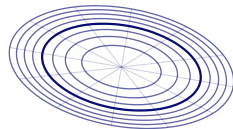
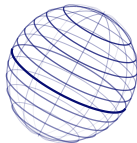
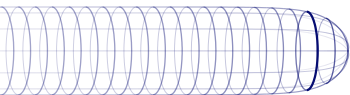


Structure and Rigidity of Gradient Ricci Solitons

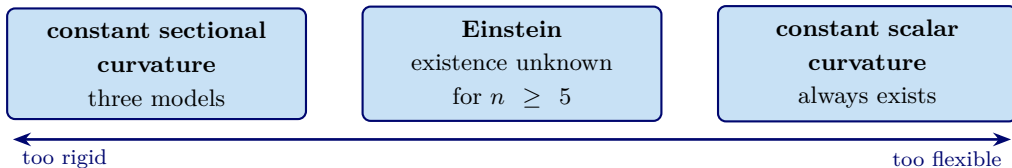


Matheus Horácio

Joint work with Valter Borges (UFPA) and João Paulo dos Santos (UnB),
and with Wenqi Li (East China Normal), Jianyu Ou (Xiamen),
Guoqiang Wu (Zhejiang Sci-Tech) and Detang Zhou (UFF).

The quest for the best metric

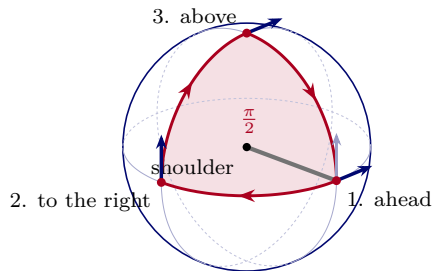
“It is geometers’ dream [...] to find a canonical metric g_{best} on a given smooth manifold V so that all topology of V will be captured by geometry.” Gromov [30]



- On closed manifolds: Yamabe gives constant scalar curvature in every conformal class [47]; for $n \geq 5$, no closed manifold is known to admit no Einstein metric [9].
- Two ways to control the Ricci tensor below: a **parallel structure**, and the **soliton equation**.

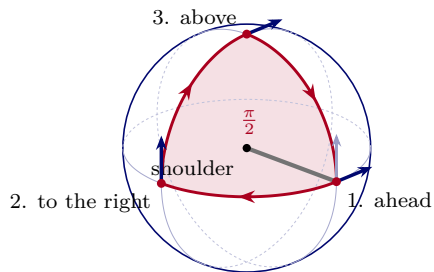
Holonomy you can feel

1. Extend your arm straight ahead and point your thumb upward.



The sphere described by your hand, seen
from the front.

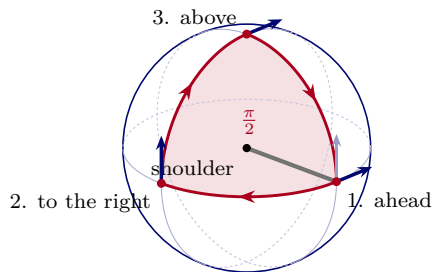
Holonomy you can feel



The sphere described by your hand, seen
from the front.

1. Extend your arm straight ahead and point your thumb upward.
2. Sweep your arm to the right, parallel to your chest; your thumb still points up.

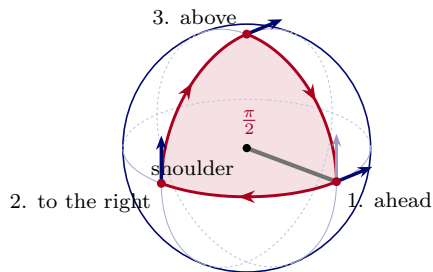
Holonomy you can feel



The sphere described by your hand, seen
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1. Extend your arm straight ahead and point your thumb upward.
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3. Raise your arm so your hand is above your head. Finally, bring it down to the starting position.

Holonomy you can feel

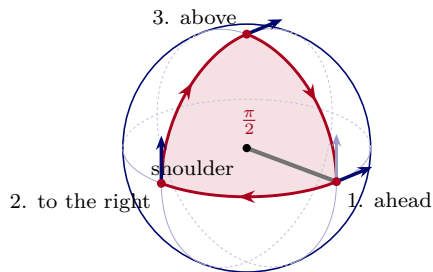


The sphere described by your hand, seen from the front.

1. Extend your arm straight ahead and point your thumb upward.
2. Sweep your arm to the right, parallel to your chest; your thumb still points up.
3. Raise your arm so your hand is above your head. Finally, bring it down to the starting position.

You will notice that your thumb no longer points where it began: it now points left!

Holonomy you can feel



The sphere described by your hand, seen from the front.

1. Extend your arm straight ahead and point your thumb upward.
2. Sweep your arm to the right, parallel to your chest; your thumb still points up.
3. Raise your arm so your hand is above your head. Finally, bring it down to the starting position.

You will notice that your thumb no longer points where it began: it now points left!

Your arm has fixed length and rotates about a single point, your shoulder, so your hand moves on a sphere, along three arcs of great circles.

Berger's holonomy classification

Theorem (Berger [7, 9]). For simply connected, irreducible, non-locally-symmetric (M^n, g) :

Hol(g)	n	the metric is
SO(n)	n	generic
U(m)	$2m \geq 4$	Kähler, not necessarily Einstein
SU(m)	$2m \geq 4$	Kähler, Ricci-flat (Calabi–Yau)
Sp(k)	$4k \geq 8$	hyperkähler, Ricci-flat
Sp(k)Sp(1)	$4k \geq 8$	quaternionic-Kähler: Einstein, $R \neq 0$
G ₂	7	Ricci-flat
Spin(7)	8	Ricci-flat

Except for generic holonomy, only U(m) allows non-Einstein metrics.

What Kähler holonomy gives us

Definition

A **Kähler manifold** is a Riemannian manifold (M^{2m}, g) with a field J of endomorphisms of TM such that $J^2 = -I$, $g(JX, JY) = g(X, Y)$ and $\nabla J = 0$. Then J is integrable: M is a complex manifold.

- A tensor field is parallel if and only if its value at p is invariant under $\text{Hol}_p(g)$. So $\text{Hol}(g) \subset \text{U}(m)$ if and only if g is **Kähler** for some J . Holonomy exactly $\text{U}(m)$ is the case with no further reduction.
- The Kähler form $\omega = g(J\cdot, \cdot)$ and the Ricci form $\rho = \text{Ric}(J\cdot, \cdot)$ are closed, and $[\rho] = 2\pi c_1(M)$.
- Locally, $\omega = i\partial\bar{\partial}\varphi$ and $\rho = -i\partial\bar{\partial}\log \det(g_{j\bar{k}})$: one function gives the metric, another its Ricci form. The Ricci tensor is free, but tied to the metric by a scalar equation.

Calabi–Yau theorem [13, 52]

On a compact Kähler manifold, prescribing a closed real Ricci form in $2\pi c_1(M)$ determines a unique metric in each Kähler class.

$$\omega_\varphi = \omega + i\partial\bar{\partial}\varphi, \quad \omega_\varphi^m = e^F \omega^m.$$

- $c_1 < 0$: Kähler–Einstein (Aubin–Yau [6, 52]).
- $c_1 = 0$: Ricci-flat in every Kähler class (Yau).
- $c_1 > 0$: existence has obstructions; it is equivalent to K-polystability [16, 8].

The point: a geometric tensor equation becomes a scalar equation.

