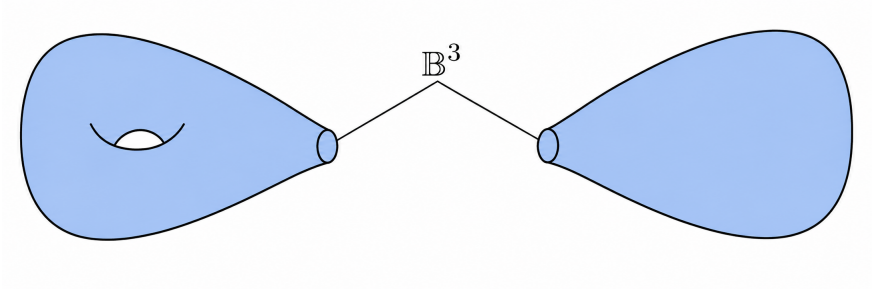
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- Hamilton could not show that such a process does not get stuck in a situation like Zeno's paradox. Perelman introduced notions of surgery that avoided such a situation.

The first significant breakthrough

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A very particular case of the Poincaré conjecture

Let $\mathcal{M}^3$ be a closed differentiable 3-manifold that admits a Riemannian metric of strictly positive Ricci curvature. Then the universal cover of $\mathcal{M}$ is $\mathbb{S}^3$. In particular, if $\mathcal{M}$ is simply connected, then $\mathcal{M}$ is $\mathbb{S}^3$.

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- Before Hamilton's work, it was not known whether 3-manifolds with $\sec > 0$ (resp. 4-manifolds with positive curvature operator) were space forms. To this day, there is no proof of this fact that does not use the Ricci flow!

The first significant breakthrough

- To approach the proof of this result, we will first need some preliminary notions. The idea is to evolve the initial metric so that its Ricci curvature is "pinched" and converges to a metric of constant Ricci curvature.
- We will closely follow (up to only a few slight changes of notation) the approach presented by Peter Topping in [7].

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Uniform curvature "blow-up"

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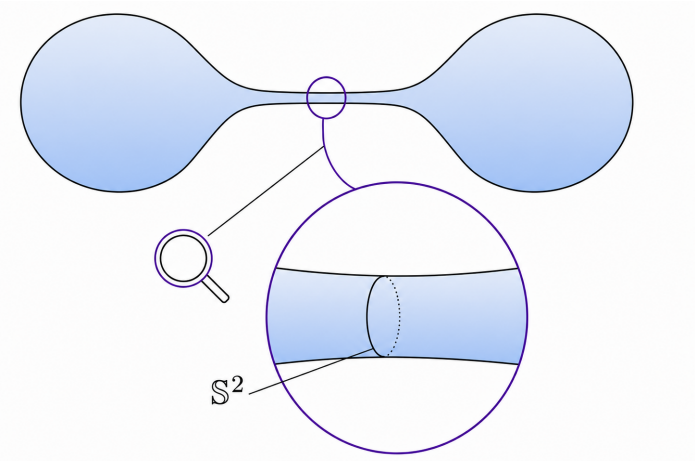
In particular, if the flow is defined on $[0, T)$ and $\mathrm{Scal} \geq \delta > 0$ at the initial time $t = 0$, then $T \leq \frac{n}{2\delta}$.

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We will apply rescalings (or "inflations") to the manifold in order to decrease the curvature. The analysis of what happens in such an inflationary limit will allow us to identify the topology of $\mathcal{M}^3$.

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Convergence of manifolds

- An exhaustion of a manifold $\mathcal{M}$ is a sequence of open sets $(U_k)_{k\in\mathbb{N}}$ such that $\overline{U_k}$ is compact, $\overline{U_k} \subset U_{k+1}$ $\forall k$ and $\bigcup_{k\in\mathbb{N}} U_k = \mathcal{M}$.

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- In particular, if $K \subset \mathcal{M}$ is a compact subset, there exists $n_0 \in \mathbb{N}$ such that $K \subset U_n$ for every $n \geq n_0$. Consequently, if $\mathcal{M}$ is compact then $U_n = \mathcal{M}$ for every sufficiently large n.

Convergence of manifolds

We say that a sequence $(\mathcal{M}_i, g_i, p_i)$ of complete pointed Riemannian manifolds *converges* (smoothly) to the complete pointed Riemannian manifold $(\mathcal{M}, g, p)$ as $i \to \infty$ if there exist

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such that

$$\phi_i^* g_i \to g$$

smoothly as $i \to \infty$, in the sense that for any compact set $K \subset \mathcal{M}$, the tensor $\phi_i^* g_i - g$ and all its covariant derivatives of all orders (with respect to any fixed initial connection) converge uniformly to zero on K.