Where solitons come from



Richard Hamilton

Ricci flow [31]: $\partial_t g = -2 \operatorname{Ric}_g$. Self-similar solutions evolve by diffeomorphisms and scaling. In the gradient case:

Gradient Ricci soliton

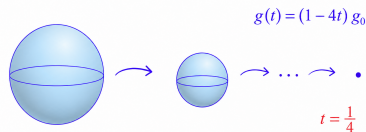
$$\operatorname{Ric} + \nabla^2 f = \lambda g.$$

Shrinking: $\lambda > 0$; steady: $\lambda = 0$; expanding: $\lambda < 0$.

Solitons supply models for high-curvature regions of the flow.

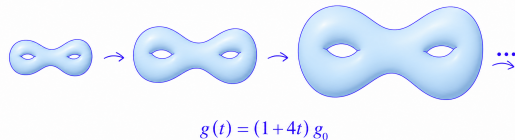
Examples under the flow

Round sphere (shrinking):



shrinks to a point in finite time.

$\mathbb{H}^3/\Gamma$ (expanding):



expands forever.

Einstein: f constant. **Gaussian:** $\mathbb{R}^k$ flat, $f = \frac{\lambda}{2}|x|^2$. **Products:** $N \times \mathbb{R}^k$ with N Einstein,
 $\text{Ric}_N = \lambda g_N$.

The Ricci flow preserves the Kähler condition and becomes a parabolic complex Monge–Ampère equation.

Cao's Kähler–Ricci flow theorem [14]

On a compact Kähler manifold:

- $c_1 = 0$: convergence to the Ricci-flat metric in the initial class;
- $c_1 < 0$: convergence to Kähler–Einstein, starting in the normalized class $-c_1$;
- $c_1 > 0$: global existence of the normalized flow in the class c_1 , but convergence is not automatic.

Cao returns below through his rigidity conjecture.

Definition (Petersen–Wylie [44])

A gradient Ricci soliton is **rigid** if it is a quotient $N \times_{\Gamma} \mathbb{R}^k$ of a product of an Einstein manifold N , $\text{Ric}_N = \lambda g_N$, with the Gaussian soliton $\mathbb{R}^k$, $f = \frac{\lambda}{2}|x|^2$.

Theorem (Petersen–Wylie [44, Theorem 1.2])

A complete gradient Ricci soliton with $\lambda \neq 0$ is rigid if and only if

$$R \text{ is constant} \quad \text{and} \quad \text{Rm}(X, \nabla f) \nabla f = 0 \quad \text{for every } X \quad (\text{radial flatness}).$$

- In dimensions 2 and 3, every complete shrinker is rigid (see [22]). From dimension 4 on, this fails: Koiso [35] and Cao [15] found non-rigid compact Kähler shrinkers, and Feldman, Ilmanen and Knopf [26] non-compact ones.

The Weyl tensor and harmonic Weyl curvature

- For $n \geq 3$, the **Schouten tensor** is $\text{Sch} = \frac{1}{n-2} \left(\text{Ric} - \frac{R}{2(n-1)} g \right)$. Here $\oslash$ is the Kulkarni–Nomizu product; on a space form, $\text{Rm} = \frac{c}{2} g \oslash g$. The normalization makes the Ricci contraction of $g \oslash \text{Sch}$ equal to Ric . The **Weyl tensor** is what remains:

$$\text{Rm} = W + g \oslash \text{Sch}$$

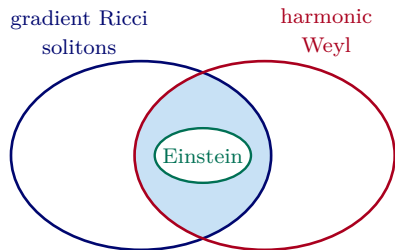
W is totally trace-free; for $n \geq 4$, $W = 0$ if and only if g is locally conformally flat.

- Harmonic Weyl curvature** means $\delta W = 0$. For $n \geq 4$,

$$\delta W = 0 \iff \text{Sch is a Codazzi tensor: } (\nabla_X \text{Sch})(Y, Z) = (\nabla_Y \text{Sch})(X, Z).$$

- A Codazzi tensor controls its eigendistributions (Derdziński [23]): where the multiplicities are constant, they are integrable, with totally umbilical leaves. These are the **eigendistributions of Ric**, and they will carry the geometry.

Cutting the space of metrics from two sides



- Both classes contain Einstein metrics. We study their intersection.
- **First cut:** solitons with $\delta W = 0$. Locally on an open dense set, with at most two factors:

$$ds^2 + h_1(s)^2 g_{N_1} + h_2(s)^2 g_{N_2},$$

with N_1, N_2 Einstein.

- **Second cut:** constant scalar curvature. Cao conjectured that it forces rigidity on complete shrinkers.

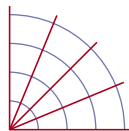
I. Structure under harmonic Weyl curvature

Level sets of a function

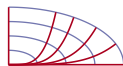
Definition (Wang [49])

A smooth function f on a Riemannian manifold (M, g) is **transnormal** if $|\nabla f|^2 = b(f)$ for a smooth function b , so that $|\nabla f|$ is constant on each level set. It is **isoparametric** if, in addition, $\Delta f = a(f)$ for a continuous function a .

- Where $\nabla f \neq 0$, let $\nu = \nabla f / |\nabla f|$. Then $\nabla_\nu \nu = \nabla^\Sigma |\nabla f| / |\nabla f|$. Locally near a regular point, transnormality is equivalent to the integral curves of ν being unit speed geodesics. The regular levels are locally **equidistant** [10].
- For transnormal f , $\Delta f = \frac{1}{2} b'(f) + \sqrt{b(f)} H$ on the regular levels, H their mean curvature: f is isoparametric exactly when every regular level has **constant mean curvature**.



$f = |x|^2$
transnormal



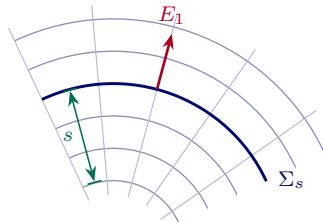
$f = x^2 + 4y^2$
not transnormal

Blue: levels. Red: gradient lines. On the right, $|\nabla f|$ varies along the levels, and the lines $y = Cx^4$ are not geodesics.

On a soliton, everything depends on one variable

Let (M^n, g, f) , $n \geq 4$, be a gradient Ricci soliton with $\delta W = 0$ and f nonconstant. On a regular region with constant Ricci eigenvalue multiplicities:

- $E_1 = \nabla f / |\nabla f|$ is an eigenvector of Ric [27]. We write D_a for the other eigendistributions of Ric, which are tangent to the level sets of f ;
- R , $|\nabla f|$ and the Ricci eigenvalues $\lambda_1, \dots, \lambda_n$ are constant on connected level components [27, 32, 33], so they are functions of the arc length s along E_1 only.



The level sets Σ_s

Since $\nabla^2 f = \lambda g - \text{Ric}$, the level shape operator restricts to $\xi_a I$ on D_a , with $\xi_a = (\lambda - \lambda_a)/f'$, where $f' = |\nabla f|$. The principal curvatures are constant on each level set, so f is **isoparametric**, with levels of constant principal curvatures.

Theorem (Kim [32] for $n = 4$, [33] for $n \geq 5$)

Let (M^n, g, f) , $n \geq 4$, be a gradient Ricci soliton with harmonic Weyl curvature and nonconstant f . Then Ric has **at most three distinct eigenvalues** at each point: one for E_1 , at most two tangent to the levels.