- In particular, if $\mathcal{M}$ is compact, $\mathcal{M}_i$ is diffeomorphic to $\mathcal{M}$ for every sufficiently large i (a priori we would have that $\mathcal{M}$ is diffeomorphic only to its image under ϕ_i, but since each ϕ_i is a diffeomorphism and $\mathcal{M}$ is trivially simultaneously open and closed in $\mathcal{M}$, their images are simultaneously open and closed in $\mathcal{M}_i$, and therefore in the case where $\mathcal{M}$ is compact, $\mathcal{M}$ is diffeomorphic to $\mathrm{img}(\phi_i) = \mathcal{M}_i$).

Convergence of manifolds

One can prove that two consequences of the convergence $(\mathcal{M}_i, g_i, p_i) \to (\mathcal{M}, g, p)$ are that

- for any $s > 0$ and $k \in \{0\} \cup \mathbb{N}$,

$$\sup_{i \in \mathbb{N}} \sup_{B_{g_i}(p_i, s)} \|\nabla^k \mathrm{Rm}(g_i)\| < \infty$$

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- In fact, the conditions above are sufficient for subconvergence.

"Arzelà-Ascoli" for Cheeger-Gromov convergence

Suppose that $(\mathcal{M}_i, g_i, p_i)$ is a sequence of complete pointed Riemannian manifolds (all of the same dimension n) satisfying the two previous conditions. Then there exists a complete pointed Riemannian manifold $(\mathcal{M}, g, p)$ (also of dimension n) such that after passing to a subsequence in i,

$$(\mathcal{M}_i, g_i, p_i) \to (\mathcal{M}, g, p)$$

The following estimates give us half of what we need to deal with the case of positive initial Ricci curvature:

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BBS estimates

Suppose that $M > 0$ is a constant and that $g(t)$ is a Ricci flow on a closed manifold $\mathcal{M}^n$, where $t \in [0, M^{-1}]$. Then for any $k \in \mathbb{N}$, there exists a constant $C = C(n, k)$ such that if $\|\mathrm{Rm}\| \leq M$ on $\mathcal{M} \times [0, M^{-1}]$, then for any $t \in [0, M^{-1}]$, we have

$$\|\nabla^k \mathrm{Rm}\| \leq \frac{CM}{t^{\frac{k}{2}}}$$

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The estimate on the injectivity radius is also satisfied (see [7]).

Convergence and compactness of flows

- Consider $(\mathcal{M}_i, g_i(t))$ a sequence of smooth families of complete Riemannian manifolds for $t \in (a, b)$, where $-\infty \leq a < 0 < b \leq \infty$, $p_i \in \mathcal{M}_i$ points of $\mathcal{M}_i$ for each i, $(\mathcal{M}, g(t))$ a smooth family of complete Riemannian manifolds and $p \in \mathcal{M}$ a point of $\mathcal{M}$.

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- We say that

$$(\mathcal{M}_i, g_i(t), p_i) \to (\mathcal{M}, g(t), p)$$

as $i \to \infty$ if there exist:

Convergence and compactness of flows

- a sequence of compact sets $\Omega_i \subset \mathcal{M}$ exhausting $\mathcal{M}$ and satisfying $p \in \mathrm{int}(\Omega_i)$ for each i

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such that

$$\phi_i^* g_i(t) \to g(t)$$

as $i \to \infty$ in the sense that $\phi_i^* g(t) - g(t)$ and its derivatives of all orders (where here we mean both the time derivatives and the spatial covariant derivatives with respect to any fixed initial connection) converge uniformly to zero on every compact subset of $\mathcal{M} \times (a, b)$.

Convergence and compactness of flows

Let $\mathcal{M}_i$ be a sequence of manifolds of dimension n, and let $p_i \in \mathcal{M}_i$ be points of $\mathcal{M}_i$ for each i. Suppose that $g_i(t)$ is a sequence of complete Ricci flows on $\mathcal{M}_i$ for $t \in (a,b)$, where $-\infty \le a < 0 < b \le \infty$. Suppose that

- $$\sup_{i\in\mathbb{N}} \sup_{\substack{x\in\mathcal{M}_i \\ t\in(a,b)}} \|\mathrm{Rm}(g_i(t))\|(x) < \infty; \text{ and}$$

Then there exist a manifold $\mathcal{M}$ of dimension n, a complete Ricci flow $g(t)$ on $\mathcal{M}$ for $t \in (a,b)$, and a point $p \in \mathcal{M}$ such that, after passing to a subsequence in i,

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The inflationary limits

- We are interested in applying the theorems seen above to analyze rescalings of Ricci flows near their singularities. Let $(\mathcal{M}, g(t))$ be a solution of the Ricci flow with $\mathcal{M}$ closed on the maximal interval $[0, T)$. We have already seen that

$$\sup_{x \in \mathcal{M}} \|\mathrm{Rm}\|(x,t) \to \infty$$

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  as $t \uparrow T$.

- We take points (p_i, t_i) that maximize $\|\mathrm{Rm}\|$ on the compact set $\mathcal{M} \times \left[0, T - \frac{1}{i}\right]$, that is

$$\|\mathrm{Rm}\|(p_i, t_i) = \sup_{\substack{x \in \mathcal{M} \\ t \in [0, T-1/i]}} \|\mathrm{Rm}\|(x,t)$$

- In particular, $\|\mathrm{Rm}\|(p_i, t_i) \to \infty$ as $i \to \infty$. We define rescaled (and translated) flows $g_i(t)$ by

$$g_i(t) = \|\mathrm{Rm}\|(p_i, t_i) g\left(t_i + \frac{t}{\|\mathrm{Rm}\|(p_i, t_i)}\right)$$

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- These flows are defined on the intervals

$$-t_i \|\mathrm{Rm}\|(p_i, t_i) \leq t \leq (T - t_i)\|\mathrm{Rm}\|(p_i, t_i)$$

  Moreover, for each i, we have $\|\mathrm{Rm}_{g_i(0)}\|(p_i) = 1$.

The inflationary limit of singularities

Suppose that $\mathcal{M}^n$ is a closed manifold, and $g(t)$ is a Ricci flow on a maximal interval $[0,T)$ with $T < \infty$. Then there exist sequences $p_i \in \mathcal{M}$ and $t_i \uparrow T$ satisfying

$$\|\mathrm{Rm}\|(p_i,t_i) = \sup_{\substack{x\in\mathcal{M}\\ t\in[0,t_i]}} \|\mathrm{Rm}\|(x,t) \to \infty$$

such that, defining

$$g_i(t) = \|\mathrm{Rm}\|(p_i,t_i)\, g\left(t_i + \frac{t}{\|\mathrm{Rm}\|(p_i,t_i)}\right)$$

there exist $b = b(n) > 0$, a complete Ricci flow $(\mathcal{N},\hat{g}(t))$ defined for $t \in (-\infty, b)$, and $p_\infty \in \mathcal{N}$ such that $(\mathcal{M}, g_i(t), p_i) \to (\mathcal{N},\hat{g}(t),p_\infty)$ as $i \to \infty$. Moreover, $\|\mathrm{Rm}_{\hat{g}(0)}\|(p_\infty) = 1$, and $\|\mathrm{Rm}_{\hat{g}(t)}\| \le 1$ for every $t \le 0$.

Evolution of the eigenvalues

- In dimension three, let us diagonalize the curvature operator $\mathrm{Rm} : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M})$, that is, let us choose an orthonormal basis $\{\varphi^\alpha\}$ of $\Lambda^2(\mathcal{M})$ such that

$$\left(\mathrm{Rm}\left(\varphi^\alpha, \varphi^\beta\right)\right)_{1\leq\alpha,\beta\leq 3} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

  where, without loss of generality ,

$$\lambda \geq \mu \geq \nu.$$

Evolution of the eigenvalues

- One can show that the evolution of the eigenvalues of $\mathrm{Rm}(t) : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M})$ is determined by the system

$$\begin{aligned}\frac{\partial \lambda}{\partial t} &= \lambda^2 + \mu\nu \\ \frac{\partial \mu}{\partial t} &= \mu^2 + \lambda\nu \\ \frac{\partial \nu}{\partial t} &= \nu^2 + \lambda\mu\end{aligned}$$

Evolution of the eigenvalues

- Moreover, since

$$\frac{\partial}{\partial t}(\lambda-\mu)=(\lambda-\mu)(\lambda+\mu-\nu)$$
$$\frac{\partial}{\partial t}(\mu-\nu)=(\mu-\nu)(-\lambda+\mu+\nu)$$

the inequality $\lambda(0)\geq\mu(0)\geq\nu(0)$ is also preserved, that is, $\lambda(t)\geq\mu(t)\geq\nu(t)$. It is also straightforward to verify that

$$\mathrm{Ric}=\frac{1}{2}\begin{pmatrix}\mu+\nu & 0 & 0\\ 0 & \lambda+\nu & 0\\ 0 & 0 & \lambda+\mu\end{pmatrix}$$

from which we also get

$$\mathrm{Scal}=\mathrm{tr}(\mathrm{Ric})=\lambda+\mu+\nu$$

Control in dimension three

For any constants $0 < \beta < B < \infty$ *such that*

$$\beta g(0) \leq \mathrm{Ric}_{g(0)} \leq B g(0),$$

there exist constants $A = A(\beta, B) > 0$ *and* $\theta \in \left(\frac{1}{2}, 1\right)$ *such that*

$$\lambda - \nu \leq A(\lambda + \mu)^{\theta}$$

along the Ricci flow.