In a multiply warped model $g = ds^2 + \sum_i h_i(s)^2 g_{N_i}$, with $\xi_i = h'_i/h_i$:

- **The soliton equation ties fibers to eigenvalues:** $\lambda_i = \lambda - f'\xi_i$ on N_i . Distinct fibers (warping functions not proportional) are distinct tangent Ricci eigenvalues: the bound says **at most two fibers**.
- **Harmonic Weyl curvature gives one ODE for all of them:**
 $\xi'_i + \xi_i^2 = -\frac{\lambda_1}{n-1} \iff h''_i = -\frac{\lambda_1}{n-1} h_i.$
- With constant R , at most three distinct Ricci eigenvalues force rigidity (Fernández-López-García-Río [28]).

What we wanted

Kim's proof for $n \geq 5$ uses refined adapted frames and long computations with connection coefficients. We wanted a shorter route, without special frames:

adapted coordinates $\longrightarrow$ multiply warped product $\longrightarrow$ a polynomial $\longrightarrow$ the bound.

In coordinates adapted to all the Ricci eigendistributions,

$$g = ds^2 + h_1(s)^2 g_{N_1} + \cdots + h_k(s)^2 g_{N_k}, \quad h'_i/h_i = \xi_i,$$

with *any* number k of fibers. Then count the fibers. This is the approach of the first versions of our papers, October 2025. (**Borges, dos Santos, _____**)

Step 2 works: a polynomial of degree two

$k_a = \text{Ric}_{\Sigma_s}(E_a, E_a)$ is the **intrinsic** Ricci curvature of the level. **Lemma (Borges, dos Santos, _____)**:

$$k'_a + 2\xi_a k_a = P_s(\xi_a), \quad P_s(x) = Bx^2 + (B' + 2\lambda)x + (C - \lambda)B + C'.$$

B, C depend only on s ; P_s cannot vanish identically throughout an open interval.

The cancellation: Gauss gives $k_a = C + B\xi_a - \xi_a^2$; Riccati gives $\xi'_a = -\xi_a^2 + C - \lambda$. Differentiate k_a and add $2\xi_a k_a$: the cubic terms cancel.

Why no interval of zero polynomials: all coefficients zero give $B = B' = 0$ and $\lambda = 0$. Put $H = \sum_a \xi_a$ and recall $B = H - f'$. Riccati and the normal soliton equation give

$$H = f', \quad H' = f'' - \sum_a \xi_a^2.$$

Hence $\sum_a \xi_a^2 = 0$, so every $\xi_a = 0$ and $f' = H = 0$, contradicting the regular region.

Step 2 works: a polynomial of degree two

$k_a = \text{Ric}_{\Sigma_s}(E_a, E_a)$ is the **intrinsic** Ricci curvature of the level. **Lemma (Borges, dos Santos, _____)**:

$$k'_a + 2\xi_a k_a = P_s(\xi_a), \quad P_s(x) = Bx^2 + (B' + 2\lambda)x + (C - \lambda)B + C'.$$

B, C depend only on s ; P_s cannot vanish identically throughout an open interval.

Warped: $k_i = c_i/h_i^2$, c_i constant, and $\xi_i = h'_i/h_i$ give

$$k'_i + 2\xi_i k_i = 0 = P_s(\xi_i).$$

A nonzero quadratic has at most two roots. Since $\lambda_i = \lambda - f'\xi_i$ and $f' \neq 0$, this bounds the tangent Ricci eigenvalues by two.

Why the bound extends: three distinct values at any remaining point would persist on an interval by continuity. Then P_s would have three roots and be the zero polynomial on that whole interval, a contradiction.

Step 1 has a gap

A **net** on M is a splitting $TM = D_1 \oplus \cdots \oplus D_r$ into smooth distributions; a chart is **adapted** to it if each D_i is spanned by coordinate fields, and the net is **locally decomposable** if such charts exist around every point (Reckziegel–Schaaf [46]). Here, $\langle E_1 \rangle$ and the D_a form the **Ricci net**.

What the first versions assumed

Each eigendistribution is integrable, so the Ricci net is locally decomposable.

$$\underbrace{[D_i, D_i] \subset D_i \text{ for each } i}_{\text{integrability: a chart for each } D_i} \quad \text{versus} \quad \underbrace{[D_i, D_j] \subset D_i \oplus D_j \text{ for all } i, j}_{\text{decomposability: one chart for all}}$$

The second condition is strictly stronger, even for mutually orthogonal distributions.

A referee pointed this out, and the referee was right.

We hit a wall

- What we needed: for a harmonic Weyl soliton, is $D_a \oplus D_\alpha$ integrable?
- We asked on MathOverflow [\[11\]](#). Robert Bryant answered that, for conformally flat metrics without a soliton equation, it is false in general: $\delta W = 0$ alone is not enough, and the soliton equation has to enter.
- With at most two tangent eigendistributions there is nothing to prove: the only obstruction is a component of $[X, Y]$, $X \in D_a$, $Y \in D_\alpha$, in a *third* one.

The equivalence theorem: four central conditions

Theorem (Borges, dos Santos, _____ [1], v2)

For a harmonic-Weyl gradient Ricci soliton, $n \geq 4$, on a regular region with constant eigenvalue multiplicities:

- (i) The Ricci net is locally decomposable.
- (ii) $g = ds^2 + \sum_i h_i(s)^2 g_{N_i}$, with Einstein fibers, including possible one-dimensional factors.
- (iv) $k'_a + 2\xi_a k_a = 0$ for each tangent eigendistribution.
- (vii) At most two tangent Ricci eigenvalues; at most three in all.

These four conditions are equivalent.

From adapted coordinates to a multiply warped metric

Adapted chart: $\partial_1 = E_1$, $a, b \in [a]$, $\alpha \notin [a]$, $\alpha \geq 2$; on the regular region, $R = R(s)$, $\xi_i = \xi_i(s)$ and $\xi_a \neq \xi_\alpha$ [1, Lemma 3.2].

Along E_1 : $[\partial_1, \partial_a] = 0$ and $\nabla_{\partial_a} E_1 = \xi_a \partial_a$ give

$$\partial_1 g_{ab} = \langle \nabla_{\partial_a} E_1, \partial_b \rangle + \langle \partial_a, \nabla_{\partial_b} E_1 \rangle = \boxed{2\xi_a g_{ab}}.$$

Why the derivatives can be exchanged: coordinate fields commute and the Levi-Civita connection has no torsion, so $\nabla_{E_1} \partial_a = \nabla_{\partial_a} E_1$.

Integrating in s gives $g_{ab} = h_i(s)^2 \widehat{g}_{ab}(x_2, \dots, x_n)$, where $h'_i/h_i = \xi_i$.

What is still missing

The remaining tensor could depend on the coordinates of *all* fibers. We still need $\partial_\alpha g_{ab} = 0$ for a different block. This is where harmonic Weyl enters.

From adapted coordinates to a multiply warped metric

Adapted chart: $\partial_1 = E_1$, $a, b \in [a]$, $\alpha \notin [a]$, $\alpha \geq 2$; on the regular region, $R = R(s)$, $\xi_i = \xi_i(s)$ and $\xi_a \neq \xi_\alpha$ [1, Lemma 3.2].