Curvature control

For any constants $0 < \beta < B < \infty$ *such that*

$$\beta g_0 \leq \mathrm{Ric}_{g(0)} \leq B g_0$$

and for any $\varepsilon > 0$*, there exists a constant* $C_\varepsilon = C(\beta, B, \varepsilon)$ *such that*

$$\left\| \mathrm{Ric} - \frac{1}{3}\mathrm{Scal} \cdot g \right\|_{g(t)} \leq \varepsilon \cdot \mathrm{Scal}_{g(t)} + C_\varepsilon$$

The first classification in dimension three

Let $(\mathcal{M}^3, g_0)$ be a Riemannian 3-manifold with positive Ricci curvature. Then the Ricci flow $g(t)$ of $\mathcal{M}$ defined on a maximal interval $[0, T)$ becomes round in the following sense: there exist

- a metric g_∞ on $\mathcal{M}$ of constant positive sectional curvature,

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such that, defining new Ricci flows $g_i(t)$ for $t \leq 0$ by

$$g_i(t) = \|\mathrm{Rm}\|(p_i, t_i) \cdot g\left(t_i + \frac{t}{\|\mathrm{Rm}\|(p_i, t_i)}\right)$$

we have

$$(\mathcal{M}, g_i(t), p_i) \to (\mathcal{M}, (c-t)g_\infty, p_\infty)$$

Proof

- Using the theorems seen above, we obtain a constant $b > 0$, sequences $\{p_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$, $\{t_i \uparrow T\}_{i\in\mathbb{N}}$, rescaled flows $g_i(t)$ and (for $t \in (-\infty, b)$) a Ricci flow limit $(\mathcal{N}, \hat{g}(t))$ with a base point $p_\infty \in \mathcal{N}$. As we have also already seen, for any $t \in [0, T)$ and $\varepsilon > 0$, we have

$$\left\| \mathrm{Ric} - \frac{1}{3}\mathrm{Scal} \cdot g \right\|_{g(t)} \leq \varepsilon \cdot \mathrm{Scal}_{g(t)} + C_\varepsilon$$

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$$\left\| \mathrm{Ric} - \frac{1}{3}\mathrm{Scal} \cdot g \right\|_{g(t)} \leq \varepsilon \cdot \mathrm{Scal}_{g(t)} + C_\varepsilon$$

- Since $g_i(t)$ differs from $g(t)$ only by a time translation and a rescaling, with respect to $g_i(0)$ we have

$$\left\| \mathrm{Ric} - \frac{1}{3}\mathrm{Scal} \cdot g_i(0) \right\|_{g_i(0)} \leq \varepsilon \cdot \mathrm{Scal}_{g_i(0)} + \frac{C_\varepsilon}{\|\mathrm{Rm}\|(p_i, t_i)}$$

Proof

- Letting $i \to \infty$, we obtain

$$\left\| \mathrm{Ric} - \frac{1}{3}\mathrm{Scal} \cdot \hat{g}(0) \right\|_{\hat{g}(0)} \leq \varepsilon \cdot \mathrm{Scal}_{\hat{g}(0)}$$

- Letting $i \to \infty$, we obtain

$$\left\| \mathrm{Ric} - \frac{1}{3}\mathrm{Scal} \cdot \hat{g}(0) \right\|_{\hat{g}(0)} \leq \varepsilon \cdot \mathrm{Scal}_{\hat{g}(0)}$$

- It follows from the arbitrariness of $\varepsilon > 0$ that

$$\mathrm{Ric}_{\hat{g}(0)} - \frac{1}{3}\mathrm{Scal}_{\hat{g}(0)} \cdot \hat{g}(0) \equiv 0$$

and therefore $(\mathcal{N}, \hat{g}(0))$ is an Einstein manifold, so that $\mathrm{Scal}_{\hat{g}(0)}$ is constant.

Proof

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- Since $\mathrm{Scal}_{g(t)} > 0$, it follows that $\mathrm{Scal}_{g_i(t)} > 0$ and therefore $\mathrm{Scal}_{\hat{g}(0)} \geq 0$.