Across different blocks: the harmonic Weyl equation on a soliton is

$$\text{Rm}(\nabla f, X, Y, Z) = \frac{Y(R)g(X, Z) - Z(R)g(X, Y)}{2(n-1)}.$$

Since $\partial_a R = \partial_\alpha R = 0$, symmetry and commuting coordinates give, for every z ,

$$0 = \text{Rm}(\partial_a, \partial_\alpha, E_1, \partial_z) = (\xi_a - \xi_\alpha) \langle \nabla_{\partial_\alpha} \partial_a, \partial_z \rangle.$$

Distinct blocks give $\xi_a - \xi_\alpha \neq 0$, so every component vanishes: $\nabla_{\partial_\alpha} \partial_a = 0$ and $\boxed{\partial_\alpha g_{ab} = 0}$.

Only s and the block's own coordinates remain. Together with $\partial_1 g_{ab} = 2\xi_a g_{ab}$, this gives

$$\boxed{g = ds^2 + \sum_i h_i(s)^2 g_{N_i}(x_i)}, \quad \frac{h'_i}{h_i} = \xi_i.$$

The proof of the cycle

- **The first arrow is now proved.** The two coordinate identities give the multiply warped metric; substituting in the soliton equation gives Einstein fibers.
- **The middle arrows are the polynomial calculation.** $k_i = c_i/h_i^2$ gives $k'_i + 2\xi_i k_i = 0 = P_s(\xi_i)$, hence at most two tangent eigenvalues.
- **The bound removes the obstruction.** Each D_a is integrable, $[E_1, D_a] \subset D_a$, and

$$\langle [X, Y], E_1 \rangle = 0 \quad (X \in D_a, Y \in D_\alpha).$$

No third tangent block: $[X, Y] \in D_a \oplus D_\alpha$.

Four more faces of the same theorem

The condition we use next (Borges, dos Santos, _____ [1])

Each regular level has parallel intrinsic Ricci tensor:

$$\nabla^{\Sigma_s} \text{Ric}_{\Sigma_s} = 0.$$

Also equivalent: totally geodesic complements; vanishing mixed sectional curvatures of the levels; isoparametric levels with at most two principal curvatures. **Scope.** With decomposability, the polynomial gives a short proof of Kim's bound. Kim's theorem supplies the unconditional conclusion. This is not a new unconditional proof of his theorem.

Almost Ricci solitons: where the project started

Definition (Pigola–Rigoli–Rimoldi–Setti [45])

A **gradient almost Ricci soliton** satisfies $\text{Ric} + \nabla^2 f = \lambda g$ with $\lambda \in C^\infty(M)$; for instance, gradient Schouten solitons, $\lambda = \frac{R}{2(n-1)} + \tau$.

- **The original program:** Kim's bound for almost solitons with $\delta W = 0$ in every dimension ($n = 4$: Kim [34]). Our first proof used the same coordinates, with the same gap.
- **The repair reverses the order:** first the bound, adapting Kim's refined frames (λ cancels, and the contracted Bianchi identity forces it to be constant where Kim needed it); then the coordinates.
- J. Geom. Anal. (2026) [2]: at most three Ricci eigenvalues; a multiply warped product with at most two fibers; complete Schouten solitons with $\delta W = 0$ are rigid. (Borges, dos Santos, _____)

II. Ricci stationarity and Cao's conjecture

Something we noticed along the way

In the warped model $g = ds^2 + \sum_i h_i(s)^2 g_{N_i}$, the level Σ_s carries

$$g_s = \sum_i h_i(s)^2 g_{N_i},$$

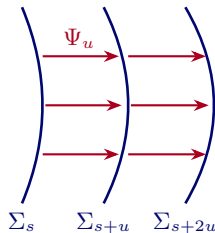
a Riemannian product with each $h_i(s)$ constant on Σ_s .
Constant scaling preserves the Levi-Civita connection, so $\text{Ric}_c g = \text{Ric}_g$ as a covariant tensor. Thus

$$\text{Ric}_{g_s} = \sum_i \text{Ric}_{g_{N_i}} \quad \text{for every } s.$$

With $\Psi_u(s, x) = (s + u, x)$, the flow of $E_1 = \partial_s$:

$$\Psi_u^* \text{Ric}_{\Sigma_{s+u}} = \text{Ric}_{\Sigma_s}$$

Differentiating in u gives back condition (iv): the stationarity of Ric_{Σ_s} is *equivalent* to Kim's bound, and so it always holds.



The next question

A question suggested by the levels

Let Φ_t be the flow of ∇f . What if the Ricci tensor of **the whole manifold** is stationary along it?

$$\Phi_t^* \text{Ric} = \text{Ric} \quad \text{for all } t \quad \Longleftrightarrow \quad \mathcal{L}_{\nabla f} \text{Ric} = 0.$$

- A new question, **not** a consequence: the ambient $\text{Ric}|_{T\Sigma_s} = \sum_a \lambda_a(s) h_a(s)^2 g_{N_a}$ does change with s .
- With $\tau = 1 - 2\lambda t$ and $\sigma = -(2\lambda)^{-1} \log \tau$, the soliton generates the Ricci flow $g(t) = \tau(t) \Phi_{\sigma(t)}^* g$, and

$$\text{Ric}_{g(t)} = \Phi_{\sigma(t)}^* \text{Ric}, \quad \partial_t \text{Ric}_{g(t)} = \frac{1}{\tau(t)} \Phi_{\sigma(t)}^* (\mathcal{L}_{\nabla f} \text{Ric}).$$

- The chain rule gives $\sigma' = 1/\tau$. Thus $\mathcal{L}_{\nabla f} \text{Ric} = 0$ says: the covariant **Ricci tensor is constant in time along the Ricci flow**. Rigid solitons satisfy it. Are they the only ones?

Constant scalar curvature on a soliton

- Two identities hold on every gradient Ricci soliton:

$$\nabla R = 2 \operatorname{Ric}(\nabla f), \quad \Delta_f R = 2\lambda R - 2|\operatorname{Ric}|^2, \quad \Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle.$$

- If R is constant, then $\operatorname{Ric}(\nabla f) = 0$ and $|\operatorname{Ric}|^2 = \lambda R$.
- Compact:** integrate $\Delta f = n\lambda - R$ to get $\Delta f = 0$; a harmonic function on a closed connected manifold is constant. Hence Einstein [25].
- Steady:** $|\operatorname{Ric}|^2 = 0$, so the metric is Ricci-flat.
- What is left is the complete, non-compact case with $\lambda \neq 0$. There, after adding a constant to f , $|\nabla f|^2 = 2\lambda f$ and $\Delta f = n\lambda - R$: f is **isoparametric** [28].
- By Petersen–Wylie, rigid $\iff$ constant R **and** radially flat. The question is therefore whether the first condition implies the second.

Conjecture (Huai-Dong Cao, 2019; recorded in [22])

Let (M^n, g, f) , $n \geq 4$, be a complete gradient shrinking Ricci soliton. If (M, g) has constant scalar curvature, then it is rigid, that is, a finite quotient of

$$N^k \times \mathbb{R}^{n-k}$$

for some Einstein manifold N of positive scalar curvature.

- Raised at the IX Workshop on Differential Geometry, Maceió, 2019.
- By Petersen–Wylie: on a complete shrinker, constant scalar curvature alone should force radial flatness.

What is known

Complete gradient shrinkers with constant scalar curvature R ; status in October 2026, to the best of our knowledge.

every n	$R \in \{0, \lambda, \dots, n\lambda\}$; $R = 0, n\lambda$ Einstein; $R = \lambda$ impossible; $R = (n - 1)\lambda$ rigid; at most three Ricci eigenvalues rigid [28, 44]
$n = 4$	settled [22, 43]
$n = 5$	$R = 3\lambda$ [38], $R = 4\lambda$ [28]; $R = 2\lambda$ under bounded curvature [39]
$n \geq 6$	conditional results for $R = (n - 2)\lambda$ [51, 40]
special classes	$\delta W = 0$ (no assumption on R) [27, 41]; a level with parallel second fundamental form and parallel Ricci tensor [5]
Kähler	every dimension [4, 3]

open	$n \geq 5$, $R \in \{2\lambda, \dots, (n - 2)\lambda\}$, outside these classes; first case: $n = 5$, $R = 2\lambda$, without a curvature bound
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The rigidity criterion

Theorem (_____, Li, Ou, Wu, Zhou [3])

Let (M^n, g, f) be a complete connected gradient Ricci soliton with $\lambda \neq 0$. If

$$\mathcal{L}_{\nabla f} \text{Ric} \geq 0 \text{ everywhere} \quad \text{or} \quad \mathcal{L}_{\nabla f} \text{Ric} \leq 0 \text{ everywhere,}$$

then $\mathcal{L}_{\nabla f} \text{Ric} = 0$ and the soliton is rigid.

No assumption on R : its constancy is a conclusion. Hamilton's evolution equation:

$\partial_t \text{Ric} = \Delta_L \text{Ric}$. On a soliton,

$$\mathcal{L}_{\nabla f} \text{Ric} = \Delta_L \text{Ric}.$$

Flow interpretation: a Ricci tensor monotone in time must be stationary, and this forces rigidity.

The proof, step 1: R is constant along the flow lines

$T = \mathcal{L}_{\nabla f} \text{Ric}$; $\gamma' = \nabla f$ on all of $\mathbb{R}$ (Zhang [53]).

$$T = \nabla_{\nabla f} \text{Ric} + 2\lambda \text{Ric} - 2 \text{Ric}^2, \quad (R \circ \gamma)'' = 2T(\nabla f, \nabla f).$$

If $T \leq 0$: $R \circ \gamma$ is concave and bounded below [53].

Why this forces constancy

A nonzero derivative of a concave function on $\mathbb{R}$ forces it to tend to $-\infty$ in one time direction. The lower bound rules this out.

Thus $\langle \nabla f, \nabla R \rangle = 0$ in this case.

The proof, step 1: R is constant along the flow lines

$T = \mathcal{L}_{\nabla f} \text{Ric}$; $\gamma' = \nabla f$ on all of $\mathbb{R}$ (Zhang [53]).

$$T = \nabla_{\nabla f} \text{Ric} + 2\lambda \text{Ric} - 2 \text{Ric}^2, \quad (R \circ \gamma)'' = 2T(\nabla f, \nabla f).$$

If $T \geq 0$: transport a unit vector U parallel along γ and put

$$a = \text{Ric}(U, U), \quad \beta = \max\{0, \lambda\}.$$

$$a' = T(U, U) + 2|\text{Ric}(U)|^2 - 2\lambda a \geq 2a(a - \lambda).$$

Why the inequality holds: $T(U, U) \geq 0$ and Cauchy–Schwarz gives $|\text{Ric}(U)|^2 \geq a^2$.

If $b = a - \beta > 0$, then $a(a - \lambda) \geq b^2$ for either sign of λ ; hence $b' \geq 2b^2 > 0$. Once positive, b stays positive.

The proof, step 1: R is constant along the flow lines

$T = \mathcal{L}_{\nabla f} \text{Ric}$; $\gamma' = \nabla f$ on all of $\mathbb{R}$ (Zhang [53]).

$$T = \nabla_{\nabla f} \text{Ric} + 2\lambda \text{Ric} - 2 \text{Ric}^2, \quad (R \circ \gamma)'' = 2T(\nabla f, \nabla f).$$

Rule out $a(t_0) > \beta$: set $b = a - \beta$, $b_0 = b(t_0) > 0$ and $t_* = t_0 + 1/(2b_0)$.

$$(1/b)' \leq -2 \implies b(t) \geq \frac{b_0}{1 - 2b_0(t - t_0)} \longrightarrow +\infty \quad (t \uparrow t_*).$$

Why blow-up is impossible: γ exists on all of $\mathbb{R}$; Ric and the parallel field U are smooth. Thus $a(t)$ is continuous and bounded on the compact interval $[t_0, t_*]$.

So $a(t_0) \leq \beta$. Since t_0 and the unit vector are arbitrary, $\text{Ric} \leq \beta g$ and $R \leq n\beta$.

Now $R \circ \gamma$ is convex and bounded above on $\mathbb{R}$, hence constant:

$$\boxed{\langle \nabla f, \nabla R \rangle = 0}.$$

The proof, step 2: R is constant, and $T = 0$

Lemma (_____, Li, Ou, Wu, Zhou [3]). On a complete connected soliton with $\lambda \neq 0$, $\langle \nabla f, \nabla R \rangle = 0$ forces R constant.

Reach a critical point. Along $\gamma' = \nabla f$, the soliton equation gives

$$\frac{d}{dt} |\nabla f|^2 = 2\lambda |\nabla f|^2 - \langle \nabla f, \nabla R \rangle = 2\lambda |\nabla f|^2, \quad |\nabla f|(\gamma(t)) = e^{\lambda t} |\nabla f|(p).$$

For $\lambda > 0$, integrate the speed as $t \rightarrow -\infty$; for $\lambda < 0$, as $t \rightarrow +\infty$.

Why a limit exists

The remaining tail length tends to zero, so the orbit is Cauchy. Metric completeness gives a limit $q \in M$; the speed tends to zero, so continuity gives $\nabla f(q) = 0$.

At q , $\nabla R = 2 \operatorname{Ric}(\nabla f) = 0$ and $R(p) = R(q)$: **every value of R is critical.**

The proof, step 2: R is constant, and $T = 0$

Lemma (_____, Li, Ou, Wu, Zhou [3]). On a complete connected soliton with $\lambda \neq 0$, $\langle \nabla f, \nabla R \rangle = 0$ forces R constant.

Use Sard and connectedness. Every value is critical, so Sard's theorem gives measure zero for $R(M)$.

Why measure zero is enough here

M is connected and R is continuous, so $R(M)$ is an interval. An interval containing two distinct values has positive length. Therefore $R(M)$ is a single point.

Then $T = 0$. Taking the trace of the identity for T gives

$$\mathrm{tr}_g T = \langle \nabla f, \nabla R \rangle + 2\lambda R - 2|\mathrm{Ric}|^2 = \Delta R = 0.$$

A semidefinite tensor has eigenvalues of one sign. Their sum is zero, so all vanish: $\boxed{T = 0}$.

The proof, step 3: the spectrum of the Ricci tensor

With $T = 0$ and R constant, write $A = g^{-1} \text{Ric}$ and $D = \nabla_{\nabla f}$:

$$DA = 2A(A - \lambda I), \quad \text{tr } A^2 = \lambda R \text{ is constant.}$$

Differentiate the moments. The product rule and cyclicity of trace give

$$D \text{tr } A^j = j \text{tr}(A^{j-1} DA).$$

$$0 = D \text{tr } A^2 = 4(\text{tr } A^3 - \lambda \text{tr } A^2),$$

$$0 = D \text{tr } A^3 = 6(\text{tr } A^4 - \lambda \text{tr } A^3).$$

Why the second derivative is zero: the first line makes $\text{tr } A^3 = \lambda \text{tr } A^2$ constant. No choice of smoothly varying eigenvectors is needed.