Proof

- Since $\dim(\mathcal{M}) = 3$, it follows that $\hat{g}(0)$ has constant sectional curvature.
- Since $\mathrm{Scal}_{g(t)} > 0$, it follows that $\mathrm{Scal}_{g_i(t)} > 0$ and therefore $\mathrm{Scal}_{\hat{g}(0)} \geq 0$.
- We then see that the constant value taken by the sectional curvature of $\hat{g}(0)$ is nonnegative. But recalling that $\|\mathrm{Rm}(\hat{g}(0))\|(p_\infty) = 1$, we conclude that in fact $\hat{g}(0)$ is a metric of strictly positive constant sectional curvature.

Proof

- By the Bonnet-Myers theorem, it follows that $\mathcal{N}$ is a closed manifold. As we have already seen, $\mathcal{N}$ is then diffeomorphic to $\mathcal{M}$.

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- By the Bonnet-Myers theorem, it follows that $\mathcal{N}$ is a closed manifold. As we have already seen, $\mathcal{N}$ is then diffeomorphic to $\mathcal{M}$.
- By uniqueness of the flow, we see that $\hat{g}(t) = (c-t)g_\infty$, where g_∞ is a positive multiple of $\hat{g}(0)$ and $c > 0$ is a constant. Therefore

$$(\mathcal{M}, g_i(t), p_i) \to (\mathcal{M}, (c-t)g_\infty, p_\infty)$$

as we wanted to show.

Decomposition by the Hodge star

Let $(\mathcal{M}^4, g)$ be an oriented Riemannian manifold of dimension four. Then the bundle $\Lambda^2(\mathcal{M})$ of 2-forms on $\mathcal{M}$ admits the following decomposition into $\star$-invariant eigenbundles:

$$\Lambda^2(\mathcal{M}) = \Lambda^2_+(\mathcal{M}) \oplus \Lambda^2_-(\mathcal{M})$$

where $\Lambda^2_\pm(\mathcal{M}) = \{\omega \in \Lambda^2(\mathcal{M}) \mid \star\,\omega = \pm\omega\}$.

Decomposition of Rm

The block decomposition of the curvature operator corresponding to the decomposition of $\Lambda^2(\mathcal{M})$ via the Hodge star $\star$ is given by

$$\mathrm{Rm} = \left(\begin{array}{c|c} \left(\mathcal{W}^+ + \frac{\mathcal{S}}{12}\right)\Big|_{\Lambda^2_+(\mathcal{M})} & \mathcal{E}|_{\Lambda^2_-(\mathcal{M})} \\ \hline \mathcal{E}|_{\Lambda^2_+(\mathcal{M})} & \left(\mathcal{W}^- + \frac{\mathcal{S}}{12}\right)\Big|_{\Lambda^2_-(\mathcal{M})} \end{array}\right)$$

Dimension four

- After appropriate rescalings, the same strategy used in dimension three can be used to show that a closed 4-manifold $\mathcal{M}^4$ with positive curvature operator admits a metric g_∞ whose curvature operator is given by

$$\mathrm{Rm} = \begin{pmatrix} 2 \cdot \mathrm{Id} & 0 \\ 0 & 2 \cdot \mathrm{Id} \end{pmatrix}$$

  In particular, Rm is a multiple of $g \owedge g$ and therefore $\mathcal{M}^4$ has constant sectional curvature. In even dimension the only such manifolds are spheres and projective spaces.

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  In particular, Rm is a multiple of $g \owedge g$ and therefore $\mathcal{M}^4$ has constant sectional curvature. In even dimension the only such manifolds are spheres and projective spaces.
- Hamilton conjectured that closed manifolds of any dimension with positive curvature operator are space forms. In [6], Böhm and Wilking confirmed this conjecture.

In [2], Huai-Dong Cao proved the following:

A particular case of the Calabi conjecture

Let $\mathcal{M}$ be a compact Kähler manifold with definite first Chern class $c_1(\mathcal{M})$. If $c_1(\mathcal{M}) = 0$, then for any initial Kähler metric g_0, the solution to the normalized Ricci flow exists for all time and converges to a Ricci-flat metric as $t \to \infty$. If $c_1(\mathcal{M}) < 0$ and the initial metric g_0 is chosen as a representative of $-c_1(\mathcal{M})$, then the solution to the normalized Ricci flow exists for all time and converges to an Einstein metric of negative scalar curvature as $t \to \infty$. If $c_1(\mathcal{M}) > 0$ and the initial metric g_0 is chosen as a representative of $c_1(\mathcal{M})$, then the solution to the normalized Ricci flow exists for all time ***(but in general does not converge to an Einstein metric, there are obstructions!)***.

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