The proof, step 3: the spectrum of the Ricci tensor

With $T = 0$ and R constant, write $A = g^{-1} \text{Ric}$ and $D = \nabla \nabla f$:

$$DA = 2A(A - \lambda I), \quad \text{tr } A^2 = \lambda R \text{ is constant.}$$

Combine the two moment identities:

$$|A(A - \lambda I)|^2 = \text{tr } A^4 - 2\lambda \text{tr } A^3 + \lambda^2 \text{tr } A^2 = 0.$$

Why this determines every eigenvalue: A is symmetric, so the left side is $\sum_i a_i^2(a_i - \lambda)^2$. Each term is nonnegative; their sum is zero, so $a_i \in \{0, \lambda\}$.

Thus $0 \leq \text{Ric} \leq \lambda g$ for $\lambda > 0$, and $\lambda g \leq \text{Ric} \leq 0$ for $\lambda < 0$.

With R constant, Petersen–Wylie’s Proposition 1.3(2) [44] gives radial flatness; Theorem 1.2 gives **rigidity**. This final step is a geometric theorem.

Proposition (_____, Li, Ou, Wu, Zhou [3]; for constant R , _____ [4])

On any gradient Kähler–Ricci soliton,

$$(\mathcal{L}_{\nabla f} \text{Ric})(X, Y) = \frac{1}{2}(\nabla^2 R(X, Y) + \nabla^2 R(JX, JY))$$

- Cartan gives $\mathcal{L}_{\nabla f} \rho = d(\iota_{\nabla f} \rho) = \frac{1}{2} dd^c R$, where $\rho = \text{Ric}(J\cdot, \cdot)$ and $d^c R = -dR \circ J$.
- **Why this gives a Ricci identity:** $\mathcal{L}_{\nabla f} J = 0$, so differentiating $\rho = \text{Ric}(J\cdot, \cdot)$ introduces no extra J term.
- If R is constant, then $\mathcal{L}_{\nabla f} \text{Ric} = 0$, and the criterion applies. The Kähler structure supplies exactly the stationarity the criterion needs.

Theorem (_____ [4]; in the form below, _____, Li, Ou, Wu, Zhou [3])

Let (M^{2m}, g, f, J) be a complete connected non-steady gradient Kähler–Ricci soliton, $\lambda \neq 0$, with **constant scalar curvature**. Then $R = 2k\lambda$ for some $k \in \{0, \dots, m\}$, and the universal cover is, holomorphically and isometrically, $N^{2k} \times \mathbb{C}^{m-k}$, with N Kähler–Einstein, $\text{Ric}_{g_N} = \lambda g_N$. If $\lambda > 0$, the quotient is trivial: $M \cong N^{2k} \times \mathbb{C}^{m-k}$.

- Cao's conjecture for Kähler–Ricci shrinkers, in every dimension; and the analogue for expanders.

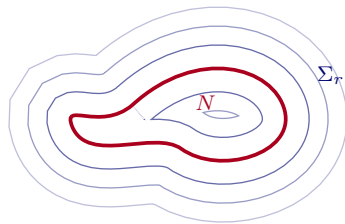
III. Minimum-set geometry and mean curvature flow

The minimum set of the potential

On a complete non-compact shrinker with constant $R = k\lambda$, normalized so that $|\nabla f|^2 = 2\lambda f$, we have $\Delta f = q\lambda$, $q = n - k$: f is **isoparametric**, and $N = f^{-1}(0)$ is its focal variety, smooth [49] and **minimal** [29].

Theorem (Cheng–Zhou [22, Theorem 2.2])

- N is compact and connected, and $f = \frac{\lambda}{2} d(\cdot, N)^2$;
 - on N , $\nabla^2 f$ is 0 on TN and λI on the normal spaces, so $R = \lambda \dim N$;
 - the regular levels are the tubes Σ_r around N .
-
- **Minimal is not totally geodesic.** On the rigid model $N \times \mathbb{R}^q$, the minimum set $N \times \{0\}$ is totally geodesic, and the levels are $N \times \mathbb{S}^{q-1}(r)$. Is the converse true?



The tubes around N (schematic).

The Euclidean model: Segre

- **Segre [48]:** Euclidean isoparametric hypersurfaces are open subsets of hyperplanes, spheres or tubes around affine subspaces.
- In particular, if on a connected open subset of $\mathbb{R}^n$

$$|\nabla u|^2 = 4u \quad \text{and} \quad \Delta u = 2q,$$

then $u = d(\cdot, L)^2$ for an affine subspace L of codimension q . Bochner gives $|\nabla^2 u|^2 = 4q$. Flatness gives $\Delta \nabla^2 u = \nabla^2 \Delta u = 0$, so

$$0 = \frac{1}{2} \Delta |\nabla^2 u|^2 = |\nabla^3 u|^2 + \langle \nabla^2 u, \Delta \nabla^2 u \rangle = |\nabla^3 u|^2.$$

Why quadratic: all third derivatives vanish, so the Hessian is constant on the connected domain. The focal set L is **affine, hence totally geodesic**.

- On our shrinker, $u = 2f/\lambda$ satisfies the **same two equations**, and $|\nabla^2 u|^2 = 4q$ still holds. The second step fails: on a curved manifold, $\Delta \nabla^2 u$ has curvature terms.

Why the space-form argument stops

Cartan's comparison [12]: for the shape operator $S = \nabla \nu$ of parallel hypersurfaces,

$$\text{space form: } S' = -S^2 - cI,$$

$$\text{our tubes: } S' = -S^2 - \text{Rm}(\cdot, \nu)\nu.$$

In a space form, differentiating the mean curvature determines successive power sums of S .
On a constant-scalar-curvature shrinker, we know

$$\text{Ric}(\nu, \nu) = 0, \quad \text{tr } S = \frac{q-1}{r}, \quad \text{tr } S^2 = \frac{q-1}{r^2}.$$

$$(\text{tr } S^2)' = -2 \text{tr } S^3 - 2 \text{tr } (S \text{Rm}(\cdot, \nu)\nu).$$

Why the argument stops: trace zero of the radial curvature operator does not imply that its product with S has trace zero.

Rigidity is decided at the minimum set

Theorem (_____, in preparation)

Let (M^n, g, f) be a complete gradient shrinking Ricci soliton with constant scalar curvature $R = k\lambda$ and nonconstant potential, and let N be the minimum set of f . The following are equivalent:

- (i) M is rigid;
- (ii) N is totally geodesic;
- (iii) $\nabla \text{Ric} = 0$ at every point of N .

- In particular, Cao's conjecture is equivalent to the statement that the minimum set is always totally geodesic.
- If N is a component of the fixed point set of an isometry, or of the zero set of a Killing field, then M is rigid. On a Kähler shrinker, $J\nabla f$ is such a Killing field: one more proof of the Kähler case.

The same picture for self-shrinkers

Mean curvature flow: $\partial_t x = \mathbf{H}$, the negative gradient flow of area. A self-shrinker satisfies $\mathbf{H} = -x^\perp$; $\sqrt{-2t}x$ evolves by MCF up to reparametrization.

$$f = \frac{1}{2}|x|^2, \quad K(X, Y) = \langle A(X, Y), \mathbf{H} \rangle, \quad \boxed{\nabla^2 f + K = g.}$$

K and $\text{tr } K = |\mathbf{H}|^2$ play the roles of Ric and R .

Theorem (_____, in preparation)

For a complete connected Euclidean self-shrinker, these are equivalent:

- $\nabla^\perp \mathbf{H} = 0$;
- $|\mathbf{H}|^2$ is constant and the minimum set of f is totally geodesic in M ;
- $\mathcal{L}_{\nabla f} K \geq 0$ everywhere; equivalently, $\mathcal{L}_{\nabla f} K \leq 0$ everywhere.

Up to an ambient orthogonal transformation, $x(p, y) = (F(p), y)$ on $N^k \times \mathbb{R}^{n-k}$, with F complete minimal in $\mathbb{S}^{k+p-1}(\sqrt{k})$, $k > 0$. For $k = 0$: a plane through the origin.

The same picture for self-shrinkers: three curvature quantities

For a Euclidean self-shrinker, the Gauss equation gives

$$R_M = |\mathbf{H}|^2 - |A|^2.$$

- Any two of these quantities being constant force the third to be constant.
- Constant $|\mathbf{H}|^2$ alone does not give this conclusion by algebra.
- Our comparison with Cao uses $|\mathbf{H}|^2$, the trace of K , in the role of scalar curvature.

The constant- $|A|^2$ classification is a separate question; references are in the backup slides.

A counterpart of Cao's conjecture

Conjecture

For a complete connected Euclidean self-shrinker with $|\mathbf{H}|^2$ constant, the minimum set of $f = \frac{1}{2}|x|^2$ is **totally geodesic in M** .

Ricci shrinker	Euclidean self-shrinker
$\text{Ric} + \nabla^2 f = \lambda g$	$K + \nabla^2 f = g$
$R = k\lambda$ constant	$ \mathbf{H} ^2 = k$ constant
minimum set has dimension k	minimum set has dimension k
rigid $\iff N$ totally geodesic	split $\iff N$ totally geodesic

The minimum-set equivalences are results **in preparation**.

The common question: must the minimum set be totally geodesic?

Thank you

Harmonic Weyl solitons

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Rigidity and Cao's conjecture

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Thank you!

References

- [1] **Borges, V.; Horácio, M. A. R. M.; dos Santos, J. P.** *The Ricci tensor of a gradient Ricci soliton with harmonic Weyl tensor.* arXiv:2510.11939; v1, 13 October 2025; v2, with the equivalence theorem, 6 October 2026.
- [2] **Borges, V.; Horácio, M. A. R. M.; dos Santos, J. P.** *Local structure of gradient almost Ricci solitons with harmonic Weyl tensor.* J. Geom. Anal. **36** (2026), Paper No. 256. arXiv:2510.13057.
- [3] **Horácio, M.; Li, W.; Ou, J.; Wu, G.; Zhou, D.** *Rigidity of Kähler–Ricci solitons with constant scalar curvature.* arXiv:2610.11935.
- [4] **Horácio, M. A. R. M.** *Rigidity of nonsteady gradient Kähler–Ricci solitons with constant scalar curvature.* arXiv:2609.29359.

References

- [5] **Horácio, M. A. R. M.** *On Ricci solitons whose level hypersurfaces have parallel second fundamental form.* arXiv:2610.11209.
- [6] **Aubin, T.** *Équations du type Monge–Ampère sur les variétés kählériennes compactes.* Bull. Sci. Math. (2) **102** (1978), 63–95.
- [7] **Berger, M.** *Sur les groupes d’holonomie homogène des variétés à connexion affine et des variétés riemanniennes.* Bull. Soc. Math. France **83** (1955), 279–330.
- [8] **Berman, R. J.** *K-polystability of $\mathbb{Q}$ -Fano varieties admitting Kähler–Einstein metrics.* Invent. Math. **203** (2016), 973–1025.
- [9] **Besse, A. L.** *Einstein manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **10**, Springer, 1987.

References

- [10] **Bolton, J.** *Transnormal systems*. Quart. J. Math. Oxford Ser. (2) **24** (1973), 385–395.
- [11] **Bryant, R.** Answer to *Integrability of direct sum of distinct Ricci eigendistributions*. MathOverflow, question 503724 (2025).
- [12] **Cartan, É.** *Familles de surfaces isoparamétriques dans les espaces à courbure constante*. Ann. Mat. Pura Appl. **17** (1938), 177–191.
- [13] **Calabi, E.** *On Kähler manifolds with vanishing canonical class*. In: Algebraic Geometry and Topology, a symposium in honor of S. Lefschetz, Princeton Univ. Press (1957), 78–89.
- [14] **Cao, H.-D.** *Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds*. Invent. Math. **81** (1985), 359–372.

References

- [15] **Cao, H.-D.** *Existence of gradient Kähler–Ricci solitons*. Elliptic and parabolic methods in geometry (Minneapolis, 1994), A K Peters (1996), 1–16.
- [16] **Chen, X.; Donaldson, S.; Sun, S.** *Kähler–Einstein metrics on Fano manifolds, I, II, III*. J. Amer. Math. Soc. **28** (2015), 183–197, 199–234, 235–278.
- [17] **Chen, Y.; Li, T.** *Complete self-shrinkers with bounded the second fundamental form in $\mathbb{R}^{n+1}$* . arXiv:2306.10343.
- [18] **Cheng, Q.-M.; Li, Z.; Wei, G.** *Complete self-shrinkers with constant norm of the second fundamental form*. arXiv:2003.11464.
- [19] **Cheng, Q.-M.; Li, Z.; Wei, G.** *A classification of complete 3-dimensional self-shrinkers in the Euclidean space $\mathbb{R}^4$* . arXiv:2204.11386.

References

- [20] Cheng, Q.-M.; Ogata, S. *2-dimensional complete self-shrinkers in $\mathbb{R}^3$* . arXiv:1504.02225.
- [21] Chern, S.-S.; do Carmo, M.; Kobayashi, S. *Minimal submanifolds of a sphere with second fundamental form of constant length*. Functional Analysis and Related Fields, Springer (1970), 59–75.
- [22] Cheng, X.; Zhou, D. *Rigidity of four-dimensional gradient shrinking Ricci solitons*. J. Reine Angew. Math. **802** (2023), 255–274. arXiv:2105.10744.
- [23] Derdziński, A. *Classification of certain compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor*. Math. Z. **172** (1980), 273–280.

References

- [24] Ding, Q.; Xin, Y. L. *The rigidity theorems of self-shrinkers*. Trans. Amer. Math. Soc. **366** (2014), 5067–5085.
- [25] Eminenti, M.; La Nave, G.; Mantegazza, C. *Ricci solitons: the equation point of view*. Manuscripta Math. **127** (2008), 345–367.
- [26] Feldman, M.; Ilmanen, T.; Knopf, D. *Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons*. J. Differential Geom. **65** (2003), 169–209.
- [27] Fernández-López, M.; García-Río, E. *Rigidity of shrinking Ricci solitons*. Math. Z. **269** (2011), 461–466.

References

- [28] **Fernández-López, M.; García-Río, E.** *On gradient Ricci solitons with constant scalar curvature.* Proc. Amer. Math. Soc. **144** (2016), 369–378. arXiv:1409.3359.
- [29] **Ge, J.; Tang, Z.** *Geometry of isoparametric hypersurfaces in Riemannian manifolds.* Asian J. Math. **18** (2014), 117–126. arXiv:1006.2577.
- [30] **Gromov, M.** *Spaces and questions.* GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal., Special Volume, Part I (2000), 118–161.
- [31] **Hamilton, R. S.** *Three-manifolds with positive Ricci curvature.* J. Differential Geom. **17** (1982), 255–306.
- [32] **Kim, J.** *On a classification of 4-d gradient Ricci solitons with harmonic Weyl curvature.* J. Geom. Anal. **27** (2017), 986–1012.

References

- [33] **Kim, J.** *Classification of gradient Ricci solitons with harmonic Weyl curvature.* J. Geom. Anal. **35** (2025), no. 5, 139.
- [34] **Kim, J.** *Four-dimensional gradient almost Ricci solitons with harmonic Weyl curvature.* Math. Nachr. **294** (2021), 774–793.
- [35] **Koiso, N.** *On rotationally symmetric Hamilton's equation for Kähler–Einstein metrics.* Adv. Stud. Pure Math. **18-I** (1990), 327–337.
- [36] **Li, H.; Wei, Y.** *Classification and rigidity of self-shrinkers in the mean curvature flow.* J. Math. Soc. Japan **66** (2014), 709–734.
- [37] **Li, Z.; Wang, R.; Wei, G.** *The rigidity theorem for complete Lagrangian self-shrinkers.* J. Geom. Anal. **35** (2025), Paper No. 69.

References

- [38] Li, F.; Ou, J.; Qu, Y.; Wu, G. *Rigidity of five-dimensional shrinking gradient Ricci solitons with constant scalar curvature*. arXiv:2411.10712.
- [39] Li, F.; Ou, J.; Qu, Y.; Wu, G. *Rigidity of five-dimensional shrinking gradient Ricci solitons*. arXiv:2506.00887.
- [40] Li, F.; Qu, Y.; Wu, G. *Rigidity of shrinking gradient Ricci soliton with constant scalar curvature*. arXiv:2608.20040.
- [41] Munteanu, O.; Sesum, N. *On gradient Ricci solitons*. J. Geom. Anal. **23** (2013), 539–561.
- [42] Münzner, H. F. *Isoparametrische Hyperflächen in Sphären*. Math. Ann. **251** (1980), 57–71.

References

- [43] **Ou, J.; Qu, Y.; Wu, G.** *Some rigidity results on shrinking gradient Ricci soliton.* J. Geom. Anal. **35** (2025), no. 10, Paper No. 311. arXiv:2411.06395.
- [44] **Petersen, P.; Wylie, W.** *Rigidity of gradient Ricci solitons.* Pacific J. Math. **241** (2009), 329–345.
- [45] **Pigola, S.; Rigoli, M.; Rimoldi, M.; Setti, A. G.** *Ricci almost solitons.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), 757–799.
- [46] **Reckziegel, H.; Schaaf, M.** *De Rham decomposition of netted manifolds.* Results Math. **35** (1999), 175–191.
- [47] **Schoen, R.** *Conformal deformation of a Riemannian metric to constant scalar curvature.* J. Differential Geom. **20** (1984), 479–495. Completes the work of Yamabe, Trudinger and Aubin.

References

- [48] Segre, B. *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni*. Atti Accad. Naz. Lincei Rend. **27** (1938), 203–207.
- [49] Wang, Q.-M. *Isoparametric functions on Riemannian manifolds. I*. Math. Ann. **277** (1987), 639–646.
- [50] Smoczyk, K. *Self-shrinkers of the mean curvature flow in arbitrary codimension*. Int. Math. Res. Not. **2005**, no. 48, 2983–3004.
- [51] Wang, C.; Wu, G. *A note on rigidity of shrinking gradient Ricci solitons with constant scalar curvature*. Results Math. **81** (2026), Paper No. 151. arXiv:2604.23939.

- [52] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I.* Comm. Pure Appl. Math. **31** (1978), 339–411.
- [53] Zhang, Z.-H. *On the completeness of gradient Ricci solitons.* Proc. Amer. Math. Soc. **137** (2009), 2755–2759.

Backup: The curvature of a space form

- A useful first model is a space form. If the sectional curvature is the constant c , then

$$\text{Rm}(X, Y, Z, T) = c(g(X, Z)g(Y, T) - g(X, T)g(Y, Z)).$$

- The right-hand side is quadratic in g . Polarizing it gives a bilinear operation on symmetric 2-tensors, the **Kulkarni–Nomizu product**:

$$\begin{aligned}(h \mathbin{\mathbb{A}} k)(X, Y, Z, T) &= h(X, Z)k(Y, T) + h(Y, T)k(X, Z) \\ &\quad - h(X, T)k(Y, Z) - h(Y, Z)k(X, T),\end{aligned}$$

and then $\text{Rm} = \frac{c}{2} g \mathbin{\mathbb{A}} g$.

- The same operation reconstructs the part of Rm determined by the Ricci tensor. In dimension three, this is all of Rm . From dimension four on, something is left over.

Backup: Cartan, Chern, Münzner

- **Cartan [12]**: in a space form, Riccati determines successive power sums of the shape operator from the mean curvature of parallel levels.
- **Chern [21]**: for closed minimal hypersurfaces of a sphere with constant scalar curvature, the conjecture asks for a discrete set of possible values of $|S|^2$. A stronger form asks for isoparametricity.
- **Münzner [42]**: an isoparametric family in the sphere has 1, 2, 3, 4 or 6 distinct principal curvatures.

Limits of our analogy: the soliton levels need not be minimal and the ambient need not be a space form. We already know $R = \lambda \dim N$; the missing information is geometric.

Backup: The same picture for self-shrinkers: three curvature quantities

In the theorem, no properness, curvature bound or volume growth is assumed. Under bounded geometry, that $\nabla^\perp \mathbf{H} = 0$ forces the splitting goes back to Smoczyk [50].

- With A the second fundamental form, the Gauss equation gives the scalar curvature of M as $|\mathbf{H}|^2 - |A|^2$: if any two of the scalar curvature, $|A|^2$ and $|\mathbf{H}|^2$ are constant, so is the third.
- Constant $|A|^2$ is an active line of its own. A conjecture of Q.-M. Cheng and G. Wei asks that complete self-shrinkers in $\mathbb{R}^{n+1}$ with constant $|A|^2$ be $\mathbb{R}^n$, $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ or $\mathbb{S}^n(\sqrt{n})$: known for $n = 2$ [24, 20], for $n = 3$ with $\text{tr } A^3$ or $\text{tr } A^4$ constant [18, 19], and, in a preprint, for proper ones [17].

Backup: cases of the self-shrinker conjecture

- **Known** (_____, **in preparation**): codimension one; $k \in \{0, n\}$; Hamiltonian stationary Lagrangians. **Open**: codimension ≥ 2 , $1 \leq k \leq n - 1$.
- **Literature**, as far as we know: no counterexample; proved for compact M [36] and with parallel principal normal and bounded geometry [50]. **Surfaces**: complete Lagrangian self-shrinkers in $\mathbb{R}^4$ with constant $|\mathbf{H}|^2$ and bounded $|A|$ are $\mathbb{R}^2$, $\mathbb{S}^1 \times \mathbb{R}$, $\mathbb{S}^1 \times \mathbb{S}^1$ (Li–Wang–Wei [37]); without the bound, open